4

The acoustic wave equations, constitutive relations, and boundary conditions in the time Laplace-transform domain (complex frequency domain)

In a large number of cases met in practice, one is interested in the behaviour of causal acoustic wave fields in linear, time-invariant configurations. Mathematically, one can take advantage of this situation by carrying out a Laplace transformation with respect to time and considering the equations governing the acoustic wave field in the corresponding time Laplace-transform domain or *complex frequency domain.* In the complex frequency-domain relations, the time coordinate has been eliminated, and a field problem in space remains in which the Laplace-transform parameter s occurs. Causality of the field is taken into account by taking $\mathrm{Re}(s) > 0$, and requiring that all causal field quantities are, in the case where the wave field is excited by sources of finite amplitude and bounded extent, analytic functions of s in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane. The complex frequency-domain solution to a wave problem itself exhibits a number of features that are characteristic of the configuration in which the wave motion is present. If, in addition, one is interested in the actual pulse shapes of waves, one has to carry out the inverse Laplace transformation, either by analytical or by numerical methods.

In a number of wave propagation problems, the transform parameter s is profitably chosen to be real and positive. On the other hand, by taking $s = \mathrm{j}\omega$, where j is the imaginary unit and ω is real and positive, the complex steady-state representation of wave fields oscillating sinusoidally in time with angular frequency ω follows, the complex representation having the complex time factor $\exp(\mathrm{j}\omega t)$. For arbitrary complex values of s in the domain of analyticity, all complex frequency-domain wave field quantities and constitutive relaxation functions are the Laplace transforms of real-valued functions of the time coordinate t. As a consequence, the complex frequency-domain wave field quantities and constitutive relaxation functions are real-valued for real and positive values of s. On account of Schwarz's reflection principle of complex function theory, the relevant functions then take on complex conjugate values in conjugate complex points of the s plane.

In the present chapter the acoustic wave equations, constitutive relations, and boundary conditions in the complex frequency domain are given and the complex frequency-domain acoustic wave potentials and Green's functions (point-source solutions) are introduced. The notation of Appendix B is used.

4.1 The complex frequency-domain acoustic wave equations

We subject the acoustic wave equations (2.7-22) and (2.7-23) to a Laplace transformation over the interval $\mathcal{T}=\{t\in\mathcal{R};\,t>t_0\}$. For completeness, we allow a non-vanishing acoustic wave field to be present at $t=t_0$, although in the majority of cases we are interested in the causal wave field generated by sources that are switched on at the instant $t=t_0$, in which case the initial values of the acoustic wave field are taken to be zero. Since, with the use of the notation of Appendix B and the properties of the Laplace transformation,

$$\int_{t=t_0}^{\infty}\exp(-st)\partial_t\Phi_k(\boldsymbol{x},t)\,\mathrm{d}t=-\Phi_k(\boldsymbol{x},t_0)\exp(-st_0)+s\hat{\Phi}_k(\boldsymbol{x},s) \tag{4.1-1}$$

and

$$\int_{t=t_0}^{\infty}\exp(-st)\partial_t\theta(\boldsymbol{x},t)\,\mathrm{d}t=-\theta(\boldsymbol{x},t_0)\exp(-st_0)+s\hat{\theta}(\boldsymbol{x},s)\,, \tag{4.1-2}$$

we arrive at

$$\partial_k\hat{p}+s\hat{\Phi}_k=\hat{f}_k+\exp(-st_0)\Phi_k(\boldsymbol{x},t_0)\,, \tag{4.1-3}$$

$$\partial_r\hat{v}_r-s\hat{\theta}=\hat{q}-\exp(-st_0)\theta_k(\boldsymbol{x},t_0)\,. \tag{4.1-4}$$

From Equations (4.1-3) and (4.1-4) it follows that, in the complex frequency domain, one can take into account the influence of a non-vanishing initial acoustic wave field by properly incorporating its values in the complex frequency-domain volume densities of external volume force and external volume injection rate. In the remainder of our analysis, it will be tacitly understood that non-zero initial acoustic wave field values have been accounted for in this manner.

After transforming back to the time domain, the reconstructed acoustic wave field values are zero in the interval $t\in\mathcal{T}'$, where $\mathcal{T}'=\{t\in\mathcal{R};\,t<t_0\}$, and equal to the actual field values when $t\in\mathcal{T}$. In addition, many of the Laplace inversion algorithms, in particular the complex Bromwich inversion integral (Appendix B, Equation (B.1-19), of which the Fourier inversion integral is a limiting case), yield half the field values at the instant $t\in\partial\mathcal{T}$, where $\partial\mathcal{T}=\{t\in\mathcal{R};t=t_0\}$. Notationally, this can be expressed by employing the characteristic function $\chi_{\mathcal{T}}=\chi_{\mathcal{T}}(t)$ of the set $\mathcal{T}$, which is defined as

$$\chi_{\mathcal{T}}=\{1,\tfrac{1}{2},0\}\qquad\text{for } t\in\{\mathcal{T},\partial\mathcal{T},\mathcal{T}'\}\,. \tag{4.1-5}$$

With this notation, we have for the standard inversion applied to the Laplace transform $\hat{f}(\boldsymbol{x},s)$ of any space–time function $f=f(\boldsymbol{x},t)$ the result

$$\text{Inverse Laplace transform of } \hat{f}(\boldsymbol{x},s)=\chi_{\mathcal{T}}(t)\,f(\boldsymbol{x},t)\,. \tag{4.1-6}$$

Exercises

Exercise 4.1-1

(a) What volume density of external volume force corresponds in the complex frequency domain to the initial field $\Phi_k(\boldsymbol{x},t_0)$?
(b) What would be the corresponding volume density of external volume force in the space–time domain?

Answers: (a) $\exp(-st_0)\Phi_k(\boldsymbol{x},t_0)$. (b) $\Phi_k(\boldsymbol{x},t_0)\delta(t-t_0)$.

Exercise 4.1-2

(a) What volume density of external volume injection rate corresponds in the complex frequency domain to the initial field $\theta(\boldsymbol{x},t_0)$?
(b) What would be the corresponding volume density of external volume injection rate in the space–time domain?

Answers: (a) $-\exp(-st_0)\theta(\boldsymbol{x},t_0)$; (b) $-\theta(\boldsymbol{x},t_0)\delta(t-t_0)$.

4.2 The complex frequency-domain constitutive relations; the Kramers–Kronig causality relations for a fluid with relaxation

With regard to the time Laplace transformation of the constitutive relations we discuss: fluids with relaxation, instantaneously reacting fluids, and fluids whose acoustic behaviour is described by the frictional-force/bulk-viscosity loss mechanism given in Section 2.9.

Fluid with relaxation

The constitutive relations for a linear, time-invariant, locally reacting fluid with relaxation are, in their low-velocity linearised approximation, given by (see Equations (2.7-26) and (2.7-27))

$$\Phi_k(\boldsymbol{x},t) = \int_{t'=0}^{\infty} \mu_{k,r}(\boldsymbol{x},t')v_r(\boldsymbol{x},t-t')\,\mathrm{d}t' \tag{4.2-1}$$

and

$$\theta(\boldsymbol{x},t) = -\int_{t'=0}^{\infty} \chi(\boldsymbol{x},t')p(\boldsymbol{x},t-t')\,\mathrm{d}t' \,. \tag{4.2-2}$$

Mathematically, the right-hand sides of Equations (4.2-1) and (4.2-2) are *convolutions in time.* (The notion that convolutions in time can serve as the mathematical description of mechanical relaxation goes back to Boltzmann (see Boltzmann, 1876).) From this it can be expected that the Laplace transformation possibly reveals additional properties of the relaxation functions.

Carrying out the Laplace transformation of Equations (4.2-1) and (4.2-2) over the interval $t \in \mathcal{R}$, we obtain, assuming the acoustic wave field to be of a transient nature,

$$\hat{\Phi}_k(\boldsymbol{x},s) = \hat{\mu}_{k,r}(\boldsymbol{x},s)\hat{v}_r(\boldsymbol{x},s) \, , \tag{4.2-3}$$

$$\hat{\theta}(\boldsymbol{x},s) = -\hat{\chi}(\boldsymbol{x},s)\hat{p}(\boldsymbol{x},s) \, , \tag{4.2-4}$$

respectively, where, in view of Equations (B.1-12) and (2.5-7),

$$\hat{\mu}_{k,r}(\boldsymbol{x},s) = \int_{t'=0}^{\infty} \exp(-st')\mu_{k,r}(\boldsymbol{x},t')\,\mathrm{d}t' \, , \tag{4.2-5}$$

$$\hat{\chi}(\boldsymbol{x},s) = \int_{t'=0}^{\infty} \exp(-st')\chi(\boldsymbol{x},t')\,\mathrm{d}t' \, . \tag{4.2-6}$$

Evidently, the quantities $\{\hat{\mu}_{k,r},\hat{\chi}\}$ are the Laplace transforms of causal functions of time. Therefore, as is shown in Appendix B, Section B.3, their real and imaginary parts for imaginary values of $s = \mathrm{j}\omega$, with $\omega \in \mathcal{R}$, introduced according to

$$\{\hat{\mu}_{k,r},\hat{\chi}\}(\boldsymbol{x},\mathrm{j}\omega) = \{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) - \mathrm{j}\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) \, , \tag{4.2-7}$$

satisfy the Kramers–Kronig relations (see Equations (B.3-18) and (B.3-19))

$$\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) = -\frac{1}{\pi} ⨍_{\omega'=-\infty}^{\infty} \frac{\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega')}{\omega' - \omega}\,\mathrm{d}\omega' \, , \tag{4.2-8}$$

and

$$\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) = \frac{1}{\pi} ⨍_{\omega'=-\infty}^{\infty} \frac{\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega')}{\omega' - \omega}\,\mathrm{d}\omega' \, , \tag{4.2-9}$$

Equations (4.2-8) and (4.2-9) imply that $\{\mu'_{k,r},\chi'\}$ and $\{\mu''_{k,r},\chi''\}$ form pairs of Hilbert transforms. Another property of $\{\hat{\mu}_{k,r},\hat{\chi}\}(\boldsymbol{x},\mathrm{j}\omega)$ is that (see Equations (B.3-6) and (B.3-7))

$$\{\mu'_{k,r},\chi'\}(\boldsymbol{x},-\omega) = \{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) \qquad \text{for all } \omega \in \mathcal{R}, \tag{4.2-10}$$

and

$$\{\mu''_{k,r},\chi''\}(\boldsymbol{x},-\omega) = -\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) \qquad \text{for all } \omega \in \mathcal{R}, \tag{4.2-11}$$

i.e. $\mu'_{k,r}$ and χ' are even functions of ω and $\mu''_{k,r}$ and χ'' are odd functions of ω for $\omega \in \mathcal{R}$. Using these properties in the right-hand sides of Equations (4.2-8) and (4.2-9), these relations can be rewritten as (see Equations (B.3-26) and (B.3-27))

$$\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) = -\frac{2}{\pi} ⨍_{\omega'=0}^{\infty} \frac{\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega')\omega}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}, \tag{4.2-12}$$

and

$$\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) = \frac{2}{\pi} ⨍_{\omega'=0}^{\infty} \frac{\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega')\omega'}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}. \tag{4.2-13}$$

If the right-hand sides of either Equations (4.2-8) and (4.2-9) or Equations (4.2-12) and (4.2-13) have to be evaluated numerically, the Cauchy principal values of the integrals may present a difficulty. To circumvent this problem, we can rewrite Equations (4.2-8) and (4.2-9) as (see Equations (B.3-24) and (B.3-25))

$$\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) = -\frac{1}{\pi}\int_{\omega'=-\infty}^{\infty} \frac{\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega') - \{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega)}{\omega' - \omega}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}, \tag{4.2-14}$$

and

$$\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) = \frac{1}{\pi}\int_{\omega'=-\infty}^{\infty} \frac{\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega') - \{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega)}{\omega' - \omega}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}, \tag{4.2-15}$$

and Equations (4.2-12) and (4.2-13) as (see Equations (B.3-29) and (B.3-30))

$$\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega) = -\frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{[\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega') - \{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega)]\,\omega}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}, \tag{4.2-16}$$

and

$$\{\mu'_{k,r},\chi'\}(\boldsymbol{x},\omega) = \frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{[\{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega')\omega' - \{\mu''_{k,r},\chi''\}(\boldsymbol{x},\omega)\omega]}{(\omega')^2 - \omega^2}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}. \tag{4.2-17}$$

Equations (4.2-14)–(4.2-17) are the Bode relations for the relaxation functions (see Bode, 1959); only proper integrals occur on their right-hand sides.

Instantaneously reacting fluid

For an instantaneously reacting fluid, Equations (2.7-24) and (2.7-25) lead, after time Laplace transformation, to

$$\hat{\Phi}_k(\boldsymbol{x},s) = \rho_{k,r}(\boldsymbol{x})\hat{v}_r(\boldsymbol{x},s) \tag{4.2-18}$$

and

$$\hat{\theta}(\boldsymbol{x},s) = -\kappa(\boldsymbol{x})\,\hat{p}(\boldsymbol{x},s)\,, \tag{4.2-19}$$

respectively, in which the constitutive coefficients $\rho_{k,r}$ and κ are independent of s.

Fluid with frictional-force/bulk-viscosity acoustic loss mechanism

For a fluid whose loss behaviour can be described by the frictional-force/bulk-viscosity acoustic loss mechanism (see Equations (2.9-4) and (2.9-5)), the complex frequency-domain constitutive equations become

$$\hat{\Phi}_k(\boldsymbol{x},s) = [s^{-1}K_{k,r}(\boldsymbol{x}) + \rho_{k,r}(\boldsymbol{x})]\,\hat{v}_r(\boldsymbol{x},s) \tag{4.2-20}$$

and

$$\hat{\theta}(\boldsymbol{x},s) = -\,[s^{-1}\Gamma(\boldsymbol{x}) + \kappa(\boldsymbol{x})]\,\hat{p}(\boldsymbol{x},s)\,, \tag{4.2-21}$$

in which the constitutive coefficients $K_{k,r}$, $\rho_{k,r}$, Γ and κ are independent of s. Note that, formally, Equations (4.2-20) and (4.2-21) have the same structure as Equations (4.2-3) and (4.2-4), and that the coefficients $s^{-1}K_{k,r} + \rho_{k,r}$ and $s^{-1}\Gamma + \kappa$ are analytic functions of s in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane (although they have a simple pole at $s = 0$).

Exercises

Exercise 4.2-1

In a number of cases the constitutive acoustic properties of a fluid show partly instantaneous behaviour and partly additional relaxation behaviour. Such behaviour is expressed by constitutive relations of the type

$$\rho_{k,r}(\boldsymbol{x},t) = \rho^{\infty}_{k,r}(\boldsymbol{x})\delta(t) + \mu_{k,r}(\boldsymbol{x},t)\,, \tag{4.2-22}$$

$$\kappa(\boldsymbol{x},t) = \kappa^{\infty}(\boldsymbol{x})\delta(t) + \chi(\boldsymbol{x},t)\,, \tag{4.2-23}$$

where $\rho^{\infty}_{k,r}(\boldsymbol{x})$ and $\kappa^{\infty}(\boldsymbol{x})$ are real and positive, and representative of the instantaneous behaviour, while $\mu_{k,r}(\boldsymbol{x},t)$ and $\chi(\boldsymbol{x},t)$ are causal relaxation functions. The corresponding complex frequency-domain expressions are

$$\hat{\rho}_{k,r}(\boldsymbol{x},s) = \rho^{\infty}_{k,r}(\boldsymbol{x}) + \hat{\mu}_{k,r}(\boldsymbol{x},s)\,, \tag{4.2-24}$$

$$\hat{\kappa}(\boldsymbol{x},s) = \kappa^{\infty}(\boldsymbol{x}) + \hat{\chi}(\boldsymbol{x},s)\,, \tag{4.2-25}$$

where $\hat{\rho}_{k,r}(\boldsymbol{x},s)$ and $\hat{\kappa}(\boldsymbol{x},s)$ are analytic in the right half $\{s\in\mathcal{C};\ \mathrm{Re}(s) > 0\}$ of the complex s plane. Show that in this case the Kramers–Kronig causality relations between $\rho'_{k,r}(\boldsymbol{x},\omega)$ and $\rho''_{k,r}(\boldsymbol{x},\omega)$ and between $\kappa'(\boldsymbol{x},\omega)$ and $\kappa''(\boldsymbol{x},\omega)$ according to

$$\hat{\rho}_{k,r}(\boldsymbol{x},\mathrm{j}\omega) = \rho'_{k,r}(\boldsymbol{x},\omega) - \mathrm{j}\rho''_{k,r}(\boldsymbol{x},\omega) \qquad \text{for } \omega\in\mathcal{R}, \tag{4.2-26}$$

$$\hat{\kappa}(\boldsymbol{x},\mathrm{j}\omega) = \kappa'(\boldsymbol{x},\omega) - \mathrm{j}\kappa''(\boldsymbol{x},\omega) \qquad \text{for } \omega\in\mathcal{R}, \tag{4.2-27}$$

are given by

$$\rho''_{k,r}(\boldsymbol{x},\omega) = -\frac{1}{\pi}\,⨍_{\omega'=-\infty}^{\infty} \frac{\rho'_{k,r}(\boldsymbol{x},\omega') - \rho^{\infty}_{k,r}(\boldsymbol{x}')}{\omega' - \omega}\,\mathrm{d}\omega' \qquad \text{for } \omega\in\mathcal{R}, \tag{4.2-28}$$

$$\rho'_{k,r}(\boldsymbol{x},\omega) - \rho^{\infty}_{k,r}(\boldsymbol{x}) = \frac{1}{\pi}\,⨍_{\omega'=-\infty}^{\infty} \frac{\rho''_{k,r}(\boldsymbol{x},\omega')}{\omega' - \omega}\,\mathrm{d}\omega' \qquad \text{for } \omega\in\mathcal{R}, \tag{4.2-29}$$

and

$$\kappa''(\boldsymbol{x},\omega) = -\frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa'(\boldsymbol{x},\omega') - \kappa^{\infty}(\boldsymbol{x}')}{\omega' - \omega} \,\mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}, \tag{4.2-30}$$

$$\kappa'(\boldsymbol{x},\omega) - \kappa^{\infty}(\boldsymbol{x}) = \frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa''(\boldsymbol{x},\omega')}{\omega' - \omega} \,\mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}. \tag{4.2-31}$$

4.3 The complex frequency-domain boundary conditions

The boundary conditions that have been discussed in Section 2.6 apply, in the linearised low-velocity approximation, to time-invariant boundaries. As a consequence of this, these boundary conditions can be transferred directly to the complex frequency domain. Therefore, at the interface $\mathcal{S}$ of two different fluids the following conditions hold: on account of Equation (2.6-2))

$$\hat{p} \quad \text{is continuous across } \mathcal{S}, \tag{4.3-1}$$

and on account of Equation (2.6-3)

$$\nu_r \hat{v}_r \quad \text{is continuous across } \mathcal{S}. \tag{4.3-2}$$

Furthermore, from Equation (2.6-4) the condition

$$\lim_{h \downarrow 0} \hat{p}(\boldsymbol{x}, + h\boldsymbol{\nu}, s) = 0 \text{ on the boundary of a void} \tag{4.3-3}$$

follows, and from Equations (2.6-5)–(2.6-7) the conditions

$$M^{\mathrm{R}}_{k,r} s \hat{v}^{\mathrm{R}}_r = \hat{F}^{\mathrm{R}}_k + M^{\mathrm{R}}_{k,r} v^{\mathrm{R}}_r(t_0) \exp(-st_0), \tag{4.3-4}$$

$$\hat{F}^{\mathrm{R}}_k = \int_{\boldsymbol{x}\in\mathcal{D}} \hat{f}^{\mathrm{R}}_k \,\mathrm{d}V - \int_{\boldsymbol{x}\in\partial\mathcal{D}} \hat{p}\nu_k \,\mathrm{d}A, \tag{4.3-5}$$

$$\lim_{h \downarrow 0} \nu_r \hat{v}_r(\boldsymbol{x}, + h\boldsymbol{\nu}, s) = \nu_r \hat{v}^{\mathrm{R}}_r(s) \quad \text{at the boundary of a perfectly rigid object}, \tag{4.3-6}$$

follow, where $\mathcal{D}$ is the domain occupied by the perfectly rigid object, $\partial\mathcal{D}$ is its boundary surface, and ν_r is the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$. In the limiting case $M^{\mathrm{R}}_{k,r} \to \infty$, we have $\hat{v}^{\mathrm{R}}_r \to 0$, and Equations (4.3-4)–(4.3-6) reduce to

$$\lim_{h \downarrow 0} \nu_r \hat{v}_r(\boldsymbol{x}, + h\boldsymbol{\nu}, s) = 0$$

$$\text{at the boundary of an immovable, perfectly rigid object}. \tag{4.3-7}$$

It is clear that these boundary conditions would also follow if the procedure described in Section 2.6 had been applied to the complex frequency-domain acoustic wave equations (4.1-3) and (4.1-4).

Exercises

Exercise 4.3-1

Apply the procedure given in Section 2.6 to the complex frequency-domain acoustic wave equations (4.1-3) and (4.1-4) to arrive at the boundary conditions given in Equations (4.3-1)–(4.3-7)

4.4 The complex frequency-domain coupled acoustic wave equations

In the majority of our calculations we shall substitute the constitutive relations Equations (4.2-3) and (4.2-4), or Equations (4.2-18) and (4.2-19), or Equations (4.2-20) and (4.2-21), in Equations (4.1-3) and (4.1-4) and thus obtain a system of differential equations in space in which the number of unknowns is equal to the number of equations and in which s occurs as a parameter. The relevant equations are written as

$$\partial_k \hat{p} + \hat{\zeta}_{k,r}\hat{v}_r = \hat{f}_k + \exp(-st_0)\Phi_k(\boldsymbol{x},t_0) \, , \tag{4.4-1}$$

$$\partial_r \hat{v}_r + \hat{\eta}\hat{p} = \hat{q} - \exp(-st_0)\theta(\boldsymbol{x},t_0) \, , \tag{4.4-2}$$

where

$\hat{\zeta}_{k,r}$ = Complex frequency-domain *longitudinal acoustic impedance per length* of the fluid, (4.4-3)

and

$\hat{\eta}$ = Complex frequency-domain *transverse acoustic admittance per length* of the fluid. (4.4-4)

(This terminology is borrowed from the conventional usage in electrical one-dimensional transmission-line theory.)

Equations (4.4-1) and (4.4-2) will be referred to as the "complex frequency-domain coupled acoustic wave equations". They serve as the point of departure in a number of subsequent analyses.

Fluid with relaxation

For a fluid with relaxation, Equations (4.2-3) and (4.2-4) lead to

$$\hat{\zeta}_{k,r} = s\hat{\mu}_{k,r} \tag{4.4-5}$$

and

$$\hat{\eta} = s\hat{\chi} \, . \tag{4.4-6}$$

Instantaneously reacting fluid

For an instantaneously reacting fluid, Equations (4.2-18) and (4.2-19) lead to

$$\hat{\zeta}_{k,r} = s\rho_{k,r} \tag{4.4-7}$$

and

$$\hat{\eta} = s\kappa \,. \tag{4.4-8}$$

Fluid with frictional-force/bulk-viscosity acoustic loss mechanism

For a fluid with the frictional-force/bulk-viscosity acoustic loss mechanism, Equations (4.2-20) and (4.2-21) lead to

$$\hat{\zeta}_{k,r} = K_{k,r} + s\rho_{k,r} \tag{4.4-9}$$

and

$$\hat{\eta} = \Gamma + s\kappa \,. \tag{4.4-10}$$

4.5 Complex frequency-domain acoustic scalar and vector potentials

The complex frequency-domain acoustic scalar and vector potentials in the theory of radiation from sources are introduced along the same lines as described in Section 2.10. The starting points are Equations (4.4-1) and (4.4-2) in which we incorporate the initial-value contributions in the volume source densities, i.e.

$$\partial_k \hat{p} + \hat{\zeta}_{k,r}\hat{v}_r = \hat{f}_k \,, \tag{4.5-1}$$

$$\partial_r \hat{v}_r + \hat{\eta}\hat{p} = \hat{q} \,, \tag{4.5-2}$$

and in which inertia relaxation effects, if present, have been incorporated in $\hat{\zeta}_{k,r} = s\hat{\mu}_{k,r}$ and compliance relaxation effects, if present, have been incorporated in $\hat{\eta} = s\hat{\chi}$. For instantaneously reacting fluids $\hat{\zeta}_{k,r} = s\rho_{k,r}$ and $\hat{\eta} = s\kappa$; for fluids with the frictional-force/bulk-viscosity acoustic loss mechanism $\hat{\zeta}_{k,r} = K_{k,r} + s\rho_{r,k}$ and $\hat{\eta} = \Gamma + s\kappa$.

Let $\{\hat{p},\hat{v}_r\} = \{\hat{p}^q,\hat{v}_r^q\}$ denote the acoustic wave motion that is generated by the volume injection source distribution $\hat{q} = \hat{q}(s)$, in the absence of a force source distribution, i.e. $\hat{f}_k = 0$. Then,

$$\partial_k \hat{p}^q + \hat{\zeta}_{k,r}\hat{v}_r^q = 0 \,, \tag{4.5-3}$$

$$\partial_r \hat{v}_r^q + \hat{\eta}\hat{p}^q = \hat{q} \,. \tag{4.5-4}$$

Taking advantage of the fact that the right-hand side of Equation (4.5-3) is zero, this equation is rewritten as

$$\hat{v}_r^q = -\hat{\zeta}_{r,k}^{-1}\partial_k \hat{p}^q \,, \tag{4.5-5}$$

where $\hat{\zeta}_{r,k}^{-1}$ is the tensor of rank two that is inverse to $\hat{\zeta}_{k,r}$, i.e.

$$\hat{\zeta}_{k,s}\hat{\zeta}_{s,r}^{-1} = \delta_{k,r} \,. \tag{4.5-6}$$

Substitution of Equation (4.5-5) in Equation (4.5-4) yields the second-order scalar differential equation

$$\partial_r(\hat{\zeta}_{r,k}^{-1}\partial_k \hat{p}^q) - \hat{\eta}\hat{p}^q = -\hat{q} \,. \tag{4.5-7}$$

By analogy with Equation (2.10-11) we now introduce the complex frequency-domain acoustic scalar potential $\hat{\Psi} = \hat{\Psi}(x,s)$ as the solution to the second-order differential equation (acoustic scalar Helmholtz equation)

$$\partial_r(s\hat{\zeta}^{-1}_{r,k}\partial_k\hat{\Psi}) - s\hat{\eta}\hat{\Psi} = -\hat{q}\,, \tag{4.5-8}$$

with the volume source density of injection rate as the forcing term on the right-hand side. Comparison of Equations (4.5-8) and (4.5-7) leads to

$$\hat{p}^q = s\hat{\Psi}\,, \tag{4.5-9}$$

while Equation (4.5-5) leads to

$$\hat{v}^q_r = -s\hat{\zeta}^{-1}_{r,k}\partial_k\hat{\Psi}\,. \tag{4.5-10}$$

Now, let $\{\hat{p},\hat{v}_r\} = \{\hat{p}^f,\hat{v}^f_r\}$ denote the acoustic wave motion that is generated by the force source distribution $\hat{f}_k = \hat{f}_k(s)$, in the absence of a volume injection source distribution, i.e. $\hat{q} = 0$. Then,

$$\partial_k\hat{p}^f + \hat{\zeta}_{k,r}\hat{v}^f_r = \hat{f}_k\,, \tag{4.5-11}$$

$$\partial_r\hat{v}^f_r + \hat{\eta}\hat{p}^f = 0\,. \tag{4.5-12}$$

Taking advantage of the fact that the right-hand side of Equation (4.5-12) is zero, this equation is rewritten as

$$\hat{p}^f = -\hat{\eta}^{-1}\partial_r\hat{v}^f_r\,. \tag{4.5-13}$$

Substitution of Equation (4.5-13) in Equation (4.5-11) yields the second-order vector differential equation

$$\partial_k(\hat{\eta}^{-1}\partial_r\hat{v}^f_r) - \hat{\zeta}_{k,r}\hat{v}^f_r = -\hat{f}_k\,. \tag{4.5-14}$$

By analogy with Equation (2.10-19) we now introduce the complex frequency-domain acoustic vector potential $\hat{W}_r = \hat{W}_r(x,s)$ as the solution to the second-order differential equation (acoustic vector Helmholtz equation)

$$\partial_k(s\hat{\eta}^{-1}\partial_r\hat{W}_r) - s\hat{\zeta}_{k,r}\hat{W}_r = -\hat{f}_k\,, \tag{4.5-15}$$

with the volume source density of force as the forcing term on the right-hand side. Comparison of Equations (4.5-14) and (4.5-15) leads to

$$\hat{v}^f_r = s\hat{W}_r\,, \tag{4.5-16}$$

while Equation (4.5-13) leads to

$$\hat{p}^f = -s\hat{\eta}^{-1}\partial_r\hat{W}_r\,. \tag{4.5-17}$$

Since the total wave field is the superposition of the two constituents, i.e.

$$\{\hat{p},\hat{v}_r\} = \{\hat{p}^q + \hat{p}^f, \hat{v}^q_r + \hat{v}^f_r\}, \tag{4.5-18}$$

we end up with

$$\hat{p} = s\hat{\Psi} - s\hat{\eta}^{-1}\partial_r\hat{W}_r\,, \tag{4.5-19}$$

$$\hat{v}_r = s\hat{W}_r - s\hat{\zeta}^{-1}_{r,k}\partial_k\hat{\Psi}\,, \tag{4.5-20}$$

which are the complex frequency-domain counterparts of Equations (2.10-23) and (2.10-24).

Compatibility relations

Upon carrying out the operation $\varepsilon_{i,n,k}\partial_n$ on Equation (4.5-3), we arrive at the compatibility relation

$$\varepsilon_{i,n,k}\partial_n(\hat{\zeta}_{k,r}\hat{v}_r^q) = 0 \tag{4.5-21}$$

to be satisfied by $\hat{v}_r^q$. Substitution of the expression for $\hat{v}_r^q$ given by Equation (4.5-5) shows that Equation (4.5-21) is automatically satisfied. Upon carrying out the operation $\varepsilon_{i,n,k}\partial_n$ on Equation (4.5-11), we arrive at the compatibility relation

$$\varepsilon_{i,n,k}\partial_n(\hat{\zeta}_{k,r}\hat{v}_r^f) = \varepsilon_{i,n,k}\partial_n\hat{f}_k \tag{4.5-22}$$

to be satisfied by $\hat{v}_r^f$. Substitution of the expression for $\hat{v}_r^f$ given by Equation (4.5-16) shows that Equation (4.5-22) is automatically satisfied, provided that $\hat{W}_r$ satisfies Equation (4.5-15).

In its turn, $\hat{W}_r$ has to satisfy a compatibility relation that follows upon carrying out the operation $\varepsilon_{i,n,k}\partial_k$ on Equation (4.5-15). Since

$$\varepsilon_{i,n,k}\partial_n\partial_k(s\hat{\eta}^{-1}\partial_r\hat{W}_r) = 0\,, \tag{4.5-23}$$

we obtain

$$\varepsilon_{i,n,k}\partial_n(s\hat{\zeta}_{k,r}\hat{W}_r) = \varepsilon_{i,n,k}\partial_n\hat{f}_k\,. \tag{4.5-24}$$

Exercises

Exercise 4.5-1

(a) Give the complex frequency-domain expressions for the acoustic pressure $\hat{p}$ and the particle velocity $\hat{v}_r$ in terms of their acoustic scalar and vector potentials for an inhomogeneous, isotropic fluid with relaxation, whose longitudinal scalar acoustic impedance per length is $\hat{\zeta} = \hat{\zeta}(\boldsymbol{x},s)$ and transverse acoustic admittance per length is $\hat{\eta} = \hat{\eta}(\boldsymbol{x},s)$.
(b) Give the corresponding second-order differential equations for the acoustic scalar potential and the acoustic vector potential.

Answers:

(a) $$\hat{p} = s\hat{\Psi} - s\hat{\eta}^{-1}\partial_r\hat{W}_r\,, \tag{4.5-25}$$

$$\hat{v}_r = s\hat{W}_r - s\hat{\zeta}^{-1}\partial_r\hat{\Psi} \tag{4.5-26}$$

and

(b) $$\partial_r(s\hat{\zeta}^{-1}\partial_r\hat{\Psi}) - s\hat{\eta}\hat{\Psi} = -\hat{q}\,, \tag{4.5-27}$$

$$\partial_k(s\hat{\eta}^{-1}\partial_r\hat{W}_r) - s\hat{\zeta}\hat{W}_k = -\hat{f}_k \tag{4.5-28}$$

Exercise 4.5-2

Verify that Equations (4.5-19) and (4.5-20) satisfy Equations (4.5-1) and (4.5-2), provided that Equations (4.5-8) and (4.5-15) are satisfied.

4.6 Complex frequency-domain point-source solutions and Green's functions

The complex frequency-domain point-source solutions and the corresponding complex frequency-domain Green's functions are introduced along the same lines as described in Section 2.11. To this end the volume source density of injection rate $\hat{q} = \hat{q}(\boldsymbol{x},s)$ and the volume source density of force $\hat{f} = \hat{f}(\boldsymbol{x},s)$ are written as a continuous superposition of point sources through the representations (see Equations (2.11-1) and (2.11-2))

$$\hat{q}(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \delta(\boldsymbol{x}-\boldsymbol{x}')\hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{4.6-1}$$

and

$$\hat{f}_k(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \delta_{k,k'}\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{f}_{k'}(\boldsymbol{x}',s)\,\mathrm{d}V\,, \tag{4.6-2}$$

where $\mathcal{D}^T$ is the spatial support of the distributed sources and the sifting property of the Dirac delta distribution $\delta(\boldsymbol{x} - \boldsymbol{x}')$ operative at $\boldsymbol{x}' = \boldsymbol{x}$ has been used. Now, let the scalar function $\hat{G}^{\Psi} = \hat{G}^{\Psi}(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order scalar differential equation (see Equation (2.11-3))

$$\partial_r(s\hat{\xi}^{-1}_{r,k}\partial_k\hat{G}^{\Psi}) - s\hat{\eta}\hat{G}^{\Psi} = -\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{4.6-3}$$

and let the tensor of rank two $\hat{G}^{W}_{r,k'} = \hat{G}^{W}_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order tensorial differential equation (see Equation (2.11-4))

$$\partial_k(s\hat{\eta}^{-1}\partial_r\hat{G}^{W}_{r,k'}) - s\hat{\xi}_{k,r}\hat{G}^{W}_{r,k'} = -\delta_{k,k'}\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{4.6-4}$$

then Equations (4.5-8) and (4.5-15) are satisfied by

$$\hat{\Psi}(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}^{\Psi}(\boldsymbol{x},\boldsymbol{x}',s)\hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{4.6-5}$$

and

$$\hat{W}_r(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}^{W}_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)\hat{f}_{k'}(\boldsymbol{x}',s)\,\mathrm{d}V\,, \tag{4.6-6}$$

respectively. The proof follows by observing that the differentiations on the left-hand sides of Equations (4.5-8) and (4.5-15) are with respect to $\boldsymbol{x}$, whereas the integrations in the right-hand sides of Equations (4.6-1) and (4.6-2) and Equations (4.6-5) and (4.6-6) are with respect to $\boldsymbol{x}'$. The function $\hat{G}^{\Psi} = \hat{G}^{\Psi}(\boldsymbol{x},\boldsymbol{x}',s)$ is the complex frequency-domain (scalar) Green's function associated with the acoustic scalar potential $\hat{\Psi} = \hat{\Psi}(\boldsymbol{x},s)$; the function $\hat{G}^{W}_{r,k'} = \hat{G}^{W}_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)$ is the complex frequency-domain (tensorial) Green's function associated with the acoustic vector potential $\hat{W}_r = \hat{W}_r(\boldsymbol{x},s)$. The role that these Green's functions play in the solution of acoustic wave problems is discussed more extensively in Chapters 5 and 7.

Exercises

Exercise 4.6-1

Let $\hat{u} = \hat{u}(\boldsymbol{x},s)$ be the solution to the scalar Helmholtz equation

$$\partial_m\partial_m\hat{u} - (s^2/c^2)\hat{u} = -\hat{\rho}\,. \tag{4.6-7}$$

The spatial support of the sources with volume density $\hat{\rho} = \hat{\rho}(\boldsymbol{x},s)$ is $\mathcal{D}^T$. (a) Give the differential equation for the Green's function $\hat{G} = \hat{G}(\boldsymbol{x},\boldsymbol{x}',s)$. (b) Express $\hat{u} = \hat{u}(\boldsymbol{x},s)$ as a superposition of point-source solutions.

Answers:

(a) $$\partial_m\partial_m\hat{G} - (s^2/c^2)\hat{G} = -\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{4.6-8}$$

and

(b) $$\hat{u}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \hat{G}(\boldsymbol{x},\boldsymbol{x}',s)\hat{\rho}(\boldsymbol{x}',s)\,\mathrm{d}V\,. \tag{4.6-9}$$

References

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