Appendix A

Cartesian tensors and their properties

In this chapter the most important properties of Cartesian tensors, insofar as they are needed in the theory of the radiation and scattering of acoustic waves in fluids, elastic waves in solids and electromagnetic waves are discussed.

A.1 Introduction

According to the postulates of continuum physics the only quantities that are allowed to occur in the laws of macroscopic physics are tensorial quantities, or *tensors*. Mathematically, tensors are defined through their transformation properties when changing from one reference frame to another in the space where they are observed. Hence, the structure of a tensor is intimately related to the geometrical structure of the relevant observation space. Physically, the relevant definition expresses that the laws of physics governing the phenomena that are observed, are independent of the reference frames which different observers might employ to specify their position in space. One can therefore also express the postulate by saying that the laws of macroscopic physics are "geometric in nature" or that any macroscopic physical quantity is a "geometrical object". To handle this concept mathematically, a *tensor calculus* has been developed for manipulating tensors algebraically and applying to them the operations of differentiation and integration. For this purpose, a notation (the so-called *subscript notation*) is in use that is not only the most compact and convenient one to use when writing the pertaining expressions and equations, but that also almost effortlessly leads to the corresponding statements in any of the high-level computer programming languages. The reason behind this is that in any fixed coordinate system the arithmetic value of a tensorial quantity is represented by an array (subscripted constant or variable), the dimensions of which are obvious from the notation. Altogether, this is sufficient to motivate the use of the subscript notation rather than the so-called "direct notation" that was developed in earlier times. In the direct notation, the type of tensor one is dealing with is not distinguished by its number of subscripts (as is the case in the subscript notation), but by the use of special symbols, such as light-face symbols for scalars, Latin bold-face symbols for vectors, Greek bold-face symbols for tensors of rank two, Gothic capitals for tensors in space–time in the theory of relativity, etc. With the direct notation, the algebraic operations, especially the different multiplications, are cumbersome to express. Although in our further considerations only tensors in three-dimensional Euclidean space will be needed, we shall introduce the concept of a tensor in a space of N dimensions, where N is an integer and takes the values $N \geqslant 1$, and make a few remarks about affine space, which is more general than Euclidean space.

A.2 The summation convention

The summation convention is a shorthand notation used to indicate the sum of products of arithmetic arrays. The arrays under consideration have either the same or different dimensions, but the bounds on their subscripts are all the same, $1{:}N$ let us say, where N is an integer and takes the values $N \geqslant 1$. The convention prescribes that to every *repeated subscript* in a product of two or more arrays the values $1,2,\dots,N$ are assigned successively, while after each assignment the result is added to the previous one. Repeated subscripts are also denoted as *dummy subscripts*. *Non-repeated* subscripts are denoted as *free subscripts*; these are merely successively assigned the values $1,\dots,N$. For example, let a_m and b_m, where $m \in \{1,\dots,N\}$, denote one-dimensional arrays and let $c_{m,n}$, where $m \in \{1,\dots,N\}$ and $n \in \{1,\dots,N\}$, be a two-dimensional array. Then

$$a_m b_m \quad \text{stands for} \quad \sum_{m=1}^{N} a_m b_m \,, \tag{A.2-1}$$

$$b_m c_{m,n} \quad \text{stands for} \quad \sum_{m=1}^{N} b_m c_{m,n} \quad \text{for} \quad n \in \{1,\dots,N\} \,, \tag{A.2-2}$$

$$a_m b_n c_{m,n} \quad \text{stands for} \quad \sum_{m=1}^{N} \sum_{n=1}^{N} a_m b_n c_{m,n} \,. \tag{A.2-3}$$

For ease of manipulation the summation convention is also extended to repeated subscripts in a single array with dimensions greater than or equal to two. For example, let, $d_{m,n,p}$, with $m \in \{1,\dots,N\}$, $n \in \{1,\dots,N\}$ and $p \in \{1,\dots,N\}$, be a three-dimensional array. Then

$$d_{m,m,p} \quad \text{stands for} \quad \sum_{m=1}^{N} d_{m,m,p} \quad \text{for} \quad p \in \{1,\dots,N\} \,. \tag{A.2-4}$$

In mathematical physics the arithmetic arrays are, in a fixed coordinate system, the arithmetic representation of tensors. In that case, the subscript range N equals the number of dimensions of the space in which the observations are made.

Exercises

Exercise A.2-1

Rewrite the following expressions by employing the summation convention:

(a) $\sum_{m=1}^{N} a_m c_{m,n}$;

(b) $\sum_{m=1}^{N}\sum_{n=1}^{N}\varepsilon_{i,m,n}a_m b_n$.

Answer: (a) $a_m c_{m,n}$; (b) $\varepsilon_{i,m,n}a_m b_n$.

Exercise A.2-2

Is it allowed to use in the summation convention a certain subscript more than twice in one and the same product?

Answer: No!

Exercise A.2-3

What are the dummy subscripts in (a) Equation (A.2-1); (b) Equation (A.2-2); (c) Equation (A.2-3); (d) Equation (A.2-4)?

Answers: (a) m; (b) m; (c) m,n; (d) m.

Exercise A.2-4

What are the free subscripts in (a) Equation (A.2-1); (b) Equation (A.2-2); (c) Equation (A.2-3); (d) Equation (A.2-4)?

Answers: (a) None; (b) n; (c) none; (d) p.

A.3 Cartesian reference frames in affine space and in Euclidean space

The elements of N-dimensional *affine space* $\mathcal{A}^N$, with $N \geqslant 1$, are the real-valued ordered sequences $\{x_1,...,x_N\}$ that specify a position (or a point) in this space with respect to a given Cartesian reference frame (i.e. a reference frame with a fixed location and orientation in space) with the *origin* O and the N *base vectors* $\{\boldsymbol{e}(1),...,\boldsymbol{e}(N)\}$. The base vectors are linearly independent, but otherwise have no mutual relationship. In particular, we have no means of comparing the "length" of $\boldsymbol{e}(m)$ with the "length" of $\boldsymbol{e}(n)$ if $m \neq n$, since the scale along $\boldsymbol{e}(m)$ has, in general, nothing to do with the scale along $\boldsymbol{e}(n)$ if $m \neq n$ (Figure A.3-1).

The *position vector* $\boldsymbol{x}$ of the point with coordinates $\{x_1,...,x_N\}$ is now introduced as the following linear combination of the base vectors:

$$\boldsymbol{x} = \sum_{m=1}^{N} x_m \boldsymbol{e}(m) \,. \qquad \text{(A.3-1)}$$

Furthermore, the equation of a *straight line* $\mathcal{L}$ in this space is, by definition, given by

$$\mathcal{L} = \{\boldsymbol{x}\in\mathcal{A}^N; x_m = s a_m + b_m, s\in\mathcal{R}\} \,, \qquad \text{(A.3-2)}$$

where s is a real-valued parameter that changes along $\mathcal{L}$, b_m specifies the point of $\mathcal{L}$ where $s = 0$ and a_m specifies the orientation of $\mathcal{L}$. Now, the coordinates of the origin O are given by $\{0,...,0\}$ and the coordinates of the end points of the base vectors $\boldsymbol{e}(1),...,\boldsymbol{e}(N)$ are given by

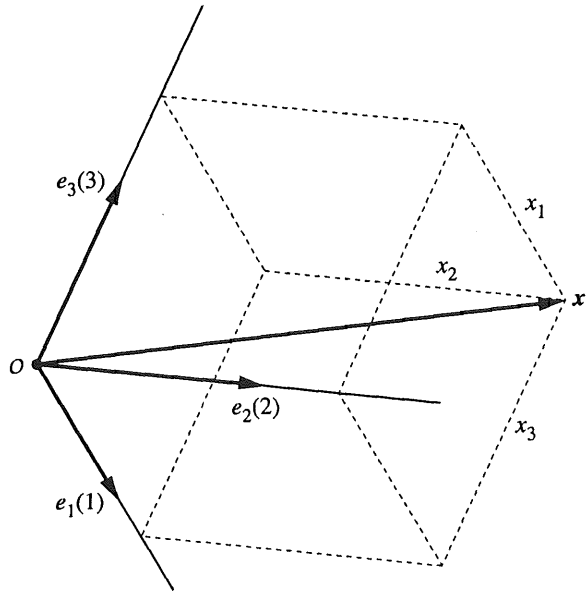

Figure A.3-1 Cartesian reference frame $\{O;\boldsymbol{e}(1),\boldsymbol{e}(2),\boldsymbol{e}(3)\}$, Cartesian coordinates $\{x_1,x_2,x_3\}$, and position vector $\boldsymbol{x}$ in three-dimensional affine space $\mathcal{A}^3$.

$\{1,0,...,0\}$, $\{0,1,...,0\}$,...,$\{0,0,...,1\}$, respectively. The straight lines through the origin and the end points of the base vectors are called the *coordinate axes* of the Cartesian reference frame.

Next, the numbers $\{x_{1,...,}x_N\}$ are subjected to a *general linear transformation* of the type

$$x_p' = \alpha_{p,m}(x_m - \lambda_m) \, , \tag{A.3-3}$$

which transformation is also known as a *point transformation*. (When $\lambda_m = 0$ for $m \in \{1,...,N\}$, the transformation is called a *homogeneous* linear transformation; such a transformation is often just called a "linear transformation".) Substitution of Equation (A.3-2) in the right-hand side of Equation (A.3-3) shows that when the numbers $\{x_1',...,x_N'\}$ are interpreted as coordinates in a primed Cartesian reference frame, the result is again a representation of a straight line; i.e. under the transformation of Equation (A.3-3) straight lines transform into straight lines. Furthermore, the numbers $\{x_1',...,x_N'\}$ can be interpreted as the Cartesian coordinates of the same observer as before, but now with respect to the primed reference frame (Figure A.3-2).

This interpretation holds for an arbitrary two-dimensional, real-valued array $\alpha_{p,m}$ and an arbitrary one-dimensional, real-valued array λ_m. Since $x_m = \lambda_m$ corresponds to $x_p' = 0$, the transformation inverse to Equation (A.3-3) can be written as

$$x_m = \lambda_m + A_{m,p} x_p' \, . \tag{A.3-4}$$

Substituting Equation (A.3-3) in the right-hand side of Equation (A.3-4) and requiring identity in $x_m - \lambda_m$, the following condition should hold:

$$A_{m,p}\alpha_{p,n} = \delta_{m,n} \, , \tag{A.3-5}$$

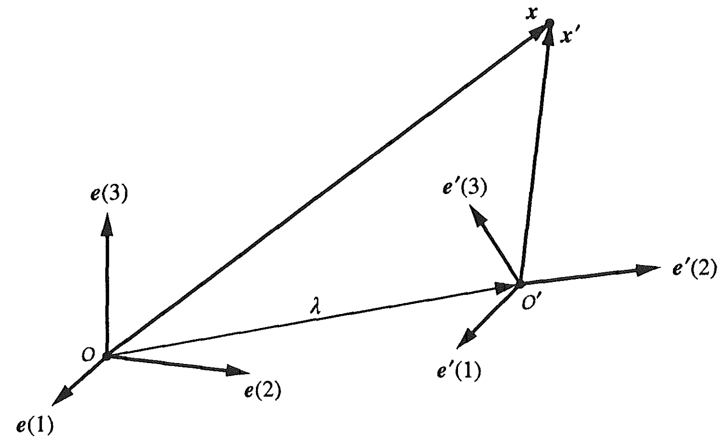

Figure A.3-2 Change of reference frame in three-dimensional affine space and the point with position vector $\boldsymbol{x}$ in $\{O;\boldsymbol{e}(1),\boldsymbol{e}(2),\boldsymbol{e}(3)\}$ and position vector $\boldsymbol{x}'$ in $\{O';\boldsymbol{e}'(1),\boldsymbol{e}'(2),\boldsymbol{e}'(3)\}$.

where $\delta_{m,n}$ denotes the Kronecker symbol: $\delta_{1,1} = \ldots = \delta_{N,N} = 1$, and $\delta_{m,n} = 0$ if $m \neq n$. On the assumption that $\alpha_{p,m}$ has a unique inverse, denoted by $\bar{\alpha}_{m,p}$, i.e. $\bar{\alpha}_{m,p}\alpha_{p,n} = \delta_{m,n}$, Equation (A.3-5) leads to

$$A_{m,p} = \bar{\alpha}_{m,p} \, . \qquad \text{(A.3-6)}$$

When $\alpha_{p,m} = \delta_{p,m}$ and $\lambda_m \neq 0$, the transformation given by Equation (A.3-3) is a pure translation. When $\alpha_{p,m} \neq 0$ and $\lambda_m = 0$, the two origins coincide and the transformation given by Equation (A.3-3) amounts to a rotation plus an extension along the coordinate axes.

The affine space becomes a *metrical space* when, corresponding to any two points with position vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, an appropriate distance function $d = d(\boldsymbol{x},\boldsymbol{y})$ is introduced. This distance function should have the following properties: (a) $d(\boldsymbol{x},\boldsymbol{y})$ is real-valued; (b) $d(\boldsymbol{x},\boldsymbol{x}) = 0$; (c) $d(\boldsymbol{x},\boldsymbol{y}) > 0$ if $\boldsymbol{x} \neq \boldsymbol{y}$; (d) $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$; (e) for any three points with position vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, the inequality $d(\boldsymbol{x},\boldsymbol{z}) \leqslant d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$ ("triangle inequality") holds (Figure A.3-3).

One distance function that satisfies these requirements is the *Riemannian distance function*

$$d_{\mathrm{R}}(\boldsymbol{x},\boldsymbol{y}) = [g_{m,n}(x_m - y_m)(x_n - y_n)]^{1/2}, \qquad \text{(A.3-7)}$$

where the metrical coefficients $g_{m,n}$ are such that the right-hand side is positive definite, i.e. it is positive for any two points with different position vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. (For the proof that Equation (A.3-7) defines a proper distance function, see Exercise A.3-5.) A space with the metric given in Equation (A.3-7) is called a *Riemannian space*. (Note that of the coefficients $\{g_{m,n} \in \mathcal{R}; m = 1,\ldots,N; n = 1,\ldots,N\}$ in Equation (A.3-7) only the sums $g_{m,n} + g_{n,m}$ contribute to the right-hand side. Therefore, $g_{m,n}$ can, without loss of generality, be assumed to be symmetric right from the beginning, i.e. $g_{m,n} = g_{n,m}$.) Equation (A.3-7) shows that, as far as its metrical properties are concerned, Riemannian space is anisotropic, i.e. equal differences in coordinates in different directions correspond to different lengths.

The Riemannian space reduces to a *Euclidean space*, denoted by $\mathcal{R}^N$, when the distance function reduces to the Euclidean one $d_{\mathrm{E}}(\boldsymbol{x},\boldsymbol{y})$, i.e. the one for which $g_{m,n} = \delta_{m,n}$. Hence, the *Euclidean distance function* is given by

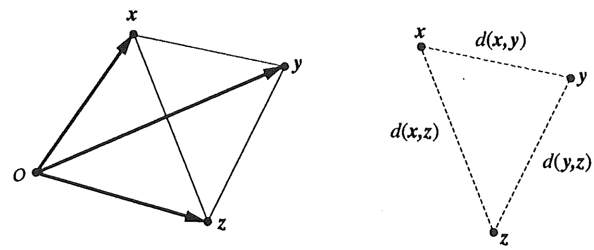

Figure A.3-3 Distance function $d(...)$ in metrical space and triangle inequality $d(x,z) \leqslant d(x,y) + d(y,z)$.

$$d_{\mathrm{E}}(x,y) = [(x_m - y_m)(x_m - y_m)]^{1/2}. \tag{A.3-8}$$

(For the proof that Equation (A.3-8) defines a proper distance function, see Exercise A.3-6.) Equation (A.3-8) shows that, as far as its metrical properties are concerned, Euclidean space is isotropic. Henceforth, we shall denote the base vectors in Euclidean space by $\{\boldsymbol{i}(1),...,\boldsymbol{i}(N)\}$.

Now, reconsider the coordinate transformation given in Equation (A.3-3). For a pure translation as well as for a pure rotation in Euclidean space the distance function $d_{\mathrm{E}}(x,y)$ for any two points should be invariant. For a pure translation this is obvious, since in the differences $x_m - y_m$ occurring in the right-hand side of Equation (A.3-8) the values of λ_m for $m \in \{1,..., N\}$ drop out. To investigate under what conditions the coefficients $\alpha_{p,m}$ are representative of a pure rotation, we substitute (Figure A.3-4)

$$x_p' = \alpha_{p,m}(x_m - \lambda_m) \tag{A.3-9}$$

and

$$y_p' = \alpha_{p,n}(y_n - \lambda_n) \tag{A.3-10}$$

where $x_n \neq y_n$ for $n = 1,...,N$, in the expression for $d_{\mathrm{E}}(x',y')$. The condition $d_{\mathrm{E}}(x',y') = d_{\mathrm{E}}(x,y)$ then leads to

$$(x_p' - y_p')(x_p' - y_p') = (x_m - y_m)(x_m - y_m)\,, \tag{A.3-11}$$

or, using Equations (A.3-9) and (A.3-10) and the properties of the Kronecker symbol,

$$\alpha_{p,m}(x_m - y_m)\alpha_{p,n}(x_n - y_n) = \delta_{m,n}(x_m - y_m)(x_n - y_n)\,. \tag{A.3-12}$$

Requiring identity in $y_n - x_n$, the following condition is obtained:

$$\alpha_{p,m}\alpha_{p,n} = \delta_{m,n}\,. \tag{A.3-13}$$

On the other hand, Equations (A.3-5) and (A.3-6) can be rewritten as

$$\bar{\alpha}_{m,p}\alpha_{p,n} = \delta_{m,n}\,. \tag{A.3-14}$$

Hence, for a change of reference frame in Euclidean space, the coefficients $\alpha_{m,p}$ must satisfy the condition

$$\bar{\alpha}_{m,p} = \alpha_{p,m}\,. \tag{A.3-15}$$

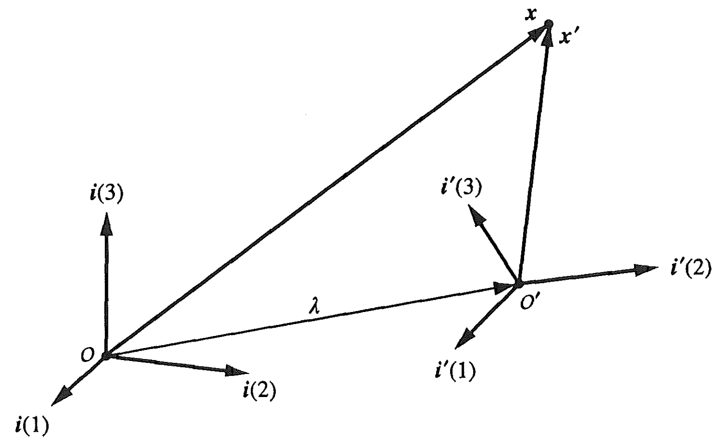

Figure A.3-4 Change of reference frame in three-dimensional Euclidean space $\mathcal{R}^3$ and the point with position vector $\boldsymbol{x}$ in $\{O;\boldsymbol{i}(1),\boldsymbol{i}(2),\boldsymbol{i}(3)\}$ and position vector $\boldsymbol{x}'$ in $\{O';\boldsymbol{i}'(1),\boldsymbol{i}'(2),\boldsymbol{i}'(3)\}$.

In the terminology of linear algebra, Equation (A.3-15) expresses that the *matrix of coefficients* $\alpha_{p,n}$ is *orthogonal* (i.e. its inverse is equal to its transpose). Equation (A.3-13) implies for $m = n$ that

$$\sum_{p=1}^{N} \alpha_{p,m}^2 = 1 \quad \text{for} \quad m = 1,\dots,N, \tag{A.3-16}$$

which shows that each column in the matrix of transformation coefficients is normalised to unity in the sense of linear algebra, while for $m \neq n$ it implies that

$$\sum_{p=1}^{N} \alpha_{p,m}\alpha_{p,n} = 0 \quad \text{for} \quad m = 1,\dots,N; \quad n = 1,\dots,N; \quad m \neq n, \tag{A.3-17}$$

which shows that any two different columns are mutually orthogonal in the sense of linear algebra. Hence, the columns of the matrix of transformation coefficients in a Euclidean space form an orthonormal system in the sense of linear algebra.

From Equations (A.3-4), (A.3-6) and (A.3-15) it also follows that for a change in reference frame in a Euclidean space the transformation inverse to Equations (A.3-9) and (A.3-10) is given by

$$x_m - \lambda_m = \alpha_{p,m}x_p' \tag{A.3-18}$$

and

$$y_m - \lambda_m = \alpha_{q,m}y_q' \, . \tag{A.3-19}$$

Substituting Equations (A.3-18) and (A.3-19) in Equation (A.3-11) and requiring identity in $y_p' - x_p'$, it follows that

$$\alpha_{p,m}\alpha_{q,m} = \delta_{p,q} \, . \tag{A.3-20}$$

Equation (A.3-20) implies for $p = q$ that

$$\sum_{m=1}^{N} a_{p,m}^2 = 1 \quad \text{for} \quad p = 1,\ldots,N, \tag{A.3-21}$$

which shows that each row in the matrix of transformation coefficients is normalised to unity in the sense of linear algebra, while for $p \neq q$ it implies that

$$\sum_{m=1}^{N} a_{p,m}a_{q,m} = 0 \quad \text{for} \quad p = 1,\ldots,N; \quad q = 1,\ldots,N; \quad p \neq q, \tag{A.3-22}$$

which shows that any two different rows in the matrix of transformation coefficients are mutually orthogonal in the sense of linear algebra. Hence, also the rows of the matrix of transformation coefficients in a Euclidean space form an orthonormal system in the sense of linear algebra.

In view of these properties the transformations defined by Equations (A.3-9) and (A.3-18) are called *orthogonal* transformations.

Since Euclidean space is isotropic, the base vectors all have the same length, which is taken as the unit of length, while their mutual orientation is the same. To distinguish them from the more general base vectors of the affine space, we have chosen to denote them by the separate symbols $\{\boldsymbol{i}(1),\ldots,\boldsymbol{i}(N)\}$ (Figure A.3-4). In Euclidean space one further introduces the cosine of the angle included between any two position vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ through the formula

$$[d_E(\boldsymbol{x},\boldsymbol{y})]^2 = [d_E(\boldsymbol{x},\boldsymbol{0})]^2 + [d_E(\boldsymbol{y},\boldsymbol{0})]^2 - 2d_E(\boldsymbol{x},\boldsymbol{0})d_E(\boldsymbol{y},\boldsymbol{0}) \cos(\boldsymbol{x},\boldsymbol{y}). \tag{A.3-23}$$

Application of this formula to the vectors $\boldsymbol{x} = \boldsymbol{i}(m)$ and $\boldsymbol{y} = \boldsymbol{i}(m)$ yields, in view of $d_E[\boldsymbol{i}(m),\boldsymbol{i}(m)] = 0$ and $d_E[\boldsymbol{i}(m),\boldsymbol{0}] = 1$,

$$\cos[\boldsymbol{i}(m),\boldsymbol{i}(m)] = 1. \tag{A.3-24}$$

Application of Equation (A.3-23) to the vectors $\boldsymbol{x} = \boldsymbol{i}(m)$ and $\boldsymbol{y} = \boldsymbol{i}(n)$, with $m \neq n$, yields, in view of $d_E[\boldsymbol{i}(m),\boldsymbol{i}(n)] = 2^{1/2}$ for $m \neq n$, $d_E[\boldsymbol{i}(m),\boldsymbol{0}] = 1$ and $d_E[\boldsymbol{i}(n),\boldsymbol{0}] = 1$:

$$\cos[\boldsymbol{i}(m),\boldsymbol{i}(n)] = 0 \qquad \text{if} \quad m \neq n. \tag{A.3-25}$$

Equations (A.3-24) and (A.3-25) show that the base vectors of Euclidean space are, geometrically speaking, each of unit length and mutually perpendicular (the latter since $\cos(\pi/2) = 0$).

Next, Equation (A.3-23) is applied to the vectors $\boldsymbol{x} = \boldsymbol{i}'(m)$ and $\boldsymbol{y} = \boldsymbol{i}(n)$ with m and n fixed. Now, $\boldsymbol{i}'(m)$ is oriented from the point with coordinates

$$\{x'_p = 0; p = 1,\ldots,N\}$$

to the point with coordinates

$$\{x'_p = \delta_{p,m}; p = 1,\ldots,N; m = 1,\ldots,N\},$$

while $\boldsymbol{i}(n)$ is oriented from the point with coordinates

$$\{y_q = 0; q = 1,\ldots,N\}$$

to the point with coordinates

$$\{y_q = \delta_{q,n}; q = 1,\ldots,N; n = 1,\ldots,N\}.$$

To construct a relationship for the angle between $\boldsymbol{i}'(m)$ and $\boldsymbol{i}(n)$ we let the two origins coincide, i.e. we take $\lambda_m = 0$ in Equation (A.3-9). Expressing everything in terms of the unprimed coordinates (see Equation (A.3-9)), we then have

$$[d_{\mathrm{E}}(\boldsymbol{x},\boldsymbol{0})]^2 = \sum_{q=1}^{N} \alpha_{m,q}^2 = 1 ,$$

where Equations (A.3-18) and (A.3-21) have been used,

$$[d_{\mathrm{E}}(\boldsymbol{y},\boldsymbol{0})]^2 = 1 \quad \text{and} \quad [d_{\mathrm{E}}(\boldsymbol{x},\boldsymbol{y})]^2 = \sum_{q=1}^{N} (\alpha_{m,q} - \delta_{q,n})(\alpha_{m,q} - \delta_{q,n}) = 2 - 2\alpha_{m,n} .$$

Substitution of these results in Equation (A.3-23) leads to

$$\alpha_{m,n} = \cos(\boldsymbol{i}_m', \boldsymbol{i}_n) . \qquad \text{(A.3-26)}$$

This relationship expresses the geometrical meaning of the transformation coefficient $\alpha_{m,n}$.

Exercises

Exercise A.3-1

Let A denote the linear orthogonal transformation with the coefficients $\alpha_{m,p}$ and B the linear orthogonal transformation with the coefficients $\beta_{n,r}$. Show that the linear transformations AB and BA are also orthogonal. (*Hint*: For the proof of the first result write $x_m = \alpha_{m,p} y_p, y_p = \beta_{p,n} z_n$ and $x_m = \gamma_{m,n} z_n$ with $\gamma_{m,n} = \alpha_{m,p}\beta_{p,n}$, and show that $\gamma_{m,r}\gamma_{m,s} = \delta_{r,s}$ if the relations $\alpha_{m,p}\alpha_{m,q} = \delta_{p,q}$ and $\beta_{n,r}\beta_{n,s} = \delta_{r,s}$ hold. For the proof of the second result write $x_m = \beta_{m,p} y_p, y_p = \alpha_{p,n} z_n$ and $x_m = \gamma_{m,n} z_n$ with $\gamma_{m,n} = \beta_{m,p}\alpha_{p,n}$, and show that $\gamma_{m,r}\gamma_{m,s} = \delta_{r,s}$ if the relations $\alpha_{m,p}\alpha_{m,q} = \delta_{p,q}$ and $\beta_{n,r}\beta_{n,s} = \delta_{r,s}$ hold.)

Exercise A.3-2

Show that when a change of reference frame in affine space is made, the relations between the base vectors $\{\boldsymbol{e}'(p); p = 1,\dots,N\}$ and $\{\boldsymbol{e}(m); m = 1,\dots,N\}$ are $\boldsymbol{e}'(p) = A_{m,p}\boldsymbol{e}(m)$; $\boldsymbol{e}(m) = \alpha_{p,m}\boldsymbol{e}'(p)$. (*Hint*: Let the two origins coincide, observe for the first result that $\boldsymbol{e}'(p)$ is a position vector with end point coordinates $\{x_q' = \delta_{q,p}; q = 1,\dots,N; p \text{ fixed}\}$ in the primed reference frame and, hence, end point coordinates $\{A_{m,q}x_q' = A_{m,q}\delta_{q,p} = A_{m,p}; m = 1,\dots,N; p \text{ fixed}\}$ in the unprimed reference frame, while for the second result observe that $\boldsymbol{e}(m)$ is a position vector with end point coordinates $\{x_n = \delta_{n,m}; n = 1,\dots,N; m \text{ fixed}\}$ in the unprimed reference frame and, hence, end point coordinates $\{\alpha_{p,n}x_n = \alpha_{p,n}\delta_{n,m} = \alpha_{p,m}; p = 1,\dots,N; m \text{ fixed}\}$ in the primed reference frame, and employ for both cases the standard bold-face notation for vectors.)

Exercise A.3-3

Show that when a change of reference frame in Euclidean space is made, the relations between the base vectors $\{\boldsymbol{i}'(p); p = 1,\dots,N\}$ and $\{\boldsymbol{i}(m); m = 1,\dots,N\}$ are $\boldsymbol{i}'(p) = \alpha_{m,p}\boldsymbol{i}(m)$; $\boldsymbol{i}(m) =$

$\alpha_{p,m}\boldsymbol{i}'(p)$. (*Hint*: Let the two origins coincide, observe for the first result that $\boldsymbol{i}'(p)$ is a position vector with end point coordinates $\{x_q' = \delta_{q,p};\ q = 1,\ldots,N;\ p \text{ fixed}\}$ in the primed reference frame and, hence, end point coordinates $\{A_{m,q}x_q' = \alpha_{m,q}\delta_{q,p} = \alpha_{p,m};\ m = 1,\ldots,N;\ p \text{ fixed}\}$ in the unprimed reference frame, while for the second result observe that $\boldsymbol{i}(m)$ is a position vector with end point coordinates $\{x_n = \delta_{n,m};\ n = 1,\ldots,N;\ m \text{ fixed}\}$ in the unprimed reference frame and, hence, end point coordinates $\{\alpha_{p,n}x_n = \alpha_{p,n}\delta_{n,m} = \alpha_{p,m};\ p = 1,\ldots,N;\ m \text{ fixed}\}$ in the primed reference frame, and employ for both cases the standard bold-face notation for vectors.)

Exercise A.3-4

Let the coefficients $\{g_{m,n}\in\mathcal{R};m = 1,\ldots,N;n = 1,\ldots,N\}$ with $g_{m,n} = g_{n,m}$ define a positive definite metric and let

$$\langle \boldsymbol{x},\boldsymbol{x}\rangle = g_{m,n}x_m y_n \tag{A.3-27}$$

denote the *inner product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ (note that $(\langle \boldsymbol{x},\boldsymbol{y}\rangle = \langle \boldsymbol{y},\boldsymbol{x}\rangle)$, with

$$|\boldsymbol{x}| = (g_{m,n}x_m x_n)^{1/2} \geqslant 0 \tag{A.3-28}$$

as the corresponding *induced norm* of $\boldsymbol{x}$. Prove that

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle \leqslant |\boldsymbol{x}|^2|\boldsymbol{y}|^2 \qquad \text{(Cauchy–Schwarz inequality).} \tag{A.3-29}$$

Proof: For either $\boldsymbol{x} = \boldsymbol{0}$, or $\boldsymbol{y} = \boldsymbol{0}$, or $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{y} = \boldsymbol{0}$, equality obviously holds. For $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$ observe that, for any $\lambda\in\mathcal{R}$, $|\boldsymbol{x} - \lambda\boldsymbol{y}|^2 \geqslant 0$, with $|\boldsymbol{x} - \lambda\boldsymbol{y}| = 0$ if and only if $\boldsymbol{x} = \lambda\boldsymbol{y}$ (i.e. if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent). As a consequence, $|\boldsymbol{x}|^2 - 2\lambda\langle \boldsymbol{x},\boldsymbol{y}\rangle + \lambda^2|\boldsymbol{y}|^2 \geqslant 0$, or $(\lambda|\boldsymbol{y}| - \langle \boldsymbol{x},\boldsymbol{y}\rangle/|\boldsymbol{y}|)^2 - \langle \boldsymbol{x},\boldsymbol{y}\rangle^2/|\boldsymbol{y}|)^2 + |\boldsymbol{x}|^2 \geqslant 0$. By choosing $\lambda = \langle \boldsymbol{x},\boldsymbol{y}\rangle/|\boldsymbol{y}|^2$, Equation (A.3-29) follows.

Exercise A.3-5

Prove that $d_R(\boldsymbol{x},\boldsymbol{y})$ as given by Equation (A.3-7) satisfies the conditions of a proper distance function: (a) $d_R(\boldsymbol{x},\boldsymbol{x}) = 0$; (b) $d_R(\boldsymbol{x},\boldsymbol{y}) > 0$ if $\boldsymbol{x} \neq \boldsymbol{y}$; (c) $d_R(\boldsymbol{x},\boldsymbol{y}) = d_R(\boldsymbol{y},\boldsymbol{x})$; (d) $d_R(\boldsymbol{x},\boldsymbol{z}) \leqslant d_R(\boldsymbol{x},\boldsymbol{y}) + d_R(\boldsymbol{x},\boldsymbol{z})$.

Proof: (a) Is obvious; (b) follows from the assumed positive definiteness of $g_{m,n}$; (c) is obvious; (d) follows from (see also Exercise A.3-4):

$$\begin{aligned}
g_{m,n}(x_m - z_m)(x_n - z_n) &= g_{m,n}(x_m - y_m + y_m - z_m)(x_n - y_n + y_n - z_n) \\
&= g_{m,n}(x_m - y_m)(x_n - y_n) + g_{m,n}(y_m - z_m)(y_n - z_n) \\
&\quad + g_{m,n}(x_m - y_m)(y_n - z_n) + g_{m,n}(y_m - z_m)(x_n - y_n) \\
&= |\boldsymbol{x} - \boldsymbol{y}|^2 + |\boldsymbol{y} - \boldsymbol{z}|^2 + 2\langle \boldsymbol{x} - \boldsymbol{y},\boldsymbol{y} - \boldsymbol{z}\rangle \leqslant |\boldsymbol{x} - \boldsymbol{y}|^2 \\
&\quad + |\boldsymbol{y} - \boldsymbol{z}|^2 + 2|\boldsymbol{x} - \boldsymbol{y}||\boldsymbol{y} - \boldsymbol{z}| = (|\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|)^2,
\end{aligned}$$

or

$$d_R(\boldsymbol{x},\boldsymbol{z}) \leqslant d_R(\boldsymbol{x},\boldsymbol{y}) + d_R(\boldsymbol{y},\boldsymbol{z})\,.$$

Here, $\langle \boldsymbol{x} - \boldsymbol{y},\boldsymbol{y} - \boldsymbol{z}\rangle \leqslant |\boldsymbol{x} - \boldsymbol{y}||\boldsymbol{y} - \boldsymbol{z}|$ has been used (see Equation A.3-29).

Exercise A.3-6

Prove that $d_E(x,y)$ as given by Equation (A.3-8) satisfies the conditions of a proper distance function: (a) $d_E(x,x) = 0$; (b) $d_E(x,y) > 0$ if $x \neq y$; (c) $d_E(x,y) = d_E(y,x)$; (d) $d_E(x,z) \leqslant d_E(x,y) + d_E(y,z)$.

Proof: (a) Is obvious; (b) is obvious; (c) is obvious; (d) follows from Exercise A.3-5 with $g_{m,n} = \delta_{m,n}$.

A.4 Definition of a Cartesian tensor

In discussing the definition and the properties of tensors we shall confine our considerations to Cartesian tensors in N-dimensional Euclidean space. For more general tensors and/or more general spaces, the reader is referred to textbooks on tensor analysis. We assume that in a bounded or unbounded domain in the Euclidean space under consideration a certain physical, or geometrical, quantity is defined. Mathematically, the definition of such a quantity has the following structure. To each point of the relevant domain a finite ordered sequence of numbers is assigned that corresponds in a unique way to the Cartesian coordinates of that point, the latter being taken with respect to a certain reference frame. The relevant procedure defines a *tensor*, and an element of the sequence is called a (Cartesian) *component* of it if, and only if, the following conditions are satisfied: (a) the sequence contains exactly N^K numbers, where N, with $N \geqslant 1$, is the dimension of the space under consideration and K, with $K \geqslant 0$, is a non-negative integer that is called the *rank* of the tensor; and (b) the value of the sequence in a certain reference frame is related to the value of the sequence in a different reference frame through a relationship that contains in a specific way the transformation coefficients interconnecting the two (Cartesian) reference frames. The structure of the latter relationship will be specified below.

Let the physical, or geometrical, quantity under consideration be denoted by the general symbol σ and let the corresponding sequence be written as $\sigma_{m_1,\dots,m_K}$, where the subscripts $m_1,\dots,m_K$ all run through the values of the sequence $\{1,\dots,N\}$. (Note that through these assignments just N^K numbers are generated.) Let us further consider two Cartesian reference frames, one with origin O and base vectors $\{\boldsymbol{i}(1),\dots,\boldsymbol{i}(N)\}$, and another with origin O' and base vectors $\{\boldsymbol{i}'(1),\dots,\boldsymbol{i}'(N)\}$, and let the Cartesian coordinates of a point of observation in the two reference frames be related through the general, linear, orthogonal transformation (Figure A.4-1)

$$x'_p = \alpha_{p,m}(x_m - \lambda_m) \quad \text{with} \quad \alpha_{p,m}\alpha_{p,n} = \delta_{m,n}\,. \tag{A.4-1}$$

Then, the quantity σ is called a (Cartesian) tensor if, and only if, the following relation exists between the value $\sigma'_{p_1,\dots,p_K}$ of the sequence defined with respect to the primed reference frame and the value $\sigma_{m_1,\dots,m_K}$ of the sequence defined with respect to the unprimed reference frame:

$$\sigma'_{p_1,\dots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\dots,m_K}\,, \tag{A.4-2}$$

i.e. for each subscript in the designation of the components of the tensor σ there is a corresponding factor α in the transformation law. (Note that in the product of the transformation

$$\boxed{\sigma'_{p_1,\ldots,p_K}(x') = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\ldots,m_K}(x)}$$

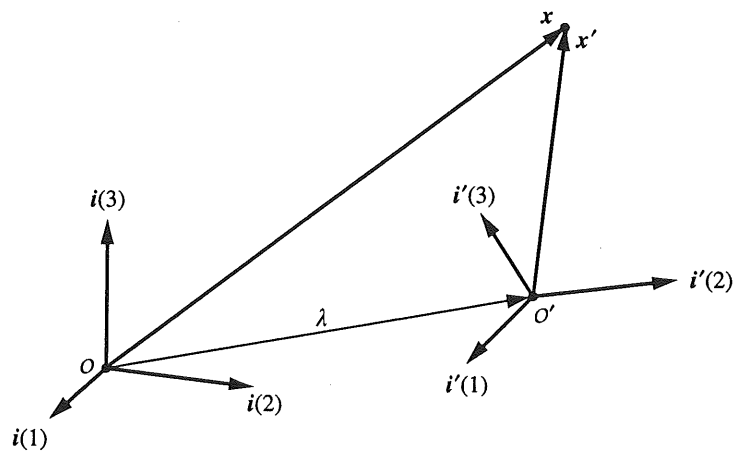

Figure A.4-1 Change of reference frame in three-dimensional Euclidean space used to define a Cartesian tensor of rank K ($K \geqslant 0$).

coefficients on the right-hand side the order of the factors is arbitrary.) In Equation (A.4-2) the use of the summation convention is understood: $\{m_1,\ldots,m_K\}$ are the dummy subscripts, $\{p_1,\ldots,p_K\}$ are the free subscripts. With regard to the coefficients $\alpha_{p,m}$, it should be noted that they satisfy the relation (see Equation (A.3-13))

$$\alpha_{p,m}\alpha_{p,n} = \delta_{m,n} , \tag{A.4-3}$$

since we are dealing with a Euclidean space. If we want to express the functional dependence on the position coordinates, we write $\sigma_{m_1,\ldots,m_K} = \sigma_{m_1,\ldots,m_K}(x_1,\ldots,x_N)$ or $\sigma_{m_1,\ldots,m_K} = \sigma_{m_1,\ldots,m_K}(x)$ for short. Equation (A.4-2) then becomes

$$\sigma'_{p_1,\ldots,p_K}(x') = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\ldots,m_K}(x) . \tag{A.4-4}$$

It is customary to call a tensor of rank zero a *scalar*, and a tensor of rank one a *vector*. From the definition it is clear that in any Cartesian reference frame in N-dimensional Euclidean space a tensor of rank K is arithmetically represented by an array of dimension K, the bounds on the subscripts being $1{:}N$ (for all of them).

(*Note*: In an affine space and in a curved space, the components of a tensor must be further distinguished as their contravariant and their covariant ones; usually, this distinction is made explicit by using superscripts to denote the contravariant components and subscripts to denote the covariant components. The relevant distinction is lost in a Euclidean space due to the orthogonality of the matrix of transformation coefficients in Equation (A.4-1). For this reason, for tensors in a Euclidean space only subscripts are usually employed, as is done in the present text.)

Exercises

Exercise A.4-1

Let ϕ denote a scalar function of position in N-dimensional Euclidean space. (a) Give the relationship between ϕ' and ϕ that corresponds to the change of reference frame given in Equation (A.4-1). (b) What is the number of components of ϕ and ϕ'?

Answer: (a) $\phi' = \phi$ (i.e. the scalar function is invariant); (b) $N^0 = 1$.

Exercise A.4-2

Let a denote a vector function of position in N-dimensional Euclidean space. (a) Give the relationship between a_p' and a_m that corresponds to the change of reference frame given in Equation (A.4-1). (b) What is the number of components of a and a'?

Answer: (a) $a_p' = \alpha_{p,m} a_m$; (b) $N^1 = N$.

Exercise A.4-3

Let τ denote a tensor function of rank two in N-dimensional Euclidean space. (a) Give the relationship between $\tau_{p,q}'$ and $\tau_{m,n}$ that corresponds to the change of reference frame given in Equation (A.4-1). (b) What is the number of components of τ and τ'?

Answer: (a) $\tau_{p,q}' = \alpha_{p,m}\alpha_{q,n}\tau_{m,n}$; (b) N^2.

A.5 Addition, subtraction and multiplication of tensors

Tensors can be subjected to the algebraic operations of addition, subtraction and multiplication. How, and under what conditions this can be done is discussed below for the different operations.

Addition and subtraction

The sum and the difference of two tensors σ and τ can be defined only if σ and τ are of equal ranks (K, let us say). Let the components of σ in a certain Euclidean reference frame be given by $\sigma_{m_1,\dots,m_K}$ and those of τ by $\tau_{m_1,\dots,m_K}$, then the components of the sum (difference) of σ and τ in the same reference frame is given by

$$(\sigma \pm \tau)_{m_1,\dots,m_K} = \sigma_{m_1,\dots,m_K} \pm \tau_{m_1,\dots,m_K} . \tag{A.5-1}$$

To prove that this procedure indeed defines a tensor, it is observed that from Equation (A.4-2) it follows that

$$\sigma'_{p_1,\dots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K} \sigma_{m_1,\dots,m_K} \tag{A.5-2}$$

and

$$\tau'_{p_1,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\tau_{m_1,\ldots,m_K} . \tag{A.5-3}$$

Consequently,

$$(\sigma \pm \tau)'_{p_1,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}(\sigma_{m_1,\ldots,m_K} \pm \tau_{m_1,\ldots,m_K}) . \tag{A.5-4}$$

Equation (A.5-4) shows that the quantity in parentheses indeed transforms like a tensor of rank K, which was to be proved.

Multiplication by a constant

Let the components of the tensor σ of rank K in a certain Euclidean reference frame be given by $\sigma_{m_1,\ldots,m_K}$. Then, the product of the constant γ and σ is defined as

$$(\gamma\sigma)_{m_1,\ldots,m_K} = \gamma\sigma_{m_1,\ldots,m_K} . \tag{A.5-5}$$

To prove that this procedure indeed defines a tensor, it is observed that from Equation (A.4-2) it follows that

$$\gamma\sigma'_{p_1,\ldots,p_K} = \gamma\alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\ldots,m_K} . \tag{A.5-6}$$

Consequently,

$$(\gamma\sigma)'_{p_1,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\gamma\sigma_{m_1,\ldots,m_K} . \tag{A.5-7}$$

Equation (A.5-7) shows that the quantity in parentheses indeed transforms like a tensor of rank K, which was to be proved.

The properties shown in Equations (A.5-4) and (A.5-7) imply that the tensors of a given rank K form a *linear space*, a property which is made explicit by

$$(\lambda\sigma + \mu\tau)_{m_1,\ldots,m_K} = \lambda\sigma_{m_1,\ldots,m_K} + \mu\tau_{m_1,\ldots,m_K} , \tag{A.5-8}$$

where λ and μ are constants, a property which follows from Equations (A.5-4) and (A.5-7).

Multiplication of two tensors

The product of two tensors can be defined for tensors of arbitrary ranks. Let σ denote a tensor of rank K and let τ denote a tensor of rank L. Furthermore, let $\sigma_{m_1,\ldots,m_K}$ and $\tau_{n_1,\ldots,n_L}$ denote, in a certain Euclidean reference frame, the components of σ and τ, respectively. Assume, for example, that $K \leqslant L$. Then, from σ and τ we can construct tensors of $K + 1$ different ranks as possible products of σ and τ; the product tensors can have the ranks $L + K$, $L + K - 2$, ..., $L - K + 2$, $L - K$.

The product of highest rank ($L + K$) is a tensor whose components in the reference frame under consideration are given by

$$(\sigma\tau)_{m_1,\ldots,m_K,n_1,\ldots,n_L} = \sigma_{m_1,\ldots,m_K}\tau_{n_1,\ldots,n_L} . \tag{A.5-9}$$

To prove that this procedure indeed defines a tensor, it is observed that from Equation (A.4-2) it follows that

$$\sigma'_{p_1,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\ldots,m_K} \tag{A.5-10}$$

and

$$\tau'_{q_1,\ldots,q_L} = \alpha_{q_1,n_1} \cdots \alpha_{q_L,n_L}\tau_{n_1,\ldots,n_L}\,. \tag{A.5-11}$$

Consequently,

$$(\sigma\tau)'_{p_1,\ldots,p_K,q_1,\ldots,q_L} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\alpha_{q_1,n_L} \cdots \alpha_{q_L,n_L}(\sigma\tau)_{m_1,\ldots,m_K,n_1,\ldots,n_L}\,. \tag{A.5-12}$$

Equation (A.5-12) shows that the quantity in parentheses indeed transforms like a tensor of rank $L + K$, which was to be proved.

Product tensors of lower rank than $L + K$ are formed by the process of *contraction*, i.e. by making, a number of times, one subscript in σ equal to one subscript in τ (thus making it a repeated subscript) and applying the summation convention to it. In this procedure the number of free subscripts is, each time, reduced by two. The positions of the repeated subscripts can be chosen arbitrarily, each choice giving rise to a different result. So, when $K \leqslant L$ there are $L(L-1)\ldots(L-K+1) = L!/(L-K)!$ product tensors of the lowest rank $L - K$. As an example, consider the tensor whose components in the Euclidean reference frame under consideration are given by

$$(\sigma\tau)_{n_{K+1},\ldots,n_L} = \sigma_{m_1,\ldots,m_K}\tau_{m_1,\ldots,m_K,n_{K+1},\ldots,n_L}\,, \tag{A.5-13}$$

where contraction is carried out over the first K subscripts of τ.To prove that this procedure indeed defines a tensor, it is observed that from Equation (A.4-2) it follows that

$$\sigma'_{p_1,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\sigma_{m_1,\ldots,m_K}\,. \tag{A.5-14}$$

and

$$\tau'_{p_1,\ldots,p_K,q_{K+1},\ldots,q_L} = \alpha_{p_1,n_1} \cdots \alpha_{p_K,n_K}\alpha_{q_{K+1},n_{K+1}} \cdots \alpha_{q_L,n_L}\tau_{n_1,\ldots,n_K,n_{K+1},\ldots,n_L}\,. \tag{A.5-15}$$

Consequently, upon using Equation (A.4-3),

$$\begin{aligned}(\sigma\tau)'_{q_{K+1},\ldots,q_L} &= \sigma'_{p_1,\ldots,p_K}\tau'_{p_1,\ldots,p_K,q_{K+1},\ldots,q_L}\\ &= \delta_{m_1,n_1} \cdots \delta_{m_K,n_K}\alpha_{q_{K+1},n_{K+1}} \cdots \alpha_{q_L,n_L}\sigma_{m_1,\ldots,m_K}\tau_{n_1,\ldots,n_K,n_{K+1},\ldots,n_L}\\ &= \alpha_{q_{K+1},n_{K+1}} \cdots \alpha_{q_L,n_L}\sigma_{m_1,\ldots,m_K}\tau_{m_1,\ldots,m_K,n_{K+1},\ldots,n_L}\\ &= \alpha_{q_{K+1},n_{K+1}} \cdots \alpha_{q_L,n_L}(\sigma\tau)_{n_{K+1},\ldots,n_L}\,.\end{aligned} \tag{A.5-16}$$

Equation (A.5-16) shows that the quantity in parentheses indeed transforms like a tensor of rank $L - K$, which was to be proved.

In the process of contraction, Equation (A.4-3) plays a vital role. In view of this relation, a contraction can also be established through the use of the Kronecker symbol. So, the contraction considered in Equation (A.5-13) could also be written as

$$(\sigma\tau)_{n_{K+1},\ldots,n_L} = \sigma_{m_1,\ldots,m_K}\tau_{n_1,\ldots,n_K,n_{K+1},\ldots,n_L}\delta_{m_1,n_1} \cdots \delta_{m_K,n_K}\,. \tag{A.5-17}$$

Notationally, the process of contraction is also extended to the subscripts of a single tensor of rank $M \geqslant 2$. (Note that in this process a product is no longer involved, but the general rule of the summation convention that a repeated subscript may occur only twice still applies.) As an example, consider the tensor whose components in the Euclidean reference frame under consideration are given by

$$\sigma_{m_1,\dots,m_{K-1},n,m_{K+1},\dots,m_{L-1},n,m_{L+1},\dots,m_M} = \sigma_{m_1,\dots,m_K,\dots,m_L,\dots,m_M}\delta_{m_K,m_L}$$
$$= \sigma_{m_1,\dots,m_K,\dots,m_L,\dots,m_M}\delta_{m_K,n}\delta_{m_L,n}\,, \qquad \text{(A.5-18)}$$

i.e. a tensor in which a single contraction is carried out over the subcripts at the Kth and Lth positions. To prove that this is indeed a tensor of rank $M-2$, it is observed that from Equation (A.4-2) it follows that

$$\sigma'_{p_1,\dots,p_{K-1},q,p_{K+1},\dots,p_{L-1},q,p_{L+1},\dots,p_M} = \sigma'_{p_1,\dots,p_K,\dots,p_L,\dots,p_M}\delta_{p_K,p_L}$$
$$= \alpha_{p_1,m_1}\cdots\alpha_{p_K,m_K}\cdots\alpha_{p_L,m_L}\cdots\alpha_{p_M,m_M}\sigma_{m_1,\dots,m_K,\dots,m_L,\dots,m_M}\delta_{p_K,p_L}$$
$$= \alpha_{p_1,m_1}\cdots\alpha_{p_{K-1},m_{K-1}}\alpha_{p_{K+1},m_{K+1}}\cdots\alpha_{p_{L-1},m_{L-1}}$$
$$\alpha_{p_{L+1},m_{L+1}}\cdots\alpha_{p_M,m_M}\sigma_{m_1,\dots,m_K,\dots,m_L,\dots,m_M}\delta_{m_K,m_L}$$
$$= \alpha_{p_1,m_1}\cdots\alpha_{p_{K-1},m_{K-1}}\alpha_{p_{K+1},m_{K+1}}\cdots\alpha_{p_{L-1},m_{L-1}}$$
$$\alpha_{p_{L+1},m_{L+1}}\cdots\alpha_{p_M,m_M}\sigma_{m_1,\dots,m_{K-1},n,m_{K+1},\dots,m_{L-1},n,m_{L+1},\dots,m_M}\,, \qquad \text{(A.5-19)}$$

where the property

$$\alpha_{p_K,m_L}\alpha_{p_L,m_L}\delta_{p_K,p_L} = \alpha_{p_K,m_K}\alpha_{p_K,m_L} = \delta_{m_K,m_L} \qquad \text{(A.5-20)}$$

has been used. Equation (A.5-19) shows that the quantity on the left-hand side indeed transforms like a tensor of rank $M-2$, which was to be proved.

Exercises

Exercise A.5-1

Which tensor products can be formed from the two scalar functions of position ϕ and ψ?

Answer: The product tensor of rank zero $\phi\psi$ (i.e. a scalar).

Exercise A.5-2

Which tensor products can be formed from the scalar functions of position ϕ and the vector function of position a_m?

Answer: The product tensor of rank one ϕa_m (i.e. a vector).

Exercise A.5-3

Which tensor products can be formed from the two vector functions of position a_m and b_n?

Answer: The product tensor of rank two $a_m b_n$ and the product tensor of rank zero $a_m b_m$ (the latter scalar is called the "inner product" of the two vectors).

Exercise A.5-4

Construct the product tensors of rank zero that can be formed from the vector a_m, the vector b_n and the tensor $\sigma_{p,q}$ of rank two.

Answer: $a_m b_n \sigma_{m,n}$, $a_m b_n \sigma_{n,m}$ and $a_m b_m \sigma_{n,n}$.

Exercise A.5-5

Prove that the inner product $\langle a_m, b_m \rangle = a_m b_m$ of two vectors satisfies the conditions of a proper inner product: (a) $\langle a_m, b_m \rangle = \langle b_m, a_m \rangle$; (b) $\langle \lambda a_m + \mu b_m, c_m \rangle = \lambda \langle a_m, c_m \rangle + \mu \langle b_m, c_m \rangle$; (c) $\langle a_m, \lambda b_m + \mu c_m \rangle = \lambda \langle a_m, b_m \rangle + \mu \langle a_m, c_m \rangle$.

Proof: (a) Is obvious; (b) follows by using the definition; (c) follows by using the definition.

Exercise A.5-6

Prove that the norm $|a_m| = (a_m a_m)^{1/2} \geqslant 0$ of the vector a_m induced by the inner product of Exercise A.5-5 leads to

$$\langle a_m, b_m \rangle^2 \leqslant |a_m|^2 |b_m|^2 \text{(Cauchy–Schwarz inequality)} . \tag{A.5-21}$$

Proof: For either $a_m = 0$, or $b_m = 0$, or $a_m = 0$ and $b_m = 0$, equality obviously holds. For $a_m \neq 0$ and $b_m \neq 0$ observe that, for any $\lambda \in \mathcal{R}$, $|a_m - \lambda b_m|^2 \geqslant 0$, with $|a_m - \lambda b_m| = 0$ if, and only if, $a_m = \lambda b_m$ (i.e. if a_m and b_m are linearly dependent). As a consequence,

$$|a_m|^2 - 2\lambda \langle a_m, b_m \rangle + \lambda^2 |b_m|^2 \geqslant 0,$$

or

$$(\lambda |b_m| - \langle a_m, b_m \rangle / |b_m|)^2 - \langle a_m, b_m \rangle^2 / |b_m|)^2 + |a_m|^2 \geqslant 0.$$

By choosing $\lambda = \langle a_m, b_m \rangle / |b_m|^2$, Equation (A.5-21) follows.

Exercise A.5-7

Prove that the norm $|a_m| = (a_m a_m)^{1/2} \geqslant 0$ of the vector a_m induced by the inner product of Exercise A.5-5 satisfies the conditions of a proper norm: (a) $|a_m| = 0$ if, and only if, $a_m = 0$; (b) $|a_m| > 0$ for any $a_m \neq 0$; (c) $|\lambda a_m| = |\lambda| |a_m|$; (d) $|a_m - b_m| \leqslant |a_m| + |b_m|$.

Proof: (a) Is obvious; (b) is obvious; (c) follows by using the definition; (d) follows from:

$$\begin{aligned}
&|a_m - b_m|^2 \\
&= (a_m - b_m)(a_m - b_m) = a_m a_m + b_m b_m - 2 a_m b_m \\
&= |a_m|^2 + |b_m|^2 - 2\langle a_m, b_m \rangle \leqslant |a_m|^2 + |b_m|^2 + 2|\langle a_m, b_m \rangle| \leqslant |a_m|^2 + |b_m|^2 + 2|a_m||b_m| \\
&= (|a_m| + |b_m|)^2,
\end{aligned}$$

or

$$|a_m - b_m| \leqslant |a_m| + |b_m| .$$

Here, the property $|\langle a_m, b_m \rangle| \leqslant |a_m||b_m|$ has been used (see Equation (A.5-21)).

Exercise A.5-8

Prove that the inner product $\langle\sigma_{m,n},\tau_{m,n}\rangle = \sigma_{m,n}\tau_{m,n}$ of two tensors of rank two satisfies the conditions of a proper inner product: (a) $\langle\sigma_{m,n},\tau_{m,n}\rangle = \langle\tau_{m,n},\sigma_{m,n}\rangle$; (b) $\langle\lambda\rho_{m,n} + \mu\sigma_{m,n},\tau_{m,n}\rangle = \lambda\langle\rho_{m,n},\tau_{m,n}\rangle + \mu\langle\sigma_{m,n},\tau_{m,n}\rangle$; (c) $\langle\rho_{m,n},\lambda\sigma_{m,n} + \mu\tau_{m,n}\rangle = \lambda\langle\rho_{m,n},\sigma_{m,n}\rangle + \mu\langle\rho_{m,n},\tau_{m,n}\rangle$.

Proof: (a) Is obvious; (b) follows by using the definition; (c) follows by using the definition.

Exercise A.5-9

Prove that the norm $|\sigma_{m,n}| = (\sigma_{m,n}\sigma_{m,n})^{1/2} \geqslant 0$ of the tensor $\sigma_{m,n}$ of rank two induced by the inner product of Exercise A.5-8 leads to

$$\langle\sigma_{m,n},\tau_{m,n}\rangle^2 \leqslant |\sigma_{m,n}|^2|\tau_{m,n}|^2 \qquad \text{(Cauchy–Schwarz inequality).} \tag{A.5-22}$$

Proof: For either $\sigma_{m,n} = 0$, or $\tau_{m,n} = 0$, or $\sigma_{m,n} = 0$ and $\tau_{m,n} = 0$, equality obviously holds. For $\sigma_{m,n} \neq 0$ and $\tau_{m,n} \neq 0$ observe that, for any $\lambda \in \mathcal{R}$, $|\sigma_{m,n} - \lambda\tau_{m,n}|^2 \geqslant 0$, with $|\sigma_{m,n} - \lambda\tau_{m,n}| = 0$ if, and only if, $\sigma_{m,n} = \lambda\tau_{m,n}$ (i.e. if $\sigma_{m,n}$ and $\tau_{m,n}$ are linearly dependent). As a consequence,

$$|\sigma_{m,n}|^2 - 2\lambda\langle\sigma_{m,n},\tau_{m,n}\rangle + \lambda^2|\tau_{m,n}|^2 \geqslant 0\,,$$

or

$$(\lambda|\tau_{m,n}| - \langle\sigma_{m,n},\tau_{m,n}\rangle/|\tau_{m,n}|)^2 - \langle\sigma_{m,n},\tau_{m,n}\rangle^2/|\tau_{m,n}|)^2 + |\sigma_{m,n}|^2 \geqslant 0\,.$$

By choosing $\lambda = \langle\sigma_{m,n},\tau_{m,n}\rangle/|\tau_{m,n}|^2$, Equation (A.5-22) follows.

Exercise A.5-10

Prove that the norm $|\sigma_{m,n}| = (\sigma_{m,n}\sigma_{m,n})^{1/2} \geqslant 0$ of the tensor $\sigma_{m,n}$ of rank two induced by the inner product of Exercise A.5-8 satisfies the conditions of a proper norm: (a) $|\sigma_{m,n}| = 0$ if, and only if, $\sigma_{m,n} = 0$; (b) $|\sigma_{m,n}| > 0$ for any $\sigma_{m,n} \neq 0$; (c) $|\lambda\sigma_{m,n}| = |\lambda||\sigma_{m,n}|$; (d) $|\sigma_{m,n} - \tau_{m,n}| \leqslant |\sigma_{m,n}| + |\tau_{m,n}|$.

Proof: (a) Is obvious; (b) is obvious; (c) follows by using the definition; (d) follows from:

$$\begin{aligned}
&|\sigma_{m,n} - \tau_{m,n}|^2 \\
&= (\sigma_{m,n} - \tau_{m,n})(\sigma_{m,n} - \tau_{m,n}) = \sigma_{m,n}\sigma_{m,n} + \tau_{m,n}\tau_{m,n} - 2\sigma_{m,n}\tau_{m,n} \\
&= |\sigma_{m,n}|^2 + |\tau_{m,n}|^2 - 2\langle\sigma_{m,n},\tau_{m,n}\rangle \;\leqslant\; |\sigma_{m,n}|^2 + |\tau_{m,n}|^2 + 2|\langle\sigma_{m,n},\tau_{m,n}\rangle| \\
&\leqslant |\sigma_{m,n}|^2 + |\tau_{m,n}|^2 + 2|\sigma_{m,n}||\tau_{m,n}| = (|\sigma_{m,n}| + |\tau_{m,n}|)^2\,,
\end{aligned}$$

or

$$|\sigma_{m,n} - \tau_{m,n}| \leqslant |\sigma_{m,n}| + |\tau_{m,n}|\,.$$

Here, the property $\langle\sigma_{m,n},\tau_{m,n}\rangle \leqslant |\sigma_{m,n}||\tau_{m,n}|$ has been used (see Equation (A.5-22)).

A.6 Symmetry properties

A tensor of at least rank two can have certain symmetry properties. Let $\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K}$ denote a tensor of rank $K \geqslant 2$. Then, this tensor is denoted as *symmetrical* in the subscripts at the positions i and j if

$$\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} = \sigma_{m_1,\ldots,m_j,\ldots,m_i,\ldots,m_K} \tag{A.6-1}$$

for all values of m_i and m_j; it is denoted as *antisymmetrical* in the subscripts at the positions i and j if

$$\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} = -\sigma_{m_1,\ldots,m_j,\ldots,m_i,\ldots,m_K} \tag{A.6-2}$$

for all values of m_i and m_j. To prove that this is indeed the property of a tensor, we observe that after a change of reference frame in Euclidean space we have

$$\sigma'_{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_i,m_i} \cdots \alpha_{p_j,m_j} \cdots \alpha_{p_K,m_K} \sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} \, . \tag{A.6-3}$$

By interchanging the free subscripts p_i and p_j, we have

$$\sigma'_{p_1,\ldots,p_j,\ldots,p_i,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_j,m_i} \cdots \alpha_{p_i,m_j} \cdots \alpha_{p_K,m_K} \sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} \, . \tag{A.6-4}$$

We now replace the dummy subscripts m_i and m_j on the right-hand side by m_j and m_i, respectively, use Equation (A.6-1), and obtain

$$\sigma'_{p_1,\ldots,p_j,\ldots,p_i,\ldots,p_K} = \alpha_{p_1,m_1} \cdots \alpha_{p_j,m_j} \cdots \alpha_{p_i,m_i} \cdots \alpha_{p_K,m_K} \sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} \, . \tag{A.6-5}$$

Upon interchanging the positions of the factors α_{p_j,m_j} and α_{p_i,m_i} on the right-hand side, it follows immediately that

$$\sigma'_{p_1,\ldots,p_j,\ldots,p_i,\ldots,p_K} = \sigma'_{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_K} \, , \tag{A.6-6}$$

which completes the proof. The proof for the antisymmetrical case runs along the same lines.

In physical applications the decomposition of a tensor into a symmetrical and an antisymmetrical part is often desirable. This decomposition is accomplished by writing, for any tensor of rank $K \geqslant 2$, and, for example, for the subscripts at the positions i and j,

$$\begin{aligned} \sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} &= \tfrac{1}{2}\left(\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} + \sigma_{m_1,\ldots,m_j,\ldots,m_i,\ldots,m_K}\right) \\ &\quad + \tfrac{1}{2}\left(\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K} - \sigma_{m_1,\ldots,m_j,\ldots,m_i,\ldots,m_K}\right) \, . \end{aligned} \tag{A.6-7}$$

The first term on the right-hand side is symmetrical in the subscripts at the positions i and j; the second term is antisymmetrical in the subscripts at the positions i and j. The procedure of Equation (A.6-7) can be carried out for any pair of subscripts.

Exercises

Exercise A.6-1

Let v_m denote a vector and let $\rho_{m,n}$ denote a tensor of rank two. Consider the scalar $w = \frac{1}{2}\rho_{m,n} v_m v_n$. Does the antisymmetrical part of $\rho_{m,n}$ contribute to the value of w?

Answer: No. (Hence, in the evaluation of w it can, from the start, be assumed that $\rho_{m,n} = \rho_{n,m}$.)

Exercise A.6-2

Let $\tau_{p,q}$ denote a tensor of rank two. Give (a) the symmetrical part of $\tau_{p,q}$, (b) the antisymmetrical part of $\tau_{p,q}$.

Answer: (a) symmetrical part: $\frac{1}{2}(\tau_{p,q}+\tau_{q,p})$; (b) antisymmetrical part: $\frac{1}{2}(\tau_{p,q}-\tau_{q,p})$.

Exercise A.6-3

The tensor $\sigma_{m_1,\ldots,m_i,\ldots,m_j,\ldots,m_K}$ of rank $K \geqslant 2$ is antisymmetrical in its subscripts at the positions i and j. What is the value of its components for which $m_i = m_j$?

Answer: Zero.

A.7 Unit tensors

There are several unit tensors that are of importance to the application of tensors in physics. Among them are:

- The symmetrical unit tensor of rank two (*Kronecker tensor*).
- The completely antisymmetrical unit tensor of rank N (*Levi–Civita tensor*), where N is the dimension of the Euclidean space under consideration.
- The diagonalizing unit tensor of rank four.
- The symmetrical unit tensor of rank four.
- The antisymmetrical unit tensor of rank four.

These tensors are discussed individually below.

Symmetrical unit tensor of rank two (Kronecker tensor) and its properties

The symmetrical unit tensor of rank two (Kronecker tensor) is denoted by $\delta_{m,n}$ and is, for any Euclidean reference frame, defined as

$$\delta_{1,1} = \ldots = \delta_{N,N} = 1; \qquad \delta_{m,n} = 0 \quad \text{if } m \neq n\,. \tag{A.7-1}$$

To prove that this is indeed a tensor, we perform a change of reference frame according to Equation (A.4-1) and obtain

$$\delta'_{p,q} = \alpha_{p,m}\alpha_{q,n}\delta_{m,n}\,. \tag{A.7-2}$$

Using

$$\alpha_{p,m}\alpha_{q,n}\delta_{m,n} = \alpha_{p,n}\alpha_{q,n}\,, \tag{A.7-3}$$

and the result of Equation (A.3-20), we obtain

$$\delta'_{p,q} = \delta_{p,q}\,, \tag{A.7-4}$$

which was to be proved. On account of Equation (A.7-4), the numerical representation of the Kronecker tensor is the same in all Cartesian reference frames in Euclidean space.

The Kronecker tensor can rightly be denoted as a unit tensor since, upon contraction over a single subscript with any tensor of rank $K \geqslant 1$, it reproduces that tensor. For example, let σ be a tensor of rank $K \geqslant 1$, then

$$\begin{aligned}\delta_{m_i,n_i}\sigma_{m_1,\dots,m_{i-1},m_i,m_{i+1},\dots,m_K} &= \sigma_{m_1,\dots,m_{i-1},n_i,m_{i+1},\dots,m_K} \\ &= \sigma_{m_1,\dots,m_K} \,. \end{aligned} \tag{A.7-5}$$

In particular, we have

$$\delta_{m,p}\delta_{p,n} = \delta_{m,n} \,, \tag{A.7-6}$$

a relation which is often used in the rearrangement of products of tensors. Furthermore,

$$\delta_{m,m} = N \tag{A.7-7}$$

(see Exercise A.7-1).

Completely antisymmetrical unit tensor of rank N (Levi–Civita tensor) and its properties

The completely antisymmetrical unit tensor of rank N (Levi–Civita tensor), where N is the dimension of the Euclidean space under consideration, is denoted by $\varepsilon_{m_1,\dots,m_N}$ and is, for any Euclidean reference frame, defined as follows:

$$\varepsilon_{m_1,\dots,m_N} = \begin{matrix}+1\\-1\end{matrix} \quad \text{when } \{m_1,\dots,m_N\} \text{ is an } \begin{matrix}\text{even}\\\text{odd}\end{matrix} \text{ permutation of } \{1,\dots,N\}\,, \tag{A.7-8}$$

while

$$\varepsilon_{m_1,\dots,m_N} = 0 \quad \begin{aligned}&\text{when not all subscripts are different,}\\ &\text{i.e. when } \{m_1,\dots,m_N\} \text{ is not a permutation of } \{1,\dots,N\}\,.\end{aligned} \tag{A.7-9}$$

To prove that $\varepsilon_{m_1,\dots,m_N}$ is indeed a tensor of rank N, we first recall that the determinant $\det(a_{p,m})$ of any square matrix with elements $a_{p,m}$ is defined as

$$\det(a_{p,m})\varepsilon_{p_1,\dots,p_N} = a_{p_1,m_1} \cdots a_{p_N,m_N}\varepsilon_{m_1,\dots,m_N} \,. \tag{A.7-10}$$

In Equation (A.7-10) we now take $a_{p,m}$ to be the coefficients that occur in the change of reference frame given in Equation (A.4-1). Then, using Equation (A.4-3), we have

$$\begin{aligned}\det(a_{p,m})\varepsilon_{p_1,\dots,p_N}a_{p_1,n_1} \cdots a_{p_N,n_N} &= a_{p_1,m_1}a_{p_1,n_1} \cdots a_{p_N,m_N}a_{p_N,n_N}\varepsilon_{m_1,\dots,m_N} \\ &= \delta_{m_1,n_1} \cdots \delta_{m_N,n_N}\varepsilon_{m_1,\dots,m_N} = \varepsilon_{n_1,\dots,n_N} \,. \end{aligned} \tag{A.7-11}$$

However, as can be inferred from Equation (A.7-10), we also have, in view of the relation (see Exercise A.7-4)

$$\varepsilon_{p_1,\dots,p_N}\varepsilon_{p_1,\dots,p_N} = N! \,, \tag{A.7-12}$$

where $N!$ (pronounced "N-factorial") is given by $N! = N(N-1) \dots 2 \,.\, 1$, the expression

$$\det(a_{p,m}) = (N!)^{-1}\varepsilon_{p_1,\dots,p_N}\varepsilon_{m_1,\dots,m_N}a_{p_1,m_1} \cdots a_{p_N,m_N} \,, \tag{A.7-13}$$

from which it follows that the procedure for evaluating a determinant is invariant against an interchange of rows and columns in the relevant matrix. Hence, Equation (A.7-10) can also be written as

$$\det(\alpha_{p,n})\varepsilon_{n_1,\ldots,n_N} = \varepsilon_{p_1,\ldots,p_N}\alpha_{p_1,n_1} \cdots \alpha_{p_N,n_N} . \tag{A.7-14}$$

Using the expression for $\varepsilon_{n_1,\ldots,n_N}$ from Equation (A.7-11) in Equation (A.7-14), it is concluded that

$$[\det(\alpha_{p,m})]^2 = 1 , \tag{A.7-15}$$

and, consequently,

$$\det(\alpha_{p,m}) = \pm 1 . \tag{A.7-16}$$

Now, if the primed and the unprimed reference frames coincide, we have $\lambda_m = 0$ and $\alpha_{p,m} = \delta_{p,m}$, and hence $\det(\alpha_{p,m}) = +1$, provided that the labelling of the base vectors of the two reference frames allows such a coincidence. (In this case, the two reference frames are said to have the same *orientation.*) Assuming that the change of reference frame given in Equation (A.4-1) is such that through a continuous variation of the coefficients the primed reference frame can be made to coincide with the unprimed one, the only solution to Equation (A.7-16) is

$$\det(\alpha_{p,m}) = 1 . \tag{A.7-17}$$

Using Equation (A.7-17) in Equation (A.7-10), and bearing in mind that the right-hand side of Equation (A.7-10) can be interpreted as $\varepsilon'_{m_1,\ldots,m_N}$, we have shown that $\varepsilon_{p_1,\ldots,p_N}$ indeed transforms like a tensor of rank N and that its arithmetic representation is the same in all Cartesian reference frames in Euclidean space, provided that the reference frames all have the same orientation.

The Levi–Civita tensor plays an important role in a number of geometrical concepts. First of all, the *volume* of the parallelepiped spanned by the ordered sequence of N linearly independent vectors in N-dimensional Euclidean space is such a concept. Let $\{\boldsymbol{a}(1),\ldots,\boldsymbol{a}(N)\} = \{a_m(1),\ldots,a_m(N)\}$, with $\{\boldsymbol{a}(1),\ldots,\boldsymbol{a}(N)\} \neq \{\mathbf{0},\ldots,\mathbf{0}\}$ being N such vectors (Figure A.7-1), and let the numbers $a_{m,n}$ be defined as

$$a_{m,n} = a_m(n) \qquad \text{for} \quad m = 1,\ldots,N; \quad n = 1,\ldots,N . \tag{A.7-18}$$

Then, the volume V_a^N of the parallelepiped spanned by $\{\boldsymbol{a}(1),\ldots,\boldsymbol{a}(N)\}$ is, by definition,

$$V_a^N = \det(a_{m,n}) , \tag{A.7-19}$$

where the right-hand side follows from the expression (see Equation (A.7-14) with $\{n_1,\ldots,n_N\} = \{1,\ldots,N\}$)

$$\det(a_{m,n}) = \varepsilon_{p_1,\ldots,p_N} a_{p_1,1} \cdots a_{p_N,N} . \tag{A.7-20}$$

The quantity V_a^N differs from zero if, and only if, the vectors $\{\boldsymbol{a}(1),\ldots,\boldsymbol{a}(N)\}$ are linearly independent (see Exercise A.7-10).

Next, the *vectorial surface area* $A^{N-1}(n)$ of the $(N-1)$-dimensional parallelepiped "not containing $\boldsymbol{a}(n)$" is introduced as

$$A_p^{N-1}(n) = \varepsilon_{p_1,\ldots,p_{n-1},p,p_{n+1},\ldots,p_N} a_{p_1,1} \cdots a_{p_{n-1},n-1} a_{p_{n+1},n+1} \cdots a_{p_N,N} . \tag{A.7-21}$$

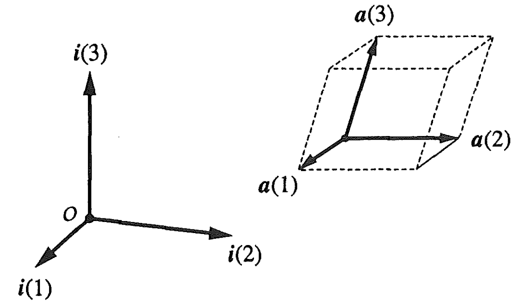

Figure A.7-1 Linearly independent vectors $\{a(1),a(2),a(3)\}$ spanning a parallelepiped in $\mathcal{R}^3$.

(Note that the free subscript p is at the nth position in the Levi–Civita tensor on the right-hand side.) Using the result of Exercise A.7-7, it then follows that

$$V_a^N = a_p(n)A_p^{N-1}(n) \qquad \text{for} \quad n = 1,\ldots,N. \tag{A.7-22}$$

Furthermore, we have

$$a_p(m)A_p^{N-1}(n) = 0 \qquad \text{for} \quad m \neq n, \tag{A.7-23}$$

since the components of $a_p(m)$ are, for $m \neq n$, contained in $A_p^{N-1}(n)$, with the consequence that the determinant corresponding to the left-hand side of Equation (A.7-23) vanishes in view of the linear dependence of its columns.

On the condition that the sequence $\{a(1),\ldots,a(N)\}$ is linearly independent, the constituent vectors can be used as a system of base vectors in affine space. Such a base finds particular application in crystal physics, where it is used to specify the atomic lattice of the pertaining crystal. To study wave propagation in such a lattice, a *reciprocal base* $\{b(1),\ldots,b(N)\} = \{b_p(1),\ldots,b_p(N)\}$ is needed that is related to the base $\{a(1),\ldots,a(N)\}$ via

$$a_p(m)b_p(n) = \delta(m,n) \qquad \text{for} \quad m = 1,\ldots,N; \quad n = 1,\ldots,N. \tag{A.7-24}$$

Here, $\delta(m,n)$ denotes the Kronecker symbol: $\delta(m,n) = 1$ if $m = n$; $\delta(m,n) = 0$ if $m \neq n$. In view of Equations (A.7-22) and (A.7-23),

$$b_p(n) = A_p^{N-1}(n)/V_a^N \qquad \text{for} \quad n = 1,\ldots,N, \tag{A.7-25}$$

satisfies Equation (A.7-24), where V_a^N is given by Equation (A.7-19). To show that the sequence $\{b(1),\ldots,b(\mathrm{N})\}$ indeed forms a base, i.e. that the constituent vectors are linearly independent, we must show that

$$\det(\beta_{p,n}) \neq 0, \tag{A.7-26}$$

with

$$\beta_{p,n} = b_p(n) \qquad \text{for} \quad p = 1,\ldots,N; \quad n = 1,\ldots,N. \tag{A.7-27}$$

To this end, we observe that, using Equation (A.7-13),

$$\det(\alpha_{p,m}\beta_{p,n}) = (N!)^{-1}\varepsilon_{m_1,\ldots,m_N}\varepsilon_{n_1,\ldots,n_N}\alpha_{p_1,m_1}\beta_{p_1,n_1}\cdots\alpha_{p_N,m_N}\beta_{p_N,n_N}. \tag{A.7-28}$$

Now, in view of the properties of the Levi–Civita tensor upon interchanging two of its subscripts, the terms on the right-hand side of Equation (A.7-28) that correspond to any equal values of the repeated subscripts $p_1,\ldots,p_N$ cancel, which implies that only the terms corresponding to unequal values of these subscripts remain. The latter values can be chosen in $N!$ ways, all leading to the same results since all the subscripts $m_1,\ldots,m_N$ and $n_1,\ldots,n_N$ are also repeated. Hence, upon reducing the subscripts $p_1,\ldots,p_N$ to the natural order $1,\ldots,N$, it follows that

$$\det(\alpha_{p,m}\beta_{p,n}) = \varepsilon_{m_1,\ldots,m_N}\alpha_{1,m_1}\cdots\alpha_{N,m_N}\varepsilon_{n_1,\ldots,n_N}\beta_{1,n_1}\cdots\alpha_{N,n_N}\,. \tag{A.7-29}$$

However, in view of Exercise A.7-6 and Equation (A.7-19),

$$\varepsilon_{m_1,\ldots,m_N}\alpha_{1,m_1}\cdots\alpha_{N,m_N} = \det(\alpha_{p,m}) = V_a^N \tag{A.7-30}$$

and, similarly,

$$\varepsilon_{n_1,\ldots,n_N}\beta_{1,n_1}\cdots\beta_{N,n_N} = \det(\beta_{p,n}) = V_b^N\,, \tag{A.7-31}$$

where V_b^N is the volume of the parallelepiped spanned by $\{\boldsymbol{b}(1),\ldots,\boldsymbol{b}(N)\}$. Using these values in Equation (A.7-29) it follows that

$$\det(\alpha_{p,m}\beta_{p,n}) = V_a^N V_b^N\,. \tag{A.7-32}$$

Using this result and

$$\det(\delta(m,n)) = 1 \tag{A.7-33}$$

in Equation (A.7-24), it finally follows that

$$V_a^N V_b^N = 1\,, \tag{A.7-34}$$

which shows that

$$V_b^N = 1/V_a^N\,, \tag{A.7-35}$$

and, hence, $V_b^N \neq 0$. Consequently, the sequence $\{\boldsymbol{b}(1),\ldots,\boldsymbol{b}(N)\}$ defined by Equation (A.7-25) indeed forms a base. Note that in Equations (A.7-24) and (A.7-34) the roles of the two bases can be interchanged, which implies that the base $\{\boldsymbol{a}(1),\ldots,\boldsymbol{a}(N)\}$ defined by

$$a_p(n) = B_p^{N-1}(n)/V_b^N\,, \tag{A.7-36}$$

with

$$B_p^{N-1}(n) = \varepsilon_{p_1,\ldots,p_{n-1},p,p_{n+1},\ldots,p_N}\beta_{p_1,1}\cdots\beta_{p_{n-1},n-1}\beta_{p_{n+1},n+1}\cdots\beta_{p_N N}\,, \tag{A.7-37}$$

is reciprocal to $\{\boldsymbol{b}(1),\ldots,\boldsymbol{b}(N)\}$.

Diagonalizing unit tensor of rank four and its properties

The diagonalizing unit tensor of rank four is denoted by $\Delta^{\delta}_{i,j,p,q}$ and is, for any Euclidean reference frame, defined as

$$\Delta^{\delta}_{i,j,p,q} = (1/N)\delta_{i,j}\delta_{p,q}\,. \tag{A.7-38}$$

The fact that $\Delta^{\delta}_{i,j,p,q}$ is indeed a tensor follows from the pertaining properties of the Kronecker tensor. Upon contracting its last two subscripts with any two different subscripts of a tensor of arbitrary rank $K \geqslant 2$, $\Delta^{\delta}_{i,j,p,q}$ diagonalises that tensor over the latter two subscripts. To show this, let σ be a tensor of rank $K \geqslant 2$ and let us carry out the contraction over the subscripts at the positions p and q, then

$$\Delta^{\delta}_{i,j,p,q}\sigma_{m_1,\ldots,m_{p-1},p,m_{p+1},\ldots,m_{q-1},q,m_{q+1},\ldots,m_K}$$
$$= (1/N)\delta_{i,j}\sigma_{m_1,\ldots,m_{p-1},p,m_{p+1},\ldots,m_{q-1},p,m_{q+1},\ldots,m_K}\,, \tag{A.7-39}$$

which is diagonal in the subscripts i and j. That $\Delta^{\delta}_{i,j,p,q}$ is indeed a unit tensor follows from the consideration that

$$\Delta^{\delta}_{i,j,p,q}\Delta^{\delta}_{p,q,r,s} = (1/N)\delta_{i,j}\delta_{p,q}(1/N)\delta_{p,q}\delta_{r,s}$$
$$= (1/N)^2\delta_{i,j}\delta_{p,p}\delta_{r,s} = (1/N)\delta_{i,j}\delta_{r,s} = \Delta^{\delta}_{i,j,r,s}\,, \tag{A.7-40}$$

in which $\delta_{p,p} = N$ (see Exercise A.7-1) has been used. Note that $\Delta^{\delta}_{i,j,p,q}$ satisfies the symmetry relations

$$\Delta^{\delta}_{i,j,p,q} = \Delta^{\delta}_{i,j,q,p} = \Delta^{\delta}_{j,i,q,p} = \Delta^{\delta}_{j,i,p,q} \quad \text{and} \quad \Delta^{\delta}_{i,j,p,q} = \Delta^{\delta}_{p,q,i,j}.$$

The diagonalizing unit tensor of rank four is particularly used to decompose a tensor of rank two into an isotropic (omnidirectional) part and a deviatoric part. Let $\sigma_{m,n}$ be a tensor of rank two, then the relevant decomposition is

$$\sigma_{m,n} = \Delta^{\delta}_{m,n,p,q}\sigma_{p,q} + \left[\sigma_{m,n} - \Delta^{\delta}_{m,n,p,q}\sigma_{p,q}\right]$$
$$= (1/N)\delta_{m,n}\sigma_{p,p} + \left[\sigma_{m,n} - (1/N)\delta_{m,n}\sigma_{p,p}\right], \tag{A.7-41}$$

where the first term on the right-hand side is the *isotropic (omnidirectional) part* and the term in brackets is the *deviatoric part*. Note in this respect that contraction of the latter part over the two subscripts yields

$$\left[\sigma_{m,n} - (1/N)\delta_{m,n}\sigma_{p,p}\right]\delta_{m,n} = \sigma_{m,m} - (1/N)\delta_{m,m}\sigma_{p,p} = \sigma_{m,m} - \sigma_{p,p} = 0\,. \tag{A.7-42}$$

Symmetrical unit tensor of rank four and its properties

The symmetrical unit tensor of rank four is denoted by $\Delta^{+}_{i,j,p,q}$ and is, for any Euclidean reference frame, defined as

$$\Delta^{+}_{i,j,p,q} = \tfrac{1}{2}\left(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}\right). \tag{A.7-43}$$

The fact that $\Delta^{+}_{i,j,p,q}$ is indeed a tensor follows from the pertaining properties of the Kronecker tensor. Upon contracting its last two subscripts with any two different subscripts of a tensor of arbitrary rank $K \geqslant 2$, $\Delta^{+}_{i,j,p,q}$ extracts out of that tensor the symmetric part in those two subscripts. To show this, let σ be a tensor of rank $K \geqslant 2$ and let us carry out the contraction over the subscripts at the positions p and q, then

$$\Delta^{+}_{i,j,p,q}\sigma_{m_1,\ldots,m_{p-1},p,m_{p+1},\ldots,m_{q-1},q,m_{q+1},\ldots,m_K}$$
$$= \tfrac{1}{2}\left[\sigma_{m_1,\ldots,m_{p-1},i,m_{p+1},\ldots,m_{q-1},j,m_{q+1},\ldots,m_K} + \sigma_{m_1,\ldots,m_{p-1},j,m_{p+1},\ldots,m_{q-1},i,m_{q+1},\ldots,m_K}\right], \tag{A.7-44}$$

which is symmetrical in the subscripts i and j.

That $\Delta^{+}_{i,j,p,q}$ is indeed a unit tensor follows from the consideration that

$$\Delta^{+}_{i,j,p,q}\Delta^{+}_{p,q,r,s} = \tfrac{1}{2}\,(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})\,\tfrac{1}{2}\,(\delta_{p,r}\delta_{q,s} + \delta_{p,s}\delta_{q,r})$$
$$= \tfrac{1}{4}\,(\delta_{i,r}\delta_{j,s} + \delta_{i,s}\delta_{j,r} + \delta_{i,s}\delta_{j,r} + \delta_{i,r}\delta_{j,s})$$
$$= \tfrac{1}{2}\,(\delta_{i,r}\delta_{j,s} + \delta_{i,s}\delta_{j,r}) = \Delta^{+}_{i,j,r,s}\,. \tag{A.7-45}$$

It is noted that $\Delta^{+}_{i,j,p,q}$ satisfies the symmetry relations $\Delta^{+}_{i,j,p,q} = \Delta^{+}_{i,j,q,p} = \Delta^{+}_{j,i,q,p} = \Delta^{+}_{j,i,p,q}$ and $\Delta^{+}_{i,j,p,q} = \Delta^{+}_{p,q,i,j}$.

The symmetrical unit tensor of rank four is particularly used to extract the symmetrical part from a tensor of rank two. Let $\sigma_{p,q}$ be a tensor of rank two, then

$$\Delta^{+}_{i,j,p,q}\sigma_{p,q} = \tfrac{1}{2}(\sigma_{i,j} + \sigma_{j,i})\,, \tag{A.7-46}$$

the right-hand side of which is indeed symmetrical in the two subscripts. The tensor $\Delta^{+}_{i,j,p,q}$ finds particular application in elastodynamics, i.e. the theory of elastic waves in solids.

Antisymmetrical unit tensor of rank four and its properties

The antisymmetrical unit tensor of rank four is denoted by $\Delta^{-}_{i,j,p,q}$ and is, for any Euclidean reference frame, defined as

$$\Delta^{-}_{i,j,p,q} = \tfrac{1}{2}\,(\delta_{i,p}\delta_{j,q} - \delta_{i,q}\delta_{j,p})\,. \tag{A.7-47}$$

The fact that $\Delta^{-}_{i,j,p,q}$ is indeed a tensor follows from the pertaining properties of the Kronecker tensor. Upon contracting its last two subscripts with any two different subscripts of a tensor of arbitrary rank $K \geqslant 2$, $\Delta^{-}_{i,j,p,q}$ extracts out of that tensor the antisymmetric part in those two subscripts. To show this, let σ be a tensor of rank $K \geqslant 2$ and let us carry out the contraction over the subscripts at the positions p and q, then

$$\Delta^{-}_{i,j,p,q}\sigma_{m_1,\ldots,m_{p-1},p,m_{p+1},\ldots,m_{q-1},q,m_{q+1},\ldots,m_K}$$
$$= \tfrac{1}{2}\left[\sigma_{m_1,\ldots,m_{p-1},i,m_{p+1},\ldots,m_{q-1},j,m_{q+1},\ldots,m_K} - \sigma_{m_1,\ldots,m_{p-1},j,m_{p+1},\ldots,m_{q-1},i,m_{q+1},\ldots,m_K}\right], \tag{A.7-48}$$

which is antisymmetrical in the subscripts i and j.

That $\Delta^{-}_{i,j,p,q}$ is indeed a unit tensor follows from the consideration that

$$\Delta^{-}_{i,j,p,q}\Delta^{-}_{p,q,r,s} = \tfrac{1}{2}\,(\delta_{i,p}\delta_{j,q} - \delta_{i,q}\delta_{j,p})\,\tfrac{1}{2}\,(\delta_{p,r}\delta_{q,s} - \delta_{p,s}\delta_{q,r})$$
$$= \tfrac{1}{4}\,(\delta_{i,r}\delta_{j,s} - \delta_{i,s}\delta_{j,r} - \delta_{i,s}\delta_{j,r} + \delta_{i,r}\delta_{j,s})$$
$$= \tfrac{1}{2}\,(\delta_{i,r}\delta_{j,s} - \delta_{i,s}\delta_{j,r}) = \Delta^{-}_{i,j,r,s}\,. \tag{A.7-49}$$

It is noted that $\Delta^{-}_{i,j,p,q}$ satisfies the symmetry relations $\Delta^{-}_{i,j,p,q} = -\Delta^{-}_{i,j,q,p} = \Delta^{-}_{j,i,q,p} = -\Delta^{-}_{j,i,p,q}$ and $\Delta^{-}_{i,j,p,q} = \Delta^{-}_{p,q,i,j}$.

The antisymmetrical unit tensor of rank four is particularly used to extract out of a tensor of rank two its antisymmetrical part. Let $\sigma_{p,q}$ be a tensor of rank two, then

$$\Delta^{-}_{i,j,p,q}\sigma_{p,q} = \tfrac{1}{2}(\sigma_{i,j} - \sigma_{j,i})\,, \tag{A.7-50}$$

the right-hand side of which is indeed antisymmetrical in the two subscripts. The tensor $\Delta^{-}_{i,j,p,q}$ finds application in electromagnetics, particularly in the four-dimensional electrodynamics of the special theory of relativity.

Exercises

Exercise A.7-1

Verify that $\delta_{m,m} = N$. (*Hint*: Observe that $\delta_{m,m} = \delta_{1,1} + \ldots + \delta_{N,N}$.)

Exercise A.7-2

Verify that $\delta_{m,n}\delta_{m,n} = N$.

Exercise A.7-3

Give the values of the Kronecker tensor in (a) one dimension, (b) two dimensions, (c) three dimensions.

Answer: (a) $\delta_{1,1} = 1$; (b) $\delta_{1,1} = \delta_{2,2} = 1$, all other elements are zero; (c) $\delta_{1,1} = \delta_{2,2} = \delta_{3,3} = 1$, all other elements are zero.

Exercise A.7-4

Verify that $\varepsilon_{m_1,\ldots,m_N}\varepsilon_{m_1,\ldots,m_N} = N!$ (*Hint*: Observe that for a non-vanishing result m_1 can be chosen in N different ways, m_2 in $N-1$ different ways, etc.)

Exercise A.7-5

Give the values of the Levi–Civita tensor in (a) one dimension, (b) two dimensions, (c) three dimensions.

Answer: (a) $\varepsilon_1 = 1$; (b) $\varepsilon_{1,2} = +1$, $\varepsilon_{2,1} = -1$, all other elements are zero; (c) $\varepsilon_{1,2,3} = \varepsilon_{2,3,1} = \varepsilon_{3,1,2} = +1$, $\varepsilon_{3,2,1} = \varepsilon_{2,1,3} = \varepsilon_{1,3,2} = -1$, all other elements are zero.

Exercise A.7-6

Show that from Equation (A.7-10) an alternative expression for $\det(\alpha_{p,m})$ follows as $\det(\alpha_{p,m}) = \alpha_{1,m_1} \cdots \alpha_{N,m_N}\varepsilon_{m_1,\ldots,m_N}$. (*Hint*: Take $p_1 = 1,\ldots,p_N = N$ and make use of $\varepsilon_{1,\ldots,N} = +1$.)

Exercise A.7-7

Show that from Equation (A.7-14) an alternative expression for $\det(\alpha_{p,n})$ follows as $\det(\alpha_{p,n}) = \varepsilon_{p_1,\ldots,p_N}\alpha_{p_1,1} \cdots \alpha_{p_N,N}$. (*Hint*: Take $n_1 = 1,\ldots,n_N = N$ and make use of $\varepsilon_{1,\ldots,N} = +1$.)

Exercise A.7-8

Show that in three-dimensional Euclidean space

$$\varepsilon_{m,p,q}\varepsilon_{m,r,s} = \delta_{p,r}\delta_{q,s} - \delta_{p,s}\delta_{q,r} = 2\Delta^{-}_{p,q,r,s} \,. \tag{A.7-51}$$

(*Hint*: Assign the values 1, 2, 3 to the dummy subscript m, write out the result and use the definitions of the Kronecker and the Levi–Civita tensors.) This equation plays an important role in many calculations with vectors in three-dimensional Euclidean space.

Exercise A.7-9

In three-dimensional Euclidean space the inner product (scalar product, dot product) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is introduced as $\boldsymbol{a} \cdot \boldsymbol{b} = a_m b_m$, and their outer product (vector product, cross product, wedge product) as $\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{m,p,q} a_p b_q$. Show with the aid of the summation convention and the unit tensors introduced in the present section that the following relations hold: (a) $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$; (b) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$; (c) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$; (d) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$; (e) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = -[\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{d})]\boldsymbol{a} + [\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{d})]\boldsymbol{b} = [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d}]\boldsymbol{c} - [(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}]\boldsymbol{d}$. (f) Evaluate $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ using the subscript notation.

Answer: (f) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \varepsilon_{p,q,r} a_p b_q c_r$.

Exercise A.7-10

Show that $V_a^N = \det(\alpha_{m,n})$, with $\alpha_{m,n} = a_m(n)$ for $m = 1,\dots,N$ and $n = 1,\dots,N$, differs from zero if, and only if, the vectors $\{a_m(1),\dots,a_m(N)\}$ are linearly independent. (*Hint*: Use the expression $\det(\alpha_{m,n}) = \varepsilon_{p_1,\dots,p_N} \alpha_{p_1,1} \cdots \alpha_{p_N,N}$ that follows from Equation (A.7-14) and observe that if $\lambda(1)\alpha_{p,1} + \dots + \lambda(N)\alpha_{p,N} = 0$ has a solution with at least two out of the coefficients $\{\lambda(1),\dots,\lambda(N)\}$ non-zero (definition of linear dependence), the right-hand side of the expression for $\det(\alpha_{m,n})$ vanishes. To show this, assume that $\lambda(1) \neq 0$ and $\{\lambda(2),\dots,\lambda(N)\} \neq \{0,\dots,0\}$, substitute $\alpha_{p,1} = -[\lambda(1)]^{-1}[\lambda(2)\alpha_{p,2} + \dots + \lambda(N)\alpha_{p,N}\}$ in the expression for $\det(\alpha_{m,n})$ and employ the symmetry in the subscript p_1 and any of the subscripts $p_2,\dots,p_N$ in the products of the elements $\alpha_{p,n}$ and the antisymmetry of the Levi–Civita tensor in these subscripts.)

Exercise A.7-11

Show, with the notation used in Exercise A.7-9, that in three-dimensional Euclidean space the base $\{\boldsymbol{b}(1),\boldsymbol{b}(2),\boldsymbol{b}(3)\}$ that is reciprocal to the base $\{\boldsymbol{a}(1),\boldsymbol{a}(2),\boldsymbol{a}(3)\}$, where $\{\boldsymbol{a}(1),\boldsymbol{a}(2),\boldsymbol{a}(3)\}$ is a sequence of linearly independent vectors, is given by $\boldsymbol{b}(1) = (1/V_a)\boldsymbol{a}(2) \times \boldsymbol{a}(3)$, $\boldsymbol{b}(2) = (1/V_a)\boldsymbol{a}(3) \times \boldsymbol{a}(1)$, $\boldsymbol{b}(3) = (1/V_a)\boldsymbol{a}(1) \times \boldsymbol{a}(2)$, in which $V_a = [\boldsymbol{a}(1) \times \boldsymbol{a}(2)] \cdot \boldsymbol{a}(3)$ is the volume of the parallelepiped spanned by $\{\boldsymbol{a}(1),\boldsymbol{a}(2),\boldsymbol{a}(3)\}$.

Exercise A.7-12

Verify that $\varepsilon_{m_1,\dots,m_i,\dots,m_j,\dots,m_n} \delta_{m_i,m_j} = 0$.

Exercise A.7-13

Verify that $\Delta^+_{i,j,p,q} \Delta^-_{p,q,r,s} = 0$ and that $\Delta^-_{i,j,p,q} \Delta^+_{p,q,r,s} = 0$.

Exercise A.7-14

Verify that $\Delta^{\delta}_{i,j,p,q}\Delta^{+}_{p,q,r,s} = \Delta^{\delta}_{i,j,r,s}$ and that $\Delta^{\delta}_{i,j,p,q}\Delta^{-}_{p,q,r,s} = 0$.

Exercise A.7-15

Show, with the aid of Equation (A.7-51), that in three-dimensional Euclidean space

$$\begin{aligned}\varepsilon_{p,m,n}\varepsilon_{q,m,n} &= \delta_{p,q}\delta_{m,m} - \delta_{p,m}\delta_{q,m} \\ &= 3\delta_{p,q} - \delta_{p,q} = 2\delta_{p,q}\,. \end{aligned} \tag{A.7-52}$$

A.8 Differentiaion of a tensor

Several definitions, theorems and rules of differential and integral calculus can conveniently be formulated by using the *Landau order notation* to indicate the relative "order of magnitude" of two functions. The two symbols are O and o. We give their definitions for functions defined over a domain in N-dimensional Euclidean space $\mathcal{R}^N$.

The Landau order symbol O

Let $f = f(x)$ and $g = g(x)$ denote two real- or complex-valued, functions that are defined over some domain $\mathcal{D} \subset \mathcal{R}^N$ ($\mathcal{D}$ need not be a proper subdomain, i.e. the case $\mathcal{D} = \mathcal{R}^N$ is allowed) and let $|f|$ and $|g|$ denote the absolute values, moduli or norms of the functions f and g, respectively. Then, we write

$$f(x) = O[g(x)] \qquad \text{for} \quad x \in \mathcal{D} \tag{A.8-1}$$

if there is a real number $A > 0$ such that

$$|f(x)| < A|g(x)| \qquad \text{for all} \quad x \in \mathcal{D}\,. \tag{A.8-2}$$

Note that the order symbol O in Equation (A.8-1) implies the absolute-value or norm signs in Equation (A.8-2). In particular, Equation (A.8-2) can hold in the limiting case upon approaching a particular point of $\mathcal{D}$. Let $\xi \in \mathcal{D}$ be that particular point, then if x approaches $\xi \in \mathcal{D}$ we have $|x - \xi| \to 0$ and we write for the relationship under consideration

$$f(x) = O[g(x)] \qquad \text{as} \quad x \to \xi\,. \tag{A.8-3}$$

Note: For the special case $g(x)$ = constant, Equation (A.8-1) implies that $|f(x)|$ is bounded for all $x \in \mathcal{D}$, while Equation (A.8-3) implies that $|f(x)|$ is bounded as $x \to \xi$.

The Landau order symbol *o*.

In contrast to the Landau order symbol O, the Landau symbol o always involves a limiting process. Again let $f = f(x)$ and $g = g(x)$ be two real- or complex-valued functions that are defined over some domain $\mathcal{D} \subset \mathcal{R}^N$ ($\mathcal{D}$ need not be a proper subdomain, i.e. the case $\mathcal{D} = \mathcal{R}^N$ is allowed) and let ξ be some point of $\mathcal{D}$. Then we write

$$f(x) = o[g(x)] \qquad \text{as} \quad x \to \xi\,, \tag{A.8-4}$$

if for any given real number $\varepsilon > 0$ (however small), we can find a real number $\delta > 0$ such that

$$|f(x)| < \varepsilon |g(x)| \qquad \text{for} \quad |x - \xi| < \delta\,. \tag{A.8-5}$$

Note: For the special case $g(x) = 1$, Equation (A.8-4) yields

$$|f(x)| = o(1) \qquad \text{as} \quad x \to \xi \tag{A.8-6}$$

which implies, in view of Equation (A.8-5), that $|f(x)| \to 0$ as $x \to \xi$.

Continuity

A function of $f = f(x)$ defined over some domain $\mathcal{D} \subset \mathcal{R}^N$ is *continuous* at the point $\xi \in \mathcal{D}$ if there exists a number F such that

$$f(x) = F + o(1) \qquad \text{as} \quad x \to \xi\,. \tag{A.8-7}$$

If Equation (A.8-7) holds, we write

$$F = \lim_{x \to \xi} f(x)\,, \tag{A.8-8}$$

or

$$F = f(\xi) \tag{A.8-9}$$

for short, which upon substitution yields

$$f(x) = f(\xi) + o(1) \qquad \text{as} \quad x \to \xi \tag{A.8-10}$$

for a continuous function.

Differentiability

A function $f = f(x)$ defined over some domain $\mathcal{D} \subset \mathcal{R}^N$ is *differentiable* at the point $\xi \in \mathcal{D}$ if for any unit vector a_m and any $h > 0$ (however small) there exists a sequence of N unique numbers $\{\partial_1 f(\xi), \ldots, \partial_N f(\xi)\}$ such that

$$f(\xi + ha) = f(\xi) + h a_m \partial_m f(\xi) + o(h) \qquad \text{as} \quad h \to 0\,. \tag{A.8-11}$$

Here, $\{\partial_1 f(\xi), \ldots, \partial_N f(\xi)\}$ are the N *partial derivatives* of f at ξ with respect to $x_1, \ldots, x_N$, respectively, and $a_m \partial_m f(\xi)$ is the *directional derivative* of f along a_m at ξ. The elements of the sequence $\{\partial_1 f(\xi),...,\partial_N f(\xi)\}$ transform like the components of a vector (see below); therefore, the subscript notation $\partial_m f(\xi)$ for the sequence is justified. The vector $\partial_m f(\xi)$ is also written as $\partial f(\xi)$ or $\nabla f(\xi)$ and is called the *gradient* of f at ξ. (Here, ∇ is the "nabla operator".) With $\boldsymbol{x} = \boldsymbol{\xi} + h\boldsymbol{a}$, Equation (A.8-11) can also be written as

$$f(\boldsymbol{x}) = f(\boldsymbol{\xi}) + (x_m - \xi_m)\partial_m f(\boldsymbol{\xi}) + o(\boldsymbol{x} - \boldsymbol{\xi}) \qquad \text{as} \quad \boldsymbol{x} \to \boldsymbol{\xi} . \tag{A.8-12}$$

Differentiation of a tensor

With regard to the differentiation of a tensor, two cases must be distinguished: differentiation with respect to a parameter (which in wave phenomena is usually the time coordinate), and differentiation with respect to the spatial (Cartesian) coordinates of the space in which the tensor is defined. For both types of differentiation the well-known rules of differential calculus apply. Only the transformation laws under the change of reference frame remain to be investigated. In the definitions we shall consider tensor functions of arbitrary rank $K \geqslant 0$ which, in general, depend on both a parameter t and the position vector $\boldsymbol{x}$ in N-dimensional Euclidean space.

Differentiation with respect to a parameter

Let σ be a tensor function of rank K in N-dimensional Euclidean space and assume that σ is at some $\boldsymbol{x}$ a differentiable function of the parameter t. Let $\sigma_{m_1,...,m_K}(\boldsymbol{x},t)$ denote the components of σ, then the partial derivative $\partial_t \sigma$ of σ with respect to t is a tensor of the same rank K, whose components are given by $\partial_t \sigma_{m_1,...,m_K}(\boldsymbol{x},t)$, where the derivative follows from Equation (A.8-11) with $N = 1$ and x_1 replaced by t. The proof that this quantity indeed satisfies the proper transformation law follows by observing that

$$\begin{aligned} \partial_t \sigma'_{p_1,...,p_K}(\boldsymbol{x}',t) &= \partial_t [\alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K} \sigma_{m_1,...,m_K}(\boldsymbol{x},t)] \\ &= \alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K} \partial_t \sigma_{m_1,...,m_K}(\boldsymbol{x},t) , \end{aligned} \tag{A.8-13}$$

where the property has been used that the coefficients $\alpha_{p,m}$ are independent of t. Derivatives of a higher order are defined in a similar manner. Note that here the subscript t on ∂_t is a reserved symbol for the parameter t on which σ depends and that this symbol is not available for use in the subscript notation and summation convention applicable to tensor components.

Differentiation with respect to one of the spatial coordinates

Let σ be a tensor function of rank K and assume that σ is a differentiable function of the spatial (Cartesian) coordinates $x_1,...,x_N$. Let $\sigma_{m_1,...,m_K} = \sigma_{m_1,...,m_K}(\boldsymbol{x},t)$ denote the components of σ, then the partial derivative $\partial_n \sigma$ of σ with respect to the spatial coordinate x_n is a tensor function of rank $K + 1$ whose components are denoted by $\partial_n \sigma_{m_1,...,m_K}(\boldsymbol{x},t)$, where ∂_n denotes the partial derivative with respect to x_n ($n = 1,...,N$) and where the derivatives follow from Equation (A.8-11). The quantity $\partial_n \sigma_{m_1,...,m_K}$ is commonly called the *gradient* of the tensor σ. To prove that $\partial_n \sigma_{m_1,\ldots,m_K}(x,t)$ indeed transforms like a tensor, it is observed that from Equation (A.3-4) it follows that (using the conventional fraction notation for differentiation)

$$\partial/\partial x'_q = (\partial x_n/\partial x'_q)\partial/\partial x_n = A_{n,q}\partial/\partial x_n = \alpha_{q,n}\partial/\partial x_n \,, \tag{A.8-14}$$

where Equations (A.3-6) and (A.3-15) have been used. On account of Equations (A.8-14) and (A.4-2) we then have (returning to our subscript notation for differentiation)

$$\begin{aligned}\partial'_q \sigma'_{p_1,\ldots,p_K}(x',t) &= \alpha_{q,n}\partial_n[\alpha_{p_1,m_1} \cdots \alpha_{p_K,m_k}\sigma_{m_1,\ldots,m_K}(x,t)] \\ &= \alpha_{q,n}\alpha_{p_1,m_1} \cdots \alpha_{p_K,m_K}\partial_n\sigma_{m_1,\ldots,m_K}(x,t) \,,\end{aligned} \tag{A.8-15}$$

where the property has been used that the coefficients $\alpha_{p,m}$ are independent of x_n. Equation (A.8-15) is the transformation law for a tensor of rank $K+1$. Derivatives of a higher order are defined in a similar manner.

In the process of forming derivatives with respect to the spatial coordinates, contractions can also be included. This can most easily be accomplished through the use of the Kronecker tensor introduced in Section A.7 (for an example, see Exercise A.8-1).

In those cases where no ambiguity arises, the indication of the arguments for t and x is usually omitted.

Exercises

Exercise A.8-1

Let $\sigma_{m_1,\ldots,m_K}$ be a tensor of rank $K \geqslant 1$ in N-dimensional Euclidean space. What is the rank of the tensor $\partial_{m_1}\sigma_{m_1,\ldots,m_K}$?

Answer: $K-1$. (*Note*: This tensor is commonly called the *divergence* of σ.)

Exercise A.8-2

Let $\tau_{m,n}$ be a tensor of rank two. What is the rank of the tensor $\partial_m\tau_{m,n}$?

Answer: 1 (i.e. it is a vector).

Exercise A.8-3

Evaluate the scalar $\partial_m x_m (= \partial_1 x_1 + \ldots + \partial_N x_N)$.

Answer: $\partial_m x_m = N$.

Exercise A.8-4

Show that $\partial_p x_m = \delta_{p,m}$.

Exercise A.8-5

In three-dimensional Euclidean space the gradient of a scalar function ϕ of position is introduced as $\operatorname{grad}\phi = \partial_m\phi$, the divergence of a vector function v of position as $\operatorname{div} v = \partial_m v_m$, and the curl (rotation) of a vector function w of position as curl $w = \varepsilon_{m,p,q}\partial_p w_q$. Show that the following relations hold: (a) div $(v \times w) = w \cdot \text{curl}\, v - v \cdot \text{curl}\, w$; (b) curl curl w = grad div $w - \Delta w$, where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator (Laplacian); (c) div $x = 3$; (d) curl $x = \mathbf{0}$.

Exercise A.8-6

Let $|x - y| = [(x_m - y_m)(x_m - y_m)]^{1/2} \geqslant 0$ and let ∂_p denote differentiation with respect to x_p. Determine:

(a) $\partial_p |x - y|^\alpha$;
(b) $\partial_q \partial_p |x - y|^\alpha$;
(c) $\partial_p \partial_p |x - y|^\alpha$.
(d) For which values of α is $\partial_p \partial_p |x - y|^\alpha = 0$?

Answers:

(a) $\alpha |x - y|^{\alpha-2}(x_p - y_p)$;
(b) $\alpha(\alpha - 2)|x - y|^{\alpha-4}(x_q - y_q)(x_p - y_p) + \alpha |x - y|^{\alpha-2}\delta_{p,q}$;
(c) $\alpha(\alpha - 2 + N)|x - y|^{\alpha-2}$;
(d) $\alpha = 0$, and $\alpha = -N + 2$ for $x \neq y$ and $N \geqslant 3$.

Exercise A.8-7

In the wave motion in a time-invariant configuration the time coordinate occurs as an independent parameter which is not (at least pre-relativistically not) related to the spatial metric of the configuration. In such configurations the function that reproduces itself after differentiation plays an important role. The relevant function is the exponential function $\phi = \exp(t)$ that satisfies the ordinary differential equation $\partial_t \phi = \phi$ and the initial condition $\phi = 1$ for $t = 0$. Show that the function $\psi = \psi(T)$ that satisfies the ordinary differential equation $\partial_t \psi - a\psi = 0$ and the initial condition $\psi = A$ for $t = 0$ is given by $\psi = A \exp(at)$.

A.9 Geometrical objects of a particular shape in *N*-dimensional Euclidean space

In the theory of the integration of tensors in N-dimensional Euclidean space in particular, a number of *domains* (open, connected sets) of specific shape are commonly encountered. Their defining relations are given in terms of the coordinates $\{x_1,...,x_N\}$ of the *points* that are their elements.

Curve in space

A *curve* $\mathcal{L}^N$ in N-dimensional Euclidean space can conveniently be described by its parametric representation (Figure A.9-1):

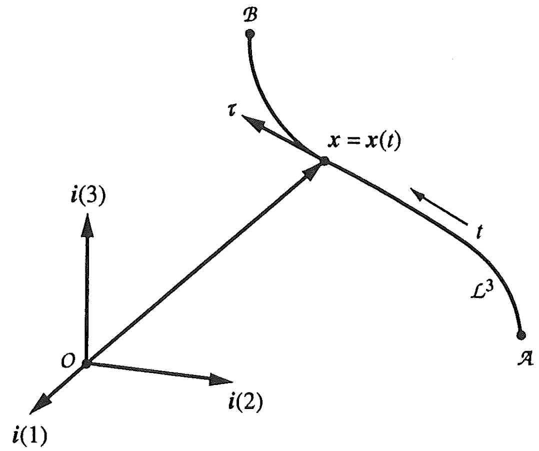

Figure A.9-1 Curve $\mathcal{L}^3$ in three-dimensional Euclidean space with parametric representation $x = x(t)$ and unit vector τ along its tangent.

$$x_m = x_m(t) \qquad \text{with} \quad t(\mathcal{A}) < t < t(\mathcal{B}) , \tag{A.9-1}$$

where t is a real-valued parameter, $t(\mathcal{A})$ corresponds to the starting point $\mathcal{A}$ of $\mathcal{L}^N$ (with coordinates $x_m[t(\mathcal{A})]$), and $t(\mathcal{B})$ corresponds to the end point $\mathcal{B}$ of $\mathcal{L}^N$ (with coordinates $x_m[t(\mathcal{B})]$). When, for example, t is the time coordinate, Equation (A.9-1) can represent the trajectory of a particle moving from $\mathcal{A}$ to $\mathcal{B}$ or the trajectory of a light ray that travels from $\mathcal{A}$ to $\mathcal{B}$.

Let $\tau_m = \tau_m(t)$ be the unit vector along the tangent to $\mathcal{L}^N$, then for two neighbouring points corresponding to the parameter values t and $t + h$, respectively, we have

$$x_m(t+h) = x_m(t) + \lambda(t)\tau_m(t)h + o(h) \qquad \text{as} \quad h \to 0 , \tag{A.9-2}$$

in which $\lambda(t)$ remains to be determined such that $\tau_m(t)\tau_m(t) = 1$. In view of the definition of the derivative we also have

$$x_m(t+h) = x_m(t) + \partial_t x_m(t)h + o(h) \qquad \text{as} \quad h \to 0 , \tag{A.9-3}$$

and, hence,

$$\lambda(t)\tau_m(t) = \partial_t x_m(t) . \tag{A.9-4}$$

From

$$[\lambda(t)\tau_m(t)][\lambda(t)\tau_m(t)] = \partial_t x_m(t)\partial_t x_m(t) \tag{A.9-5}$$

and the condition

$$\tau_m(t)\tau_m(t) = 1 \tag{A.9-6}$$

it follows that

$$\lambda(t) = [\partial_t x_m(t)\partial_t x_m(t)]^{-1/2} > 0 . \tag{A.9-7}$$

Consequently,

$$\tau_m = \frac{\partial_t x_m(t)}{[\partial_t x_n(t) \partial_t x_n(t)]^{1/2}} . \tag{A.9-8}$$

A special case arises if the arc length s along $\mathcal{L}^N$ is chosen as the parameter in the representation. Correspondingly, let

$$x_m = x_m(s) \qquad \text{where } s(\mathcal{A}) < s < s(\mathcal{B}) \tag{A.9-9}$$

be the parametric representation, then

$$x_m(s+h) = x_m(s) + \tau_m(s)h + o(h) \qquad \text{as} \quad h \to 0 , \tag{A.9-10}$$

since, by the definition of arc length,

$$[x_m(s+h) - x_m(s)][x_m(s+h) - x_m(s)] = h^2 + o(h^2) \qquad \text{as} \quad h \to 0 , \tag{A.9-11}$$

Equations (A.9-10) and (A.9-11) are compatible in view of the relationship $\tau_m(s)\tau_m(s) = 1$. Upon comparing Equation (A.9-10) with

$$x_m(s+h) = x_m(s) + [\partial_s x_m(s)]h + o(h) \qquad \text{as} \quad h \to 0 , \tag{A.9-12}$$

it now follows that

$$\tau_m(s) = \partial_s x_m(s) . \tag{A.9-13}$$

Note that the subscript s on ∂_s is now a reserved symbol for the arc-length parameter and is not available for use in the subscript notation and summation conventions applicable to tensor components.

Block or rectangle

The N-dimensional *block* or *rectangle* (N-block or N-rectangle, for short) Π^N in N-dimensional Euclidean space is defined by (Figure A.9-2)

$$\Pi^N = \{x \in \mathcal{R}^N; \, a_m < x_m < b_m \quad \text{for} \quad m = 1,\dots,N\} ; \tag{A.9-14}$$

its *edge lengths* are obviously given by $\{(b_m - a_m) \text{ for } m = 1,\dots,N\}$. Note that the edges of Π^N are parallel to the axes of the chosen reference frame.

Cube

The N-dimensional *cube* (N-cube, for short) $\mathcal{K}^N$ in N-dimensional Euclidean space is defined by

$$\mathcal{K}^N = \{x \in \mathcal{R}^N; \, a < x_m < b\} ; \tag{A.9-15}$$

the edge length, common to all edges, is obviously given by $b - a$.

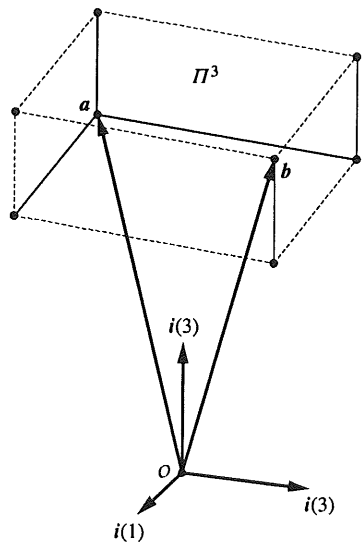

Figure A.9-2 Block or rectangle Π^3 in three-dimensional Euclidean space.

Parallelepiped or parallelogram

The N-dimensional *parallelepiped* or N-dimensional *parallelogram* (N-parallelepiped or N-parallelogram, for short) $\mathcal{P}^N$ in N-dimensional Euclidean space is defined by (Figure A.9-3)

$$\mathcal{P}^N = \left\{ x \in \mathcal{R}^N ;\ x_m = x_m(0) + \sum_{I=1}^{N} \lambda(I)\,[x_m(I) - x_m(0)] \right. \\ \left. \text{with} \quad 0 < \lambda(I) < 1, \quad \text{for } I = 1,\dots,N \right\} \tag{A.9-16}$$

Here, $\{x_m(I); I = 0,\dots,N\}$ is an ordered sequence of $(N+1)$ out of the vertices of $\mathcal{P}^N$, the ordering of which defines the orientation of $\mathcal{P}^N$. (Note that the total number of vertices of $\mathcal{P}^N$ is 2^N.)

Simplex

The N-dimensional simplex (N-simplex, for short) Σ^N in N-dimensional Euclidean space is defined by (Figure A.9-4)

$$\Sigma^N = \left\{ x \in \mathcal{R}^N ;\ x_m = \sum_{I=0}^{N} \lambda(I) x_m(I) \right. \\ \left. \text{with} \quad 0 < \lambda(I) < 1, \quad \text{for } I = 1,\dots,N, \ \text{ and } \sum_{I=0}^{N} \lambda(I) = 1 \right\}. \tag{A.9-17}$$

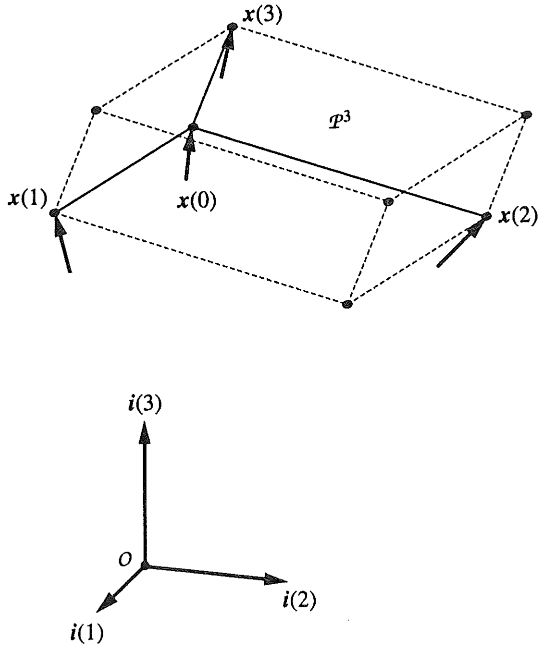

Figure A.9-3 Parallelepiped or parallelogram $\mathcal{P}^3$ in three-dimensional Euclidean space.

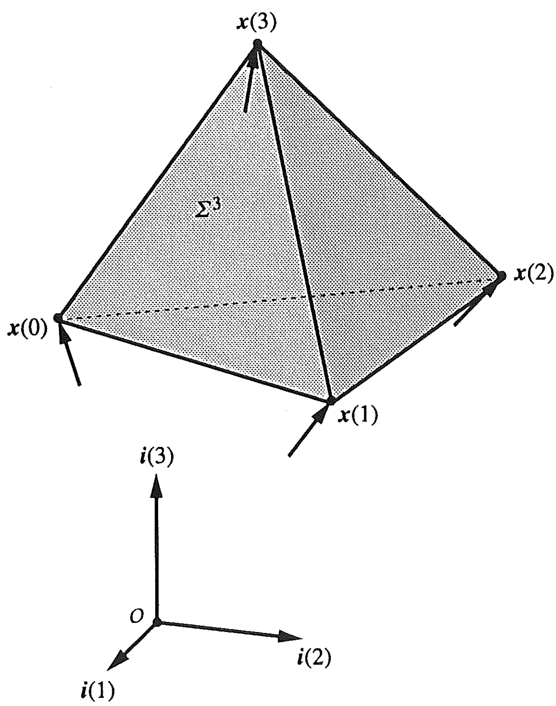

Figure A.9-4 Simplex Σ^3 in three-dimensional Euclidean space.

Here, $\{x_m(I);I = 0,...,N\}$ is the ordered sequence of its $(N + 1)$ *vertices* through which the orientation of Σ^N is defined, and $\{\lambda(I);I = 0,...,N\}$ are the *barycentric coordinates* of a point in Σ^N. Note that in (A.9-17) the vertices occur in a completely symmetrical manner. Special cases are: Σ^0 (point), Σ^1 (line segment), Σ^2 (triangle), Σ^3 (tetrahedron).

Ball

The N-dimensional *ball* (N-ball, for short) $\mathcal{B}^N$ in N-dimensional Euclidean space is defined by (Figure A.9-5)

$$\mathcal{B}^N = \{x \in \mathcal{R}^N;\ (x_m - b_m)(x_m - b_m) < r^2\}\ . \tag{A.9-18}$$

Here, b_m is the position vector of the *centre* of $\mathcal{B}^N$, and $r > 0$ is its *radius*.

Sphere

The N-dimensional *sphere* (N-sphere, for short) $\mathcal{S}^N$ in N-dimensional Euclidean space is defined by (Figure A.9-5)

$$\mathcal{S}^N = \{x \in \mathcal{R}^N;\ (x_m - b_m)(x_m - b_m) = r^2\}\ . \tag{A.9-19}$$

Here, b_m is the position vector of the its *centre*, and $r > 0$ is its *radius*.

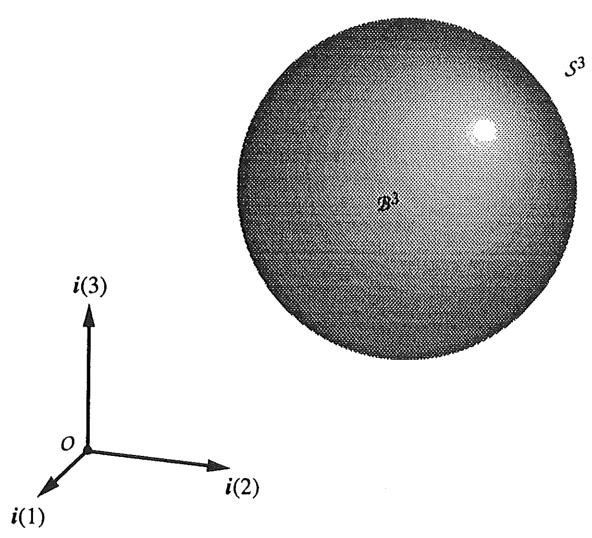

Figure A.9-5 Ball $\mathcal{B}^3$ and sphere $\mathcal{S}^3$ in three-dimensional Euclidean space.

Unit sphere

The N-dimensional *unit sphere* Ω^N in N-dimensional Euclidean space is defined by

$$\Omega^N = \{x \in \mathcal{R}^N;\ (x_m - b_m)(x_m - b_m) = 1\}\ . \tag{A.9-20}$$

Here, b_m is the position vector of the the *centre* of Ω^N; its *radius* is, obviously, unity.

Ellipsoid

The N-dimensional *ellipsoid* (N-ellipsoid, for short) $\mathcal{E}^N$ is defined by (Figure A.9-6)

$$\mathcal{E}^N = \left\{ x \in \mathcal{R}^N;\ \sum_{m=1}^{N} \frac{(x_m - b_m)^2}{a_m^2} < 1 \right\}. \tag{A.9-21}$$

Here, b_m is the position vector of its *centre*, and $\{a_m; m = 1,...,N\}$ is the collection of its *semi-axes*.

Boundary surface

In general, the *boundary surface of a domain* $\mathcal{D} \subset \mathcal{R}^N$ will be denoted by $\partial\mathcal{D}$. The representation of $\partial\mathcal{D}$ is conveniently expressed as (Figure A.9-7)

$$\partial\mathcal{D} = \{x \in \mathcal{R}^N;\ f_{\mathcal{D}}(x_1,...,x_N) = 0\}\ , \tag{A.9-22}$$

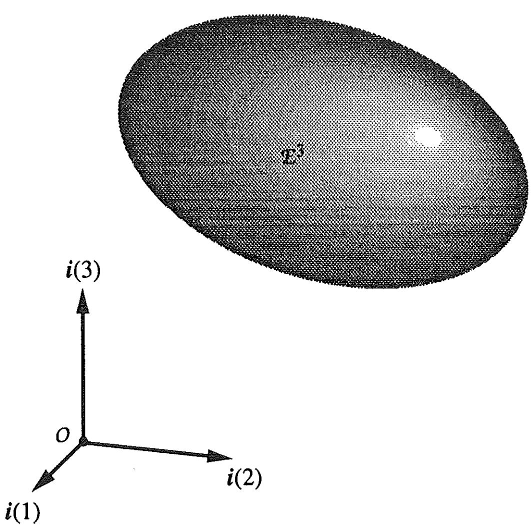

Figure A.9-6 Ellipsoid $\mathcal{E}^3$ in three-dimensional Euclidean space.

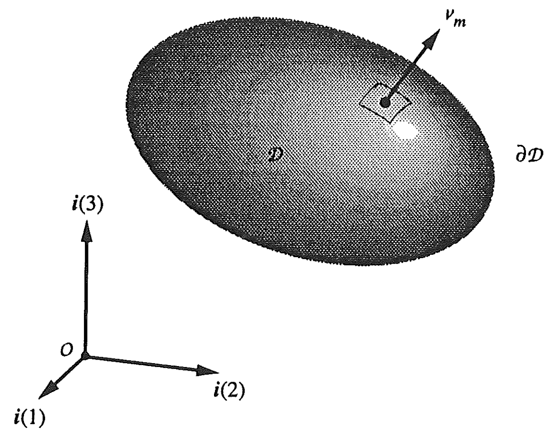

Figure A.9-7 Bounded domain $\mathcal{D}$ with closed boundary surface $\partial\mathcal{D}$ in three-dimensional Euclidean space; ν_m is the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$.

where $f_{\mathcal{D}}$ is some scalar function of position. Through the representation of Equation (A.9-22) the unit vector ν_m along the normal to $\partial\mathcal{D}$, a quantity that often occurs in the integration of a number of physical quantities over the boundary surface of a certain domain, can easily be calculated. Consider two points on $\partial\mathcal{D}$ with the position vectors x_m and $x_m + \xi_m$, respectively. Since both points are situated on $\partial\mathcal{D}$, we have

$$f_{\mathcal{D}}(x_1,\dots,x_N) = 0 \tag{A.9-23}$$

and

$$f_{\mathcal{D}}(x_1 + \xi_1,\dots,x_N + \xi_N) = 0\,, \tag{A.9-24}$$

from which it follows that

$$f_{\mathcal{D}}(x_1 + \xi_1,\dots,x_N + \xi_N) - f_{\mathcal{D}}(x_1,\dots,x_N) = 0\,. \tag{A.9-25}$$

However, for sufficiently small $|\xi|$ we have, by virtue of the definition of partial derivative (see Equation (A.8-11))

$$f_{\mathcal{D}}(x_1 + \xi_1,\dots,x_N + \xi_N) - f_{\mathcal{D}}(x_1,\dots,x_N) = \xi_m\partial_m f_{\mathcal{D}}(x_1,\dots,x_N) + o(\xi) \quad \text{as} \quad |\xi|\to 0\,. \tag{A.9-26}$$

Now, in the limit $|\xi|\to 0$, ξ_m is situated in the tangent plane to $\partial\mathcal{D}$ at the position x_m, while from Equations (A.9-25) and (A.9-26) it follows that

$$\xi_m\partial_m f_{\mathcal{D}}(x_1,\dots,x_N) = o(\xi) \qquad \text{as} \quad |\xi|\to 0\,. \tag{A.9-27}$$

Hence, in the limit $|\xi|\to 0$, the vector $\partial_m f_{\mathcal{D}}(x_1,\dots,x_N)$, i.e. the gradient of $f_{\mathcal{D}}$ at the position x_m, must be perpendicular to the tangent plane at the position x_m, which implies that it is oriented along the normal to $\partial\mathcal{D}$ at the position x_m. Therefore, we can write

$$\nu_m(x) = \alpha(x)\partial_m f_{\mathcal{D}}(x)\,, \tag{A.9-28}$$

where the coefficient α remains to be determined. Using the condition that ν_m must have unit length, i.e. $\nu_m\nu_m = 1$, we obtain

$$\alpha^2(x)\partial_m f_{\mathcal{D}}(x)\partial_m f_{\mathcal{D}}(x) = 1\,, \tag{A.9-29}$$

or

$$\alpha(x) = \pm\,[\partial_m f_{\mathcal{D}}(x)\partial_m f_{\mathcal{D}}(x)]^{-1/2}. \tag{A.9-30}$$

Hence,

$$\nu_m = \pm\frac{\partial_m f_{\mathcal{D}}}{(\partial_n f_{\mathcal{D}}\partial_n f_{\mathcal{D}})^{1/2}}\,. \tag{A.9-31}$$

The $\pm$ sign on the right-hand side indicates that ν_m can point in two opposite directions, i.e. geometrically away from $\mathcal{D}$ and towards $\mathcal{D}$.

If $\mathcal{D}$ is a *bounded domain*, the boundary surface $\partial\mathcal{D}$ is *closed* and ν_m can point either to the exterior of $\partial\mathcal{D}$ or to the interior of $\partial\mathcal{D}$.

Star-shaped domain

A domain $\mathcal{D}\subset\mathcal{R}^N$ is *star-shaped* with respect to a particular interior point if none of the straight lines joining that particular point with any other point of $\mathcal{D}$ crosses the boundary of $\mathcal{D}$ (Figure A.9-8).

Exercises

Exercise A.9-1

Determine the unit vector τ_m along the tangent to the curve $\mathcal{L}^N = \{x\in\mathcal{R}^N;\, x_m = a_m s + b_m\}$ (straight line), where s is the arc length.

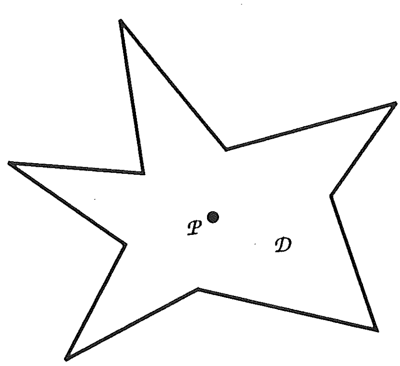

Figure A.9-8 Domain $\mathcal{D}$ that is star-shaped with respect to the point $\mathcal{P}$.

Answer: $\tau_m = a_m$ (i.e. a constant).

Exercise A.9-2

Determine the unit vector ν_m along the outward normal to the sphere $\mathcal{S}^N = \{x \in \mathcal{R}^N;\ (x_m - b_m)(x_m - b_m) = r^2\}$.

Answer: $\nu_m = (x_m - b_m)/r$.

A.10 Integration of a tensor

With regard to the integration of a tensor, again two cases have to be distinguished: integration with respect to a parameter (which in wave phenomena is usually the time coordinate), and integration over a certain manifold (curve, domain, boundary surface of a domain) in space. In those definitions or properties where no ambiguity arises, the subscripts indicating the rank of the tensor involved are omitted.

Integration with respect to a parameter

Let $\sigma = \sigma(t)$ be a real- or complex-valued tensor function of arbitrary rank of the real parameter t that is defined by the interval $t(\mathcal{A}) < t < t(\mathcal{B})$ and is continuous over this interval. Furthermore, let the interval be subdivided into NI subintervals of equal length h, of which the grid points $\{t(IJ); IJ = 0,\dots,NJ\}$, with $t(\mathcal{A}) = t(0)$ and $t(\mathcal{B}) = t(NJ)$, are the successive starting and end points. Then, we can write

$$\sigma[t(\mathcal{B})] - \sigma[t(\mathcal{A})] = \sum_{IJ=1}^{NJ} \{\sigma[t(IJ)] - \sigma[t(IJ-1)]\}\ , \qquad \text{(A.10-1)}$$

while

$$t(\mathcal{B}) - t(\mathcal{A}) = (NJ)h\ . \qquad \text{(A.10-2)}$$

In the limit $NJ \to \infty$ and $h \to 0$ such that Equation (A.10-2) remains valid, Equation (A.10-1) also remains valid, i.e. the limiting procedure can be carried out. To indicate the relevant limiting procedure we write

$$\sigma[t(\mathcal{B})] - \sigma[t(\mathcal{A})] = \int_{t=t(\mathcal{A})}^{t(\mathcal{B})} \mathrm{d}\sigma(t)\ , \qquad \text{(A.10-3)}$$

where $\int$ is the integral sign and $\mathrm{d}\sigma$ is called the *differential* of σ. Now, if in addition to the condition of continuity, $\sigma = \sigma(t)$ is differentiable over the interval $t(\mathcal{A}) < t < t(\mathcal{B})$ we can, on account of Equation (A.8-11), write

$$\sigma[t(IJ)] - \sigma[t(IJ-1)] = \partial_t\sigma\,[t(IJ-1)]\,h + o(h) \quad \text{as} \quad h \to 0\ . \qquad \text{(A.10-4)}$$

In the limit $NJ \to \infty$ and $h \to 0$, still subject to Equation (A.10-2), Equation (A.10-4) becomes

$$\mathrm{d}\sigma(t) = \partial_t\sigma(t)\,\mathrm{d}t\ , \qquad \text{(A.10-5)}$$

where the notation $h \to dt$ is justified by taking σ in Equation (A.10-4) to be the scalar function $\sigma(t) = t$. Substitution of Equation (A.10-5) in Equation (A.10-3) yields

$$\sigma[t(\mathcal{A})] - \sigma[t(\mathcal{B})] = \int_{t=t(\mathcal{A})}^{t(\mathcal{B})} \partial_t \sigma(t)\, dt\,, \tag{A.10-6}$$

which holds for any continuously differentiable tensor fuction $\sigma = \sigma(t)$. Equation (A.10-6) is one of the principal theorems of integral calculus. It holds for any tensor function of the parameter t that meets the indicated requirements.

Integration along a curve in space

Let us consider the integration of a continuous tensor function of arbitrary rank $\sigma = \sigma(\boldsymbol{x})$ positioned in N-dimensional Euclidean space along an oriented curve, and let the oriented curve $\mathcal{L}^N$ be given by its parametric representation

$$\mathcal{L}^N = \{x \in \mathcal{R}^N;\, x_p = x_p(s) \quad \text{with} \quad s(\mathcal{A}) < s < s(\mathcal{B})\}\,, \tag{A.10-7}$$

with the arc length s as the parameter, $\mathcal{L}^N$ being oriented from $\mathcal{A}$ as its starting point to $\mathcal{B}$ as its end point. A typical integral along $\mathcal{L}^N$ is then

$$\int_{x \in \mathcal{L}^N} \sigma(\boldsymbol{x})\, dx_p = \int_{s=s(\mathcal{A})}^{s(\mathcal{B})} \sigma[x_1(s),\dots,x_N(s)]\partial_s x_p(s)\, ds\,, \tag{A.10-8}$$

where ∂_s means differentiation with respect to s. (Note that in ∂_s the subscript s is a reserved symbol and is not a free subscript in the context of the summation convention.) In view of Equation (A.9-13), Equation (A.10-8) can also be written as

$$\int_{\mathcal{A}}^{\mathcal{B}} \sigma(\boldsymbol{x})\, dx_p = \int_{s=s(\mathcal{A})}^{s(\mathcal{B})} \sigma[x_1(s),\dots,x_N(s)]\tau_p(s)\, ds\,, \tag{A.10-9}$$

where $\tau_p = \tau_p(s)$ is the unit vector along the local tangent to $\mathcal{L}^N$.

A special case arises if σ is the gradient of some other tensor; the latter is then commonly called a "(scalar, vector, or tensor) potential" of σ. Let σ be of rank $K \geqslant 1$ and assume that we can write, for some $W = W(\boldsymbol{x})$,

$$\sigma_{m_1,\dots,m_{p-1},p,m_{p+1},\dots,m_K} = \partial_p W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}\,. \tag{A.10-10}$$

Substituting Equation (A.10-10) in Equation (A.10-8) and using the relationship from differential calculus

$$\partial_p W[x(s)]\,\partial_s x_p = \partial_s W[x(s)]\,, \tag{A.10-11}$$

which holds for any tensor, it follows with the aid of Equation (A.10-6) that

$$\int_{x \in \mathcal{L}^N} \sigma_{m_1,\dots,m_{p-1},p,m_{p+1},\dots,m_K}\, dx_p = \int_{s=s(\mathcal{A})}^{s(\mathcal{A})} \partial_s W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x_1(s),\dots,x_N(s)]\, ds$$

$$= W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{B})] - W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{A})]\,. \tag{A.10-12}$$

This result is evidently independent of the chosen integration path joining $\mathcal{A}$ and $\mathcal{B}$. Equation (A.10-12) has the consequence that, for tensor functions which satisfy Equation (A.10-10), integration along an oriented *closed curve* $\mathcal{C}^N$ (for which the starting point and end point coincide) yields the value zero, i.e.

$$\int_{x\in\mathcal{C}^N} \sigma(x)\, \mathrm{d}x_p = 0 \qquad \text{(A.10-13)}$$

provided that $\mathcal{C}^N$ is situated in the domain where Equation (A.10-10) holds.

Conversely, let $\mathcal{A}$ and $\mathcal{B}$ be two points located in a domain where for any closed curve Equation (A.10-13) holds, and let $\mathcal{L}^N$ be an oriented curve joining $\mathcal{A}$ and $\mathcal{B}$. Then, the application of Equation (A.10-13) to two different curves joining $\mathcal{A}$ and $\mathcal{B}$, and having the same orientation, shows that the integral

$$W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K} = \int_{x\in\mathcal{L}^N} \sigma_{m_1,\dots,m_{p-1},p,m_{p+1},\dots,m_K}\, \mathrm{d}x_p \qquad \text{(A.10-14)}$$

is independent of the chosen integration path joining $\mathcal{A}$ and $\mathcal{B}$. Let us now keep $\mathcal{A}$ fixed and consider the result of the integration as a function of the position vector $x(\mathcal{B})$. Let $\mathcal{C}$ be a point in the neighbourhood of $\mathcal{B}$, i.e.

$$x_p(\mathcal{C}) = x_p(\mathcal{B}) + ha_p\,, \qquad \text{(A.10-15)}$$

where a_p is a given unit vector and h can be arbitrarily small. Then, using Equation (A.10-14), we obtain

$$W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{C})] - W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{B})]$$
$$= \sigma_{m_1,\dots,m_{p-1},p,m_{p+1},\dots,m_K}[x(\mathcal{B})]\, ha_p + o(h) \qquad \text{as} \quad h\to 0\,. \qquad \text{(A.10-16)}$$

Using the definition of the directional derivative (see Equation (A.8-11)), we also have

$$W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{C})] - W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{B})]$$
$$= ha_p\partial_p W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}[x(\mathcal{B})] + o(h) \qquad \text{as} \quad h\to 0\,. \qquad \text{(A.10-17)}$$

Since Equations (A.10-16) and (A.10-17) hold for arbitrary unit vectors a_p, it follows that

$$\sigma_{m_1,\dots,m_{p-1},p,m_{p+1},\dots,m_K} = \partial_p W_{m_1,\dots,m_{p-1},m_{p+1},\dots,m_K}\,, \qquad \text{(A.10-18)}$$

i.e. the tensor function σ is the gradient of its potential W. Note, in this respect, that the rank of the potential W is in this case one lower than the rank of σ.

Integration over a domain and its boundary surface

For the integral of a given, continous, tensor function of a certain rank over a bounded domain $\mathcal{D}\subset\mathcal{R}^N$ we employ the standard definition of the Riemann integral. To this end, the domain $\mathcal{D}$ is covered by a grid of N-cubes $\{\mathcal{K}(IK); IK = 1,\dots,NK\}$ such that $\mathcal{D}$ is from the inside approximated by $\cup_{IK=1}^{NK}\mathcal{K}(IK)$ (Figure A.10-1).

Let $x(IK)$ be an interior point of $\mathcal{K}(IK)$ and let h be the edge length of the cubes, then

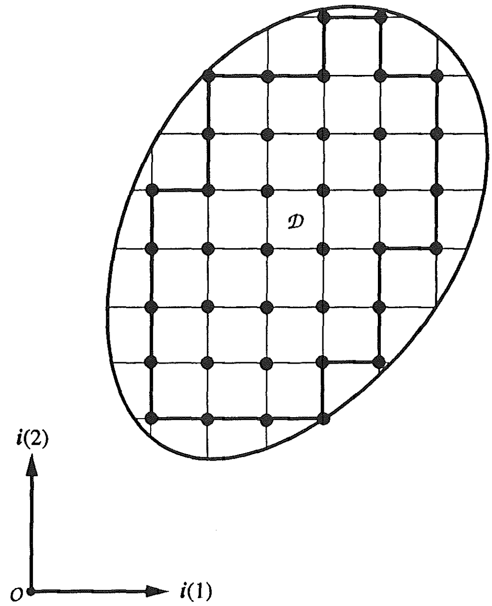

Figure A.10-1 Bounded domain $\mathcal{D}$ in two-dimensional Euclidean space and its approximation from the inside by a grid of 2-cubes with edge length h.

$$\int_{x\in\mathcal{D}} \sigma(x)\,\mathrm{d}V^N = \sum_{IK=1}^{NK} \alpha\,[x(IK)]\,h^N + o(h^N)$$
$$\text{as } h\to 0, \quad \text{keeping } (NK)h^N \text{ fixed}, \qquad \text{(A.10-19)}$$

where h^N is the volume of each N-cube.

To define the integral of a given, continuous, tensor function of a certain rank over the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$, the boundary surface is, for the moment, approximated by the union of the $(N-1)$-cubes that form the outer boundary of $\cup_{\mathrm{IK}=1}^{\mathrm{NK}} \mathcal{K}(IK)$. An important class of boundary integrals of tensor functions in physics also contains in the integrand the outward unit vector ν_p along the normal to the relevant surface. Due to the property that two surface integral contributions over common interfaces of adjacent cubes cancel (because of the assumed continuity of the tensor function, and the opposite orientations of the outward unit vectors along the normals to these interfaces), we can write

$$\int_{x\in\partial\mathcal{D}} \sigma(x)\nu_p\,\mathrm{d}A^{N-1} = \sum_{IK=1}^{NK} \int_{x\in\partial\mathcal{K}(IK)} \sigma(x)\nu_p\,\mathrm{d}A^{N-1} + o(h^N)$$
$$\text{as } h\to 0, \quad \text{keeping } (NK)h^N \text{ fixed}, \qquad \text{(A.10-20)}$$

where h^{N-1} is the surface area of a face of an N-cube.

The first special case arises for the tensor of rank zero $\sigma(x) = 1$ throughout $\mathcal{D}$. For this case, we have

$$\int_{x \in \partial \mathcal{K}(IK)} \nu_p \, \mathrm{d}A^{N-1} = 0 \qquad \text{for any} \quad p = 1,\ldots,N, \tag{A.10-21}$$

since two opposite faces perpendicular to the x_p axis of the chosen reference frame have equal areas h^{N-1} and opposite outward unit vectors along their normals. Consequently, combining Equations (A.10-21) and (A.10-20) and taking the limit $h \to 0$, it follows that

$$\int_{x \in \partial \mathcal{D}} \nu_p \, \mathrm{d}A^{N-1} = 0 \tag{A.10-22}$$

for any closed boundary surface. This result will be needed in Section A.12.

A second special case arises for the tensor of rank one $\sigma_m(x) = x_m$. For this case, we have

$$\int_{x \in \partial \mathcal{K}(IK)} x_m \nu_p \, \mathrm{d}A^{N-1} = \delta_{m,p} h^N , \tag{A.10-23}$$

since for $m \neq p$ the values of x_m on two opposite faces perpendicular to the x_p axis of the chosen reference frame have equal values, while the outward unit vectors along their normals have opposite values, whereas for $m = p$ the values of x_m on two such opposite faces differ by an amount h, the surface area of these faces being h^{N-1}. Thus,

$$\sum_{IK=1}^{NK} \int_{x \in \partial \mathcal{K}(IK)} x_m \nu_p \, \mathrm{d}A^{N-1} = (NK) h^N \delta_{m,p} , \tag{A.10-24}$$

where $(NK)h^N$ is the volume of the union of N-cubes approximating $\mathcal{D}$ from the inside. Combining Equation (A.10-24) with Equation (A.10-20) and taking the limit $h \to 0$, it follows that

$$\int_{x \in \partial \mathcal{D}} x_m \nu_p \, \mathrm{d}A^{N-1} = \delta_{m,p} V_{\mathcal{D}}^N = \delta_{m,p} \int_{x \in \mathcal{D}} \mathrm{d}V^N , \tag{A.10-25}$$

where $V_{\mathcal{D}}^N$ is the volume of $\mathcal{D}$. This result, too, will be needed in Section A.12.

The cubic grid that has been used to introduce the concept of integration over a bounded domain in N-dimensional Euclidean space indeed serves this purpose, but it fails to approximate adequately submanifolds in this space, especially the boundary surface of a domain when the surface is inclined with respect to the coordinate planes of the cubic grid. A covering that does not suffer from this deficiency, and is at the same time more fundamental from a topological point of view, is the covering with a grid of N-simplices all of which have vertices, and faces, in common. Let $\{\Sigma^N(I\Sigma); I\Sigma = 1,\ldots,N\Sigma\}$ be a collection of simplices such that the bounded domain $\mathcal{D}$ is, from the inside, approximated by $\cup_{I\Sigma=1}^{N\Sigma} \Sigma^N(I\Sigma)$ (Figure A.10-2).

Let $\sigma = \sigma(x)$ be a piecewise continuous tensor function defined on $\mathcal{D}$. Furthermore, let $\{\mathcal{P}(I\Sigma, 0),\ldots,\mathcal{P}(I\Sigma,N)\}$ denote the collection of the $N + 1$ vertices of the oriented simplex $\Sigma^N(I\Sigma)$. It is assumed that in the interior of each simplex, σ varies continuously with position, i.e. possible discontinuities in σ only occur across the common faces of two neighbouring simplices. In the interior of each of the simplices, σ is approximated by the linear interpolation between its values at the vertices, i.e.

$$\sigma(x) \approx \sum_{I=0}^{N} \lambda(I\Sigma,I) \sigma(I\Sigma,I) \qquad \text{for} \quad x \in \Sigma^N(I\Sigma) , \tag{A.10-26}$$

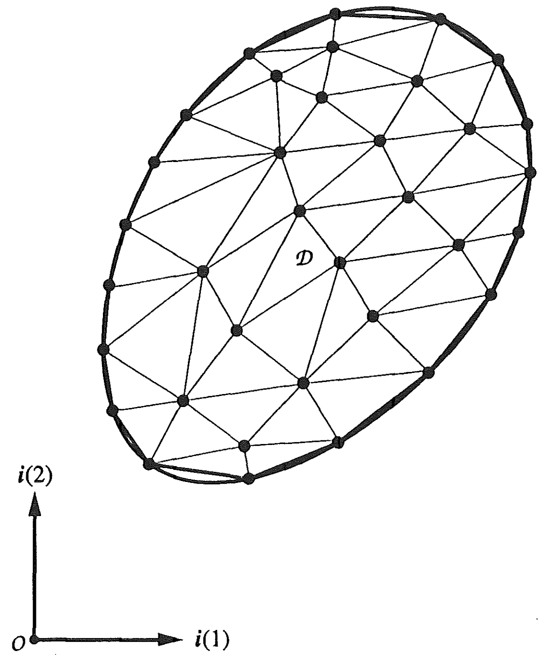

Figure A.10-2 Bounded domain $\mathcal{D}$ in two-dimensional Euclidean space and its approximation from the inside by a grid of 2-simplices.

where $\{\lambda(I\Sigma,0),\dots,\lambda(I\Sigma,N)\}$ are the barycentric coordinates of the point $\boldsymbol{x}$ in $\Sigma^N(I,\Sigma)$ (see Equation (A.9-17)) and $\sigma(I\Sigma,I)$ is the value of σ at $\mathcal{P}(I\Sigma,I)$. The barycentric coordinates satisfy the relationships $0 < \lambda(I\Sigma,I) < 1$ and $\lambda(I\Sigma,0) + \dots + \lambda(I\Sigma,N) = 1$. The integral of σ over the domain $\mathcal{D}$ is now defined as the limiting procedure

$$\int_{\boldsymbol{x}\in\mathcal{D}} \sigma(\boldsymbol{x})\,\mathrm{d}V^N = \sum_{I\Sigma=1}^{N\Sigma}\left[V_\Sigma^N(I\Sigma)\,\frac{1}{N+1}\sum_{I=0}^{N}\sigma(I\Sigma,I)\right] + o(h^N) \quad \text{as } h\to 0\,, \tag{A.10-27}$$

where $V_\Sigma^N(I\Sigma)$ is the volume of $\Sigma^N(I\Sigma)$, and h is the supremum of the maximum diameters of the simplices. If Equation (A.10-26) is exact, i.e. if σ indeed varies linearly with position in each simplex, the term in Equation (A.10-27) containing the order symbol is zero. Neglecting the term containing the order symbol, Equation (A.10-27) provides the numerical integration rule which is known as the "simplicial rule"; this is the N-dimensional counterpart of the one-dimensional "trapezoidal rule". Now, two things remain to be done: an expression needs to be obtained for $V_\Sigma^N(I\Sigma)$ in terms of the position vectors of its vertices; and it needs to be verified that for tensor functions varying linearly with position in the interior of each of the simplices, Equation (A.10-27) holds in the absence of the term containing the order symbol. These two problems are dealt with separately below.

Volume of an N-simplex

Let Σ^N be an arbitrary oriented N-simplex and let $\{\mathcal{P}(0),...,\mathcal{P}(N)\}$ be its vertices having the corresponding position vectors $\{\boldsymbol{x}(0),...,\boldsymbol{x}(N)\}$. To determine the volume V_Σ^N of Σ^N, one of the vertices, $\mathcal{P}(0)$ say, is taken to be the preferred one. Next, the vectorial edges $\boldsymbol{a}(1),...,\boldsymbol{a}(N)$ of the N-parallelepiped $\mathcal{P}^N$ leaving $\mathcal{P}(0)$ are introduced, i.e.

$$\boldsymbol{a}(J) = \boldsymbol{x}(J) - \boldsymbol{x}(0) \qquad \text{for} \quad J = 1,...,N. \tag{A.10-28}$$

The volume $V_\mathcal{P}^N$ of this N-parallelepiped is given by (see Equation (A.7-19))

$$V_\mathcal{P} N = \det[\boldsymbol{a}(1),...,\boldsymbol{a}(N)]. \tag{A.10-29}$$

Now, the parallelepiped $\mathcal{P}^N$ can be defined as $\mathcal{P}^N = \{\boldsymbol{x}\in\mathcal{R}^N; \boldsymbol{x} = \boldsymbol{x}(0) + \Sigma_{J=1}^N \lambda(J)[\boldsymbol{x}(J) - \boldsymbol{x}(0)],\ 0 < \lambda(J) < 1$ for $J = 1,...,N\}$ and, hence, its volume can be expressed as

$$\begin{aligned} V_\mathcal{P}^N &= \int_{\boldsymbol{x}\in V_\mathcal{P}^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \int_{\lambda(1)=0}^{1} \mathrm{d}\lambda(1) \ldots \int_{\lambda(N)=0}^{1} \mathrm{d}\lambda(N) \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))}, \end{aligned} \tag{A.10-30}$$

where $\partial(x_1,...,x_N)/\partial(\lambda(1),...,\lambda(N))$ denotes the N-dimensional Jacobian of the transformation from $\{x_1,...,x_N\}$ to $\{\lambda(1),...,\lambda(N)\}$. From Equations (A.10-29) and (A.10-30) it follows that

$$\frac{\partial(x_1,...,x_M)}{\partial(\lambda(1),...,\lambda(N))} = \det[\boldsymbol{a}(1),...,\boldsymbol{a}(N)]. \tag{A.10-31}$$

For the volume V_Σ^N of the simplex, the following representation now holds:

$$\begin{aligned} V_\Sigma^N &= \int_{\boldsymbol{x}\in\Sigma^N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \int_{\lambda(1)=0}^{1} \mathrm{d}\lambda(1) \ldots \int_{\lambda(N)=0}^{1-\lambda(1)-...-\lambda(N-1)} \mathrm{d}\lambda(N) \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \int_{\lambda(1)=0}^{1} \mathrm{d}\lambda(1) \ldots \int_{\lambda(N-1)=0}^{1-\lambda(1)-...-\lambda(N-2)} \frac{[1-\lambda(1)-\ldots-\lambda(N-1)]}{1!} \mathrm{d}\lambda(N-1) \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \int_{\lambda(1)=0}^{1} \mathrm{d}\lambda(1) \ldots \int_{\lambda(N-2)=0}^{1-\lambda(1)-...-\lambda(N-3)} \frac{[1-\lambda(1)-\ldots-\lambda(N-2)]^2}{2!} \mathrm{d}\lambda(N-2) \\ &= \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \frac{1}{N!}. \end{aligned} \tag{A.10-32}$$

Combining this with Equation (A.10-30), it follows that

$$V_\Sigma^N = V_\mathcal{P}^N / N!. \tag{A.10-33}$$

The same result would have been obtained if any vertex other than $\mathcal{P}(0)$ had been taken as the preferred one.

Integral over an *N*-simplex of a polynomially varying function

For the typical integral over Σ^N which contains the barycentric coordinates in a polynomial fashion in its integrand, we consider the general expression

$$INTG(n(0),...,n(N)) = \int_{x\in\Sigma^N} [\lambda(0)]^{n(0)} \dots [\lambda(N)]^{n(N)} \,dx_1 \dots dx_N , \tag{A.10-34}$$

where $\{n(0),...,n(N)\}$ are non-negative integers. Obviously, using Equations (A.10-32) and (A.10-33)

$$INTG(0,...,0) = V_\Sigma^N = V_\mathcal{P}^N/N! . \tag{A.10-35}$$

In Equation (A.10-34) we introduce $\lambda(1),...,\lambda(N)$ as the variables of integration and, with $\lambda(0) = 1 - \lambda(1) - ... - \lambda(N)$, obtain

$$INTG(n(0),...,n(N)) = \frac{\partial(x_1,...,x_N)}{\partial(\lambda(1),...,\lambda(N))} \int_{\lambda(1)=0}^{1} [\lambda(1)]^{n(1)} \,d\lambda(1) \dots$$

$$\int_{\lambda(N)=0}^{1-\lambda(1)-...-\lambda(N-1)} [1 - \lambda(1) - \dots - \lambda(N)]^{n(0)} [\lambda(N)]^{n(N)} \,d\lambda(N) . \tag{A.10-36}$$

Integration by parts in the integral with respect to $\lambda(N)$ yields, for $n(0) \geqslant 1$,

$$\int_{\lambda(N)=0}^{1-\lambda(1)-...-\lambda(N-1)} [1 - \lambda(1) - \dots - \lambda(N)]^{n(0)} [\lambda(N)]^{n(N)} \,d\lambda(N)$$

$$= \frac{n(0)}{n(N)+1} \int_{\lambda(N)=0}^{1-\lambda(1)-...-\lambda(N-1)} [1 - \lambda(1) - \dots - \lambda(N)]^{n(0)-1} [\lambda(N)]^{n(N)+1} \,d\lambda(N) . \tag{A.10-37}$$

Combining Equations (A.10-36) and (A.10-37), we obtain the recurrence relationship

$$INTG(n(0),...,n(N)) = \frac{n(0)}{n(N)+1} INTG(n(0)-1,...,n(N)+1) . \tag{A.10-38}$$

Since Equation (A.10-34) is completely symmetrical in all its parameters, Equation (A.10-38) holds for an increase in any of the parameters by one and a decrease in any other parameter by one. Using the property of the factorial function

$$(n+1)! = (n+1)n! \quad \text{with} \quad 0! = 1 , \tag{A.10-39}$$

the recurrence relationship (A.10-38) is satisfied by taking for $INTG[n(0),...,n(N)]$ the expression

$$INTG(n(0),...,n(N)) = \frac{[n(0)]! \dots [n(N)]!}{[n(0) + \dots + n(N) + p]!} p! V_\Sigma^N , \tag{A.10-40}$$

where p is, as yet, arbitrary and where Equation (A.10-35) has been taken into account. Using Equation (A.10-34) again, and bringing it into the form of Equation (A.10-36) (i.e. eliminating $\lambda(0)$), it also follows that

$$INTG(n(0)+1,n(1),\dots,n(N)) = INTG(n(0),\dots,n(N)) - INTG(n(0),n(1)+1,\dots,n(N))$$
$$- \dots - INTG(n(0),n(1),\dots,n(N)+1)\,. \qquad \text{(A.10-41)}$$

Substitution of Equation (A.10-40) in Equation (A.10-41) leads, after factoring out the expression for $INTG(n(0),\dots n(N))$, to

$$n(0)+1 = [n(0)+1+n(1)+\dots+n(N)+p] - [n(1)+1] - \dots - [n(N)+1] \qquad \text{(A.10-42)}$$

or

$$p = N\,. \qquad \text{(A.10-43)}$$

Collecting the results, we arrive at

$$INTG(n(0),\dots,n(N)) = \frac{[n(0)]!\dots[n(N)]!N!}{[n(0)+\dots+n(N)+N]!}\,V_\Sigma^N\,. \qquad \text{(A.10-44)}$$

Equation (A.10-44) is, among other things, of importance in the finite-element modelling of wave fields on a simplicial grid.

By successively taking one of integers in $\{n(0),n(1),\dots,n(N)\}$ equal to one and the others equal to zero, Equations (A.10-26) and (A.10-44) lead to Equation (A.10-27) with the order term on the right-hand side put equal to zero.

Property 1

Let $\mathcal{D}$ be a bounded subdomain of N-dimensional Euclidean space $\mathcal{R}^N$ and let $\partial\mathcal{D}$ be its closed boundary surface. Let $\mathrm{d}A_p^{N-1}$ be the outwardly oriented elementary surface area on $\partial\mathcal{D}$, then

$$\int_{x\in\partial\mathcal{D}} \mathrm{d}A_p^{N-1} = 0\,. \qquad \text{(A.10-45)}$$

To prove this result, $\mathcal{D}$ is covered by a grid of N-simplices $\mathcal{D}_\Sigma = \cup_{I\Sigma=1}^{N\Sigma}\Sigma^N(I\Sigma)$ all of which have vertices and faces in common. Let h be the supremum of the maximum diameters of these simplices. It is assumed that $\mathcal{D}$ and $\partial\mathcal{D}$ are such that

$$\int_{x\in\partial\mathcal{D}} \mathrm{d}A_p^{N-1} = \int_{x\in\partial\mathcal{D}_\Sigma} \mathrm{d}A_p^{N-1} + o(h) \quad \text{as} \quad h\to 0\,. \qquad \text{(A.10-46)}$$

Equation (A.10-46) implies that $\mathcal{D}$ and $\partial\mathcal{D}$ can, to a sufficient degree of accuracy as indicated by the order symbol, be approximated by a grid of simplices and their boundaries. Domains in this category are called as *triangulizable*. Now, since the contributions of the faces that two adjacent simplices have in common cancel,

$$\int_{x\in\partial\mathcal{D}_\Sigma} \mathrm{d}A_p^{N-1} = \Sigma_{I\Sigma=0}^{N\Sigma} \int_{x\in\partial\Sigma^N(I\Sigma)} \mathrm{d}A_p^{N-1}\,. \qquad \text{(A.10-47)}$$

To prove that the right-hand side vanishes, the following lemma is used.

Lemma

Let $\{A_p^{N-1}(0),\ldots,A_p^{N-1}(N)\}$ denote the outwardly oriented vectorial surface areas of the faces opposite the vertices $\{\mathcal{P}(0),\ldots,\mathcal{P}(N)\}$ of the simplex Σ^N, then

$$A_p^{N-1}(0) + \ldots + A_p^{N-1}(N) = 0 . \tag{A.10-48}$$

To prove Equation (A.10-48), $\mathcal{P}(0)$ is taken as a preferred vertex and the vectorial edges of Σ^N leaving $\mathcal{P}(0)$ are introduced as

$$\boldsymbol{a}(J) = \boldsymbol{x}(J) - \boldsymbol{x}(0) \qquad \text{for} \quad J = 1,\ldots,N . \tag{A.10-49}$$

In terms of these we have (see Equation (A.7-21), for the vectorial surface area of the $(N-1)$-dimensional parallelepiped not containing $\boldsymbol{a}(J)$ and having a positive component parallel to $\boldsymbol{a}(J)$),

$$A_{p_J}^{N-1}(J) = -\frac{1}{(N-1)!}\, \varepsilon_{p_1,\ldots,p_J,\ldots,p_N} a_{p_1}(1) \ldots a_{p_{J-1}}(J-1) a_{p_{J+1}}(J+1) \ldots a_{p_N}(N)$$
$$\text{for} \quad J = 1,\ldots,N , \tag{A.10-50}$$

where p_j is the free subscript. Substitution of Equation (A.10-49) into Equation (A.10-50) yields

$$A_{p_J}^{N-1}(J) = -\frac{1}{(N-1)!}\, \varepsilon_{p_1,\ldots,p_J,\ldots,p_N} \left[x_{p_1}(1) - x_{p_1}(0)\right] \ldots \left[x_{p_{J-1}}(J-1) - x_{p_{J-1}}(0)\right]$$
$$\times \left[x_{p_{J+1}}(J+1) - x_{p_{J+1}}(0)\right] \ldots \left[x_{p_N}(N) - x_{p_N}(0)\right] . \tag{A.10-51}$$

(Note that the right-hand side does not contain the position vector $\boldsymbol{x}(J)$.)

Next, we determine the similar expression for $A_p^{N-1}(0)$ that does not contain the position vector $\boldsymbol{x}(0)$. Taking $\mathcal{P}(1)$ as the preferred vertex, we obtain

$$A_{p_1}^{N-1}(0) = \frac{1}{(N-1)!}\, \varepsilon_{p_1,\ldots,p_N} \left[x_{p_2}(2) - x_{p_2}(1)\right] \ldots \left[x_{p_N}(N) - x_{p_N}(1)\right] , \tag{A.10-52}$$

where the sign has been checked to correspond to

$$\frac{1}{N}\left[x_{p_1}(1) - x_{p_1}(0)\right] A_{p_1}^{N-1}(0) = V_\Sigma^N , \tag{A.10-53}$$

which is most easily done by using the result of Exercise A.10-1. In the right–hand side of Equation (A.10-52) we now rewrite each factor according to

$$x_{p_K}(K) - x_{p_K}(1) = \left[x_{p_K}(K) - x_{p_K}(0)\right] - \left[x_{p_K}(1) - x_{p_K}(0)\right] \quad \text{for} \quad K = 2,\ldots,N . \tag{A.10-54}$$

Using the cancellation properties of the Levi–Civita tensor when contracted with two equal vectors, it then follows that

$$A_p^{N-1}(0) = -A_p^{N-1}(1) - \ldots - A_p^N(N) , \tag{A.10-55}$$

which leads to Equation (A.10-48).

As a consequence of Equation (A.10-48) the right-hand side of Equation (A.10-47) vanishes, which in its turn leads in the limit $h \rightarrow 0$ via Equation (A.10-46) to Equation (A.10-45).

Property 2

For the configuration of Property 1 we also have

$$\int_{x \in \partial \mathcal{D}} x_q \, \mathrm{d}A_p = V_{\mathcal{D}}^N \delta_{p,q} \,, \tag{A.10-56}$$

where

$$V_{\mathcal{D}}^N = \int_{x \in \mathcal{D}} \mathrm{d}V^N = \int_{x \in \mathcal{D}} \mathrm{d}x_1 \dots \mathrm{d}x_N \tag{A.10-57}$$

is the volume of $\mathcal{D}$.

To prove Equation (A.10-56), we cover, for given p, the domain $\mathcal{D}$ by adjacent N-prisms of $(N-1)$-simplicial cross-section and approximate $\partial\mathcal{D}$ by using the corresponding $(N-1)$-simplices at the top and the bottom of the prisms. (The value of x_p at the "top" is greater than the value of x_p at the "bottom".) Since for the faces of the N-prisms that are parallel to the x_p axis we automatically have $\mathrm{d}A_p = 0$, only the top and the bottom of the N-prisms yield a non-vanishing contribution to the integral over $\mathcal{D}$. At the top and the bottom, the function x_q varies linearly with position and, hence, the values of the integrals over the top and the bottom of each N-prism are equal to the values of x_q at their barycentres, multiplied by the (outwardly oriented) vectorial surface areas of that top and bottom. In view of the geometrical shape of the N-prisms, the latter are equal and opposite. If now $q \neq p$, the x_q-values at top and bottom of each N-prism are equal and the contributions cancel. If, however, $q = p$, the x_q-values differ by an amount equal to the height of each N-prism and the contributions add up to the volume of the N-prism. In the limit $h \rightarrow 0$, the results of both cases combine to give Equation (A.10-56).

Properties 1 and 2 are needed in Section A.12, where Gauss' integral theorem is derived.

Exercises

Exercise A.10-1

Let Σ^N be an arbitrary oriented N-simplex and let $\{\mathcal{P}(0),\dots,\mathcal{P}(N)\}$ be its vertices having the corresponding position vectors $\{\boldsymbol{x}(0),\dots,\boldsymbol{x}(N)\}$. Let the vectorial edge from $\mathcal{P}(0)$ to $\mathcal{P}(K)$ be denoted by $\boldsymbol{a}(K) = \boldsymbol{x}(K) - \boldsymbol{x}(0)$. Then, the volume V_Σ^N of Σ^N is given by (see Equations (A.10-29) and (A.10-33))

$$V_\Sigma^N = \frac{1}{N!} \det[\boldsymbol{a}(1),\dots,\boldsymbol{a}(N)] \,. \tag{A.10-58}$$

Show that this expression can be rewritten as

$$V_\Sigma^N = \frac{1}{N!}\begin{vmatrix} 1 & \cdots & 1 \\ x_1(0) & \cdots & x_1(N) \\ \vdots & & \vdots \\ x_N(0) & \cdots & x_N(N) \end{vmatrix} . \qquad \text{(A.10-59)}$$

A.11 The Taylor expansion

Let $\sigma = \sigma(x)$ be an M times ($M \geqslant 1$) continuously differentiable tensor function of arbitrary rank and defined in a domain that is star-shaped with respect to the point having the position vector x_p. For ease of notation we omit in this section the subscripts that indicate the rank of σ. Let a_p be an arbitrary unit vector (i.e. $a_p a_p = 1$), then all "rays" emanating from x_p have the position vector $x_p + sa_p$, where s varies over non-negative values and has the dimension of a length (in fact, s is the arc length along the "ray"). Considering $\sigma(s) = \sigma(x + sa)$ as a function of s, we have, in view of Equation (A.10-6) (Figure A.11-1),

$$\sigma(x + sa) - \sigma(x) = \int_{\xi=0}^{s} \partial_\xi \sigma(\xi)\, \mathrm{d}\xi . \qquad \text{(A.11-1)}$$

Repeated integration by parts of the right-hand side leads, with the use of the relationship

$$\int_{\xi=0}^{s} \frac{(s-\xi)^{m-1}}{(m-1)!} \partial_\xi^m \sigma(\xi)\, \mathrm{d}\xi = -\int_{\xi=0}^{s} \frac{(s-\xi)^{m-1}}{(m-1)!} \partial_\xi^m \sigma(\xi)\, \mathrm{d}(s-\xi)$$

$$= -\left[\frac{(s-\xi)^m}{m!} \partial_\xi^m \sigma(\xi)\right]_{\xi=0}^{s} + \int_{\xi=0}^{s} \frac{(s-\xi)^m}{m!} \partial_\xi^{m+1} \sigma(\xi)\, \mathrm{d}\xi$$

$$= \frac{s^m}{m!} \partial_s^m \sigma(0) + \int_{\xi=0}^{s} \frac{(s-\xi)^m}{m!} \partial_\xi^{m+1} \sigma(\xi)\, \mathrm{d}\xi \quad \text{for } m = 1,\ldots,M-1 , \qquad \text{(A.11-2)}$$

where $m! = m(m-1) \ldots 2 \,.\, 1$, to

$$\sigma(x + sa) - \sigma(x) = \sum_{m=1}^{M-1} \frac{s^m}{m!} \partial_s^m \sigma(0) + \int_{\xi=0}^{s} \frac{(s-\xi)^{M-1}}{(M-1)!} \partial_s^M \sigma(\xi)\, \mathrm{d}\xi$$

$$\text{for } M = 1,2,3,4,\ldots . \qquad \text{(A.11-3)}$$

Now,

$$\partial_s \sigma(x + sa) = a_p \partial_p \sigma(x + sa) , \qquad \text{(A.11-4)}$$

and, hence,

$$\partial_s \sigma(0) = a_p \partial_p \sigma(x) . \qquad \text{(A.11-5)}$$

Putting Equation (A.11-5) in Equation (A.11-3), replacing sa_p by h_p, and using the arbitrariness of a_p, we arrive at

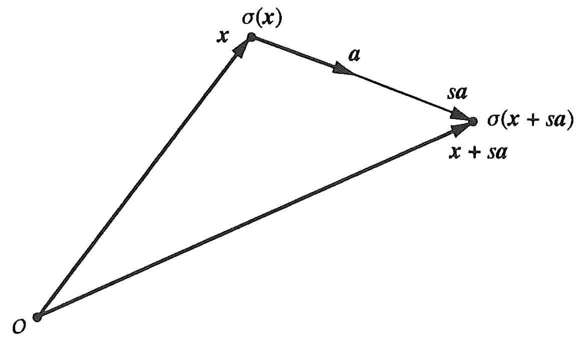

Figure A.11-1 The Taylor expansion of a tensor $\sigma = \sigma(x)$ in a star-shaped domain about x.

$$\sigma(x+h) = \sum_{m=0}^{M-1} \frac{(h_p\partial_p)^m \sigma(x)}{m!} + \int_{\xi=0}^{s} \frac{(s-\xi)^{M-1}}{(M-1)!} (a_p\partial_p)^M \sigma(x+\xi a)\, \mathrm{d}\xi$$

$$= \sum_{m=0}^{M} \frac{(h_p\partial_p)^m \sigma(x)}{m!} + \int_{\xi=0}^{s} \frac{(s-\xi)^{M-1}}{(M-1)!} \left[(a_p\partial_p)^M \sigma(x+\xi a) - (a_p\partial_p)^M \sigma(x)\right] \mathrm{d}\xi$$

$$\text{for } M = 1,2,3,\ldots, \tag{A.11-6}$$

where $(h_p\partial_p)^m\sigma(x) = \sigma(x)$ for $m = 0$ and $0! = 1$. Equation (A.11-6) is the *Taylor expansion* of σ about the point having the position vector x, for a domain that is star-shaped with respect to x. The integral on the right-hand side is the "remainder after M terms" of the expansion. Since $(a_p\partial_p)^M\sigma(x)$ is, by assumption, continuous we have

$$(a_p\partial_p)^M \sigma(x+\xi a) - (a_p\partial_p)^M \sigma(x) = o(1) \quad \text{as} \quad \xi \to 0 \tag{A.11-7}$$

and, hence, we estimate the remainder by

$$\int_{\xi=0}^{s} \frac{(s-\xi)^{M-1}}{(M-1)!} \left[(a_p\partial_p)^M \sigma(x+s\xi) - (a_p\partial_p)^M \sigma(x)\right] \mathrm{d}\xi$$

$$= o(1) \int_{\xi=0}^{s} \frac{(s-\xi)^{M-1}}{(M-1)!} \mathrm{d}\xi = o(s^M) \quad \text{as} \quad s \to 0, \tag{A.11-8}$$

which shows that the remainder in the Taylor expansion of M terms is $o(h^M)$.

Exercises

Exercise A.11-1

Give an expression for $(a_p\partial_p)^m$ for $m = 2$.

Answer: $(a_p\partial_p)^2 = (a_p\partial_p)(a_q\partial_q) = a_p a_q \partial_p \partial_q$.

A.12 Gauss' integral theorem

Let $\sigma = \sigma(x)$ be a continuously differentiable tensor function of arbitrary rank and defined in a bounded domain $\mathcal{D}$ with a piecewise smooth boundary $\partial\mathcal{D}$. Then, Gauss' integral theorem states that (Figure A.12-1)

$$\int_{x\in\mathcal{D}} \partial_p \sigma \,\mathrm{d}V = \int_{x\in\partial\mathcal{D}} \sigma \,\mathrm{d}A_p \,, \tag{A.12-1}$$

where $\mathrm{d}A_p$ is the outwardly oriented elementary surface area of $\partial\mathcal{D}$ and where, for ease of notation, we have omitted the subscripts indicating the rank of σ. To prove Equation (A.12-1), the domain $\mathcal{D}$ is covered by the union $\cup_{I\Sigma=1}^{N\Sigma}\Sigma^N(I\Sigma)$ of N-simplices that is also suitable for approximating the boundary of $\partial\mathcal{D}$ of $\mathcal{D}$. Under these conditions

$$\int_{x\in\partial\mathcal{D}} \sigma \,\mathrm{d}A_p = \int_{x\in\partial\mathcal{D}_\Sigma} \sigma \,\mathrm{d}A_p + o(h) \,, \tag{A.12-2}$$

where $\partial\mathcal{D}_\Sigma$ is the boundary of the covering, and h is the supremum of the maximum diameters of the N-simplices. Now, since σ is continuously differentiable, it can be approximated in each $\Sigma^N(I\Sigma)$ by its first-order local Taylor expansion. Let $x(I\Sigma)$ be an interior point of $\Sigma^N(I\Sigma)$, then

$$\sigma(x) = \sigma[x(I\Sigma)] + \partial_q\sigma[x(I\Sigma)][x_q - x_q(I\Sigma)] + o(h) \quad \text{as } h\to 0 \quad \text{for } x\in\Sigma^N(I\Sigma) \,. \tag{A.12-3}$$

On account of Equation (A.10-45), however,

$$\int_{x\in\partial\Sigma^N(I\Sigma)} \sigma[x(I\Sigma)] \,\mathrm{d}A_p = \sigma[x(I\Sigma)] \int_{x\in\partial\Sigma^N(I\Sigma)} \mathrm{d}A_p = 0 \,, \tag{A.12-4}$$

and

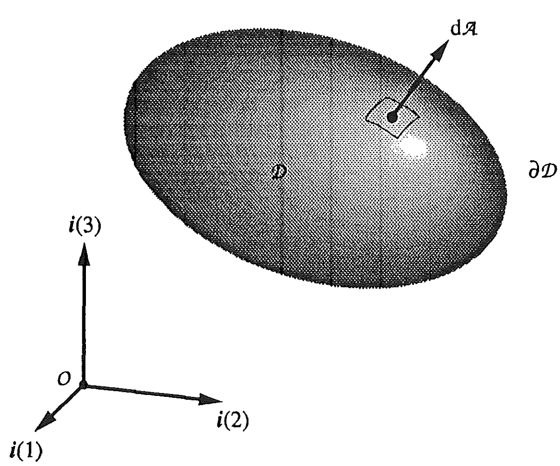

Figure A.12-1 Configuration for Gauss' integral theorem in three-dimensional Euclidean space.

$$\int_{x\in\partial\Sigma^N(I\Sigma)} \partial_q\sigma[x(I\Sigma)]\,x_q(I\Sigma)\,\mathrm{d}A_p = \partial_q\sigma[x(I\Sigma)]\,x_q(I\Sigma)\int_{x\in\partial\Sigma^N(I\Sigma)} \mathrm{d}A_p = 0\,. \tag{A.12-5}$$

Furthermore, on account of Equation (A.10-56),

$$\int_{x\in\partial\Sigma^N(I\Sigma)} \partial_q\sigma[x(I\Sigma)]\,x_q\,\mathrm{d}A_p = \partial_q\sigma[x(I\Sigma)]\int_{x\in\partial\Sigma^N(I\Sigma)} x_q\,\mathrm{d}A_p$$

$$= \partial_q\sigma[x(I\Sigma)]\delta_{q,p}V_{\Sigma^N(I\Sigma)} = \partial_p\sigma[x(I\Sigma)]V_{\Sigma^N(I\Sigma)}\,. \tag{A.12-6}$$

Equations (A.12-3)–(A.12-6) are used in the right-hand side of Equation (A.12-2). In the left-hand side of Equation (A.12-1) we now use the definition of the volume integral, i.e.

$$\int_{x\in\mathcal{D}} \partial_p\sigma\,\mathrm{d}V = \sum_{I\Sigma=1}^{N\Sigma}\int_{x\in\partial\Sigma^N(I\Sigma)} \partial_p\sigma\,\mathrm{d}V + o(1) \quad \text{as} \quad h\to 0,$$

$$= \sum_{I\Sigma=1}^{N\Sigma} \partial_p\sigma[x(I\Sigma)]V_{\Sigma^N(I\Sigma)} + o(h) + o(1) \quad as \quad h\to 0\,. \tag{A.12-7}$$

Combining Equations (A.12-2)–(A.12-7), taking into account that the contributions from the interior common interfaces of the simplicial covering to the boundary integral cancel, and taking the limit $h\to 0$, we obtain Equation (A.12-1).

Exercises

Exercise A.12-1

Let $\mathcal{D}$ be a bounded subdomain of $\mathcal{R}^N$ and let $\partial\mathcal{D}$ be its boundary. Use Gauss' integral theorem to prove that

$$\int_{x\in\partial\mathcal{D}} x_q\,\mathrm{d}A_p = \delta_{p,q}V_{\mathcal{D}}^N\,, \tag{A.12-8}$$

where

$$V_{\mathcal{D}}^N = \int_{x\in\mathcal{D}} \mathrm{d}V \tag{A.12-9}$$

is the volume of $\mathcal{D}$.

Exercise A.12-2

Let $\mathcal{D}$ be a bounded subdomain of $\mathcal{R}^N$ and let $\partial\mathcal{D}$ be its boundary. (a) Use Gauss' integral theorem to prove that

$$\int_{x\in\partial\mathcal{D}} x_p\,\mathrm{d}A_p = NV_{\mathcal{D}}^N\,, \tag{A.12-10}$$

where

$$V_{\mathcal{D}}^{N} = \int_{x \in \mathcal{D}} \mathrm{d}V \tag{A.12-11}$$

is the volume of $\mathcal{D}$. (b) Show that Equation (A.12-10) follows from Equation (A.12-8) by observing that $\delta_{p,p} = N$.