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The electromagnetic boundary conditions

Across certain boundary surfaces in a configuration, the electromagnetic properties, and hence the electromagnetic field quantities, may exhibit a discontinuous behaviour. Since at those positions the field quantities are no longer continuously differentiable, the electromagnetic field equations cease to hold and must be supplemented by boundary conditions that interrelate the field values at either side of the surface of discontinuity. Several types of boundary conditions are discussed in this chapter.

20.1 Boundary conditions at the interface of two media

In those domains in a configuration where the electromagnetic constitutive parameters change continuously with position, the electric and the magnetic field strengths are continuously differentiable functions of position and satisfy the electromagnetic field equations discussed in Chapter 18 and the constitutive relations discussed in Chapter 19. Now, in practice, it often occurs that media with different constitutive parameters are in contact along interfaces. Across such an interface, the constitutive parameters show a jump discontinuity. From the electromagnetic field equations it then follows that at least some components of the electromagnetic field quantities show a jump discontinuity across the interface as well. Consequently, the electric and the magnetic field strengths are no longer continuously differentiable in a domain that contains (part of) an interface, and Equations (18.3-1) and (18.3-2) as well as Equations (18.3-11) and (18.3-12) cease to hold. To interrelate the electromagnetic field quantities at either side of such an interface, a certain set of *boundary conditions* is needed. To arrive at these conditions, we proceed as follows.

Let $\mathcal{S}$ denote the interface and assume that $\mathcal{S}$ has everywhere a unique tangent plane. Let, further, ν_m denote the unit vector along the normal to $\mathcal{S}$ such that upon traversing $\mathcal{S}$ in the direction of ν_m, we pass from the domain $\mathcal{D}_1$ to the domain $\mathcal{D}_2$, $\mathcal{D}_1$ and $\mathcal{D}_2$ being located at either side of $\mathcal{S}$ (Figure 20.1-1).

Suppose, now, that some (or all) electromagnetic field components jump across $\mathcal{S}$. In the direction parallel to $\mathcal{S}$, all electromagnetic field components still vary in a continuously differentiable manner, and hence the partial derivatives parallel to $\mathcal{S}$ give no problem in Equations (18.3-11) and (18.3-12). The partial derivatives perpendicular to $\mathcal{S}$, on the contrary,

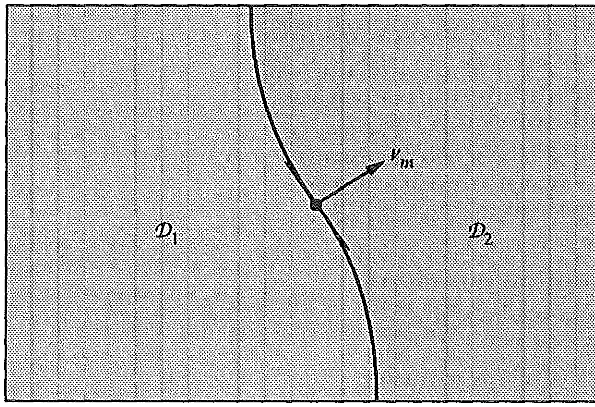

Figure 20.1-1 Interface $\mathcal{S}$ between two media with different electromagnetic properties.

meet functions that show a jump discontinuity across $\mathcal{S}$; these give rise to surface Dirac delta distributions (surface impulses) located on $\mathcal{S}$. Distributions of this kind would, however, physically be representative of surface sources located on $\mathcal{S}$. In the absence of such surface sources, the absence of surface impulses in the partial derivatives across $\mathcal{S}$ should be enforced. The latter is done by requiring that these partial derivatives only meet functions that are continuous across $\mathcal{S}$.

To investigate the consequences of this reasoning, we take a particular point on $\mathcal{S}$ and write, at that point

$$\partial_m = \nu_m(\nu_s\partial_s) + [\partial_m - \nu_m(\nu_s\partial_s)] = N_{m,s}\partial_s + T_{m,s}\partial_s \,, \tag{20.1-1}$$

in which $N_{m,s} = \nu_m\nu_s$ extracts the normal part out of a tensor with which it is contracted, and $T_{m,s} = \delta_{m,s} - \nu_m\nu_s$ extracts the tangential part out of a tensor with which it is contracted, ν_m being the unit vector along the normal to the relevant interface at the chosen point. The first term on the right-hand side of Equation (20.1-1) contains the component of ∂_m parallel to ν_m (i.e. perpendicular to $\mathcal{S}$); the second term contains the components of ∂_m perpendicular to ν_m (i.e. parallel to $\mathcal{S}$). The use of Equation (20.1-1) in Equations (18.3-11) and (18.3-12) leads to the requirements (*boundary conditions of the continuity type*)

$$\varepsilon_{k,m,p}\nu_m H_p \quad \text{is continuous across } \mathcal{S}, \tag{20.1-2}$$

and

$$\varepsilon_{j,n,r}\nu_n E_r \quad \text{is continuous across } \mathcal{S}, \tag{20.1-3}$$

respectively, from which we conclude that the components of the electric and the magnetic field strengths tangential to $\mathcal{S}$ are to be continuous across $\mathcal{S}$. (Note that, since $\nu_k\varepsilon_{k,m,p}\nu_m H_p = 0$ and $\nu_j\varepsilon_{j,n,r}\nu_n E_r = 0$, the expressions in Equations (20.1-2) and (20.1-3) contain no component along ν_m (i.e. normal to $\mathcal{S}$).)

The boundary conditions given in Equations (20.1-2) and (20.1-3) are consequences of the electromagnetic field Equations (18.3-11) and (18.3-12); they are the fundamental boundary conditions of the continuity type in any electromagnetic radiation problem. However, also from the compatibility relations, boundary conditions of the continuity type can be extracted.

Applying again the decomposition according to Equation (20.1-1), it follows from Equation (18.3-13) that on a source-free interface $\mathcal{S}$

$$\nu_k(J_k + \partial_t D_k) \quad \text{is continuous across } \mathcal{S} \tag{20.1-4}$$

and from Equation (18.3-14), together with the causality argument to derive a condition on B_j from the one on $\partial_t B_j$, that on a source-free interface $\mathcal{S}$

$$\nu_j B_j \quad \text{is continuous across } \mathcal{S}. \tag{20.1-5}$$

Equations (20.1-4) and (20.1-5) are auxiliary boundary conditions of the continuity type.

Exercises

Exercise 20.1-1

Let $\mathcal{S}$ be a surface with a uniquely defined tangent plane and let ν_m be the unit vector along the normal to $\mathcal{S}$. (a) Decompose an arbitrary vector field H_p that is defined on $\mathcal{S}$ into (b) its normal component H_p^{norm} and (c) its tangential component H_p^{tang}. Show that $\nu_p H_p^{\text{norm}} = \nu_p H_p$, $\varepsilon_{j,m,p}\nu_m H_p^{\text{tang}} = \varepsilon_{j,m,p}\nu_m H_p$, $\varepsilon_{j,m,p}\nu_m H_p^{\text{norm}} = 0$ and $\nu_p H_p^{\text{tang}} = 0$.

Answer:

(a) $H_p = H_p^{\text{norm}} + H_p^{\text{tang}}$;

(b) $H_p^{\text{norm}} = \nu_p \nu_j H_j = N_{p,j} H_j$;

(c) $H_p^{\text{tang}} = H_p - \nu_p \nu_j H_j = T_{p,j} H_j = -\varepsilon_{p,s,q}\varepsilon_{q,m,j}\nu_m H_j \nu_s$.

Exercise 20.1-2

Let $\mathcal{S}$ be a surface with a uniquely defined tangent plane and let ν_m be the unit vector along the normal to $\mathcal{S}$. (a) Decompose an arbitrary vector field E_r that is defined on $\mathcal{S}$ into (b) its normal component E_r^{norm} and (c) its tangential component E_r^{tang}. Show that $\nu_r E_r^{\text{norm}} = \nu_r E_r$, $\varepsilon_{k,n,r}\nu_n E_r^{\text{tang}} = \varepsilon_{k,n,r}\nu_n E_r$, $\varepsilon_{k,n,r}\nu_n E_r^{\text{norm}} = 0$ and $\nu_r E_r^{\text{tang}} = 0$.

Answer:

(a) $E_r = E_r^{\text{norm}} + E_r^{\text{tang}}$;

(b) $E_r^{\text{norm}} = \nu_r \nu_k E_k = N_{r,k} E_k$;

(c) $E_r^{\text{tang}} = E_r - \nu_r \nu_k E_k = T_{r,k} E_k = -\varepsilon_{r,s,q}\varepsilon_{q,m,k}\nu_m E_k \nu_s$.

Exercise 20.1-3

Let $\mathcal{S}$ be an interface between two different media and let ν_m denote the unit vector along the normal to $\mathcal{S}$. Show that the continuity of the tangential parts E_r^{tang} and H_p^{tang} of the electric and the magnetic field strengths across $\mathcal{S}$ entails the continuity of $\varepsilon_{m,r,p}\nu_m E_r H_p$ across $\mathcal{S}$. (*Hint*: Note that, making use of the results of Exercises 20.1-1 and 20.1-2,

$$\varepsilon_{m,r,p}\nu_m E_r H_p = \varepsilon_{m,r,p}\nu_m E_r^{\text{tang}} H_p^{\text{tang}}.)$$

20.2 Boundary condition at the surface of an electrically impenetrable object

A material body, occupying the domain $\mathcal{D}$ in space, is denoted as *electrically impenetrable* if it cannot sustain in its interior a non-identically vanishing electric field, while the boundary condition of the continuity of the tangential part of the electric field strength on its boundary surface $\partial\mathcal{D}$ is maintained. Consequently, the boundary condition upon approaching the surface $\partial\mathcal{D}$ of such a body via its exterior is given by (*boundary condition of the explicit type*)

$$\lim_{h\downarrow 0}\varepsilon_{j,n,r}\nu_n E_r(x+h\nu,t)=0 \qquad \text{for any} \quad x\in\partial\mathcal{D}, \tag{20.2-1}$$

where ν_m is the unit vector along the normal to $\partial\mathcal{D}$ oriented away from $\mathcal{D}$. One is not free to prescribe boundary values for the tangential part of the magnetic field strength in this case. In fact, the magnetic field strength will, in general, have a non-zero value at $\partial\mathcal{D}$, while at least for non-static fields, it is not defined in $\mathcal{D}$.

Electrically impenetrable materials arise as limiting cases of materials whose conductivity and/or permittivity go to infinity. Note that a superconductor is not an electrically impenetrable material.

Exercises

Exercise 20.2-1

An electrically conducting body with scalar conductivity σ occupies a bounded domain $\mathcal{D}$ in space. The domain $\mathcal{D}'$ exterior to the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is vacuum. Show that in the limit $\sigma\rightarrow\infty$ the boundary condition of the continuity of the tangential part of the electric field strength goes over into the boundary condition of the vanishing of the tangential part of the electric field strength, upon approaching $\partial\mathcal{D}$ via its exterior. (*Hint*: Use Equations (20.1-3) and (20.1-4), together with the requirement of the boundedness of all field quantities.)

Exercise 20.2-2

A dielectric body with scalar permittivity ε occupies a bounded domain $\mathcal{D}$ in space. The domain $\mathcal{D}'$ exterior to the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is vacuum. Show that in the limit $\varepsilon\rightarrow\infty$ the boundary condition of the continuity of the tangential part of the electric field strength goes over into the boundary condition of the vanishing of the tangential part of the electric field strength upon approaching $\partial\mathcal{D}$ via its exterior. (*Hint*: Use Equations (20.1-3) and (20.1-4), together with the requirement of the boundedness of all field quantities.)

Exercise 20.2-3

Show that, upon approaching the boundary surface $\partial\mathcal{D}$ of a electrically impenetrable body occupying the domain $\mathcal{D}$, the auxiliary boundary condition of the explicit type

$$\lim_{h\downarrow 0}\nu_j(x)B_j(x+h\nu,t)=0 \qquad \text{for any} \quad x\in\partial\mathcal{D}, \tag{20.2-2}$$

holds. (*Hint*: Note that

$$N_{pj}\partial_t B_j = -N_{p,j}\varepsilon_{j,n,r}\partial_n E_r$$

and that

$$N_{p,j}\varepsilon_{j,n,r}\partial_n = N_{p,j}\varepsilon_{j,n,r}(N_{n,s}\partial_s + T_{n,s}\partial_s) = N_{p,j}\varepsilon_{j,n,r}T_{n,s}\partial_s$$

contains only derivatives parallel to $\partial\mathcal{D}$, and use the causality argument to derive a condition for B_j from the one on $\partial_t B_j$.)

20.3 Boundary condition at the surface of a magnetically impenetrable object

A material body, occupying the domain $\mathcal{D}$ in space, is denoted as *magnetically impenetrable* if it cannot sustain in its interior a non-identically vanishing magnetic field, while the boundary condition of the continuity of the tangential part of the magnetic field strength on its boundary surface $\partial\mathcal{D}$ is maintained. Consequently, the boundary condition upon approaching the surface of such a body via its exterior is given by (*boundary condition of the explicit type*)

$$\lim_{h\downarrow 0}\varepsilon_{k,m,p}\nu_m H_p(x + h\nu,t) = 0 \qquad \text{for any} \quad x\in\partial\mathcal{D}\,, \qquad (20.3\text{-}1)$$

where ν_m is the unit vector along the outward normal to $\partial\mathcal{D}$ oriented away from $\mathcal{D}$. One is not free to prescribe boundary values for the tangential part of the electric field strength in this case. In fact, the electric field strength will, in general, have a non-zero value at $\partial\mathcal{D}$, while, at least for non-static fields, it is not defined in $\mathcal{D}$.

Magnetically impenetrable materials arise as limiting cases of materials whose permeability goes to infinity.

Exercises

Exercise 20.3-1

A magnetic body with scalar permeability μ occupies a bounded domain $\mathcal{D}$ in space. The domain $\mathcal{D}'$ exterior to the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is vacuum. Show that in the limit $\mu\to\infty$ the boundary condition of the continuity of the tangential part of the magnetic field strength goes over into the boundary condition of the vanishing of the tangential part of the magnetic field strength upon approaching $\partial\mathcal{D}$ via its exterior. (*Hint*: Use Equations (20.1-2) and (20.1-5), together with the requirement of the boundedness of all field quantities.)

Exercise 20.3-2

Show that, upon approaching the boundary surface $\partial\mathcal{D}$ of a magnetically impenetrable body occupying the domain $\mathcal{D}$, the auxiliary boundary condition of the explicit type

$$\lim_{h\downarrow 0}\nu_k(x)[J_k(x + h\nu,t) + \partial_t D_k(x + h\nu,t)] = 0 \qquad \text{for any} \quad x\in\partial\mathcal{D}\,, \qquad (20.3\text{-}2)$$

holds. (*Hint*: Note that

$$N_{r,k}(J_k + \partial_t D_k) = N_{r,k}\varepsilon_{k,m,p}\partial_m H_p$$

and that

$$N_{r,k}\varepsilon_{k,m,p}\partial_m = N_{r,k}\varepsilon_{k,m,p}(N_{m,s}\partial_s + T_{m,s}\partial_s) = N_{r,k}\varepsilon_{k,m,p}T_{m,s}\partial_s$$

contains only derivatives parallel to $\partial\mathcal{D}$.)