21

Exchange of energy in the electromagnetic field

In this chapter the exchange of energy in the electromagnetic field is discussed. Since the electric and magnetic field strengths have been defined through the force exerted on a point charge moving in vacuum, the discussion is started by considering the expression for the mechanical work done by this force. From the results, the Poynting vector is found to be the quantity that characterises the area density of the power flow in the electromagnetic field. Starting from the latter, also in the presence of matter, the electromagnetic energy theorem in stationary matter is derived. This theorem interrelates the work done by the sources with the stored electromagnetic energy, the dissipated electromagnetic power and the electromagnetic power flow.

21.1 Energy theorem for the electromagnetic field associated with the flow of a collection of electrically charged particles

The starting point for the discussion on the transfer of energy in the electromagnetic field is the expression for the mechanical work that is done by the force that an electromagnetic field exerts on an electrically charged particle moving in vacuum. Let $\{E_k,H_j\}$ denote the electric and the magnetic field strengths of the field in which the particle moves, q the electric charge of the particle, and w_k its velocity, then the force F_k exerted on the particle, is given by (see Equation (18.1-1))

$$F_k = qE_k + q\mu_0\varepsilon_{k,m,j}w_mH_j \,. \tag{21.1-1}$$

Now, the time rate $\dot{W}$ at which mechanical work is done by a force F_k acting on a particle moving with velocity w_k, is given by (Figure 21.1-1)

$$\dot{W} = F_kw_k \,. \tag{21.1-2}$$

Substitution of the expression for the force F_k given in Equation (21.1-1) into Equation (21.1-2) leads to

$$\dot{W} = qE_kw_k \,, \tag{21.1-3}$$

where the property $\varepsilon_{k,m,j}w_kw_m = 0$ has been used. As Equation (21.1-3) shows, only the electric field yields a contribution to the mechanical work; the magnetic field does not owing to the fact that the magnetic force is always oriented perpendicularly to the instantaneous velocity.

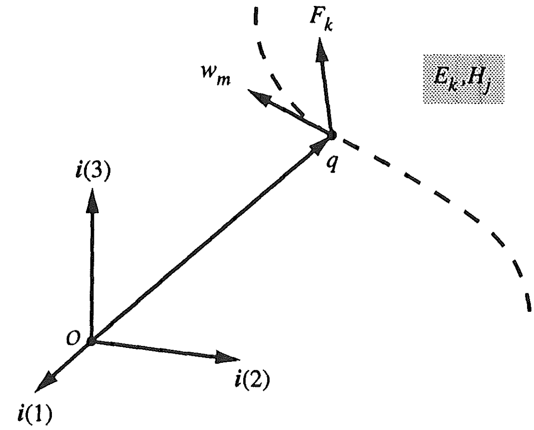

Figure 21.1-1 Mechanical work done by an electromagnetic field $\{E_k,H_j\}$ through its force F_k acting on a particle with electric charge q, moving with velocity w_m.

Now, consider a collection of electrically charged particles for which the macroscopic volume density of electric convection current can be defined (see Section 19.4). Let $N_\varepsilon = N_\varepsilon(x,t)$ be the number of particles present in the representative elementary domain $\mathcal{D}_\varepsilon = \mathcal{D}_\varepsilon(x)$, then the volume density $\dot{w} = \dot{w}(x,t)$ of the total work done by the field on the particles in $\mathcal{D}_\varepsilon$ is

$$\dot{w}(x,t) = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} q^{(p)} E_k(x^{(p)},t)\, w_k^{(p)}(t) , \tag{21.1-4}$$

where $x^{(p)} = x^{(p)}(t)$ is the position vector of the particle with label p. Since E_k is a macroscopic field quantity, we can in the right-hand side of Equation (21.1-4) replace $E_k(x^{(p)},t)$ by the electric field strength of the collection of particles that remains after the particle with label p has been removed. This remaining electric field strength $E_k(x,t)$ is a continuous function of position, and we can put, since an isolated particle in vacuum does not exert a net force on itself,

$$E_k(x^{(p)},t) \approx E_k(x,t) \qquad \text{for all} \quad p = 1,\dots,N_\varepsilon , \tag{21.1-5}$$

where x is the position vector of the (bary)centre of $\mathcal{D}_\varepsilon$. Using the definition Equation (19.4-20) for the volume density of electric convection current, Equation (21.1-4) reduces to

$$\dot{w} = E_k J_k . \tag{21.1-6}$$

Since the collection of particles is present in a vacuum domain, the pertaining electromagnetic field equations are

$$-\varepsilon_{k,m,p}\partial_m H_p + \varepsilon_0 \partial_t E_k + J_k = 0 , \tag{21.1-7}$$

$$\varepsilon_{j,n,r}\partial_n E_r + \mu_0 \partial_t H_j = 0 . \tag{21.1-8}$$

Multiplying Equation (21.1-7) by E_k, and Equation (21.1-8) by H_j, and adding the results, we obtain

$$\partial_m(\varepsilon_{m,k,j} E_k H_j) + \varepsilon_0 E_k \partial_t E_k + \mu_0 H_j \partial_t H_j + E_k J_k = 0 . \tag{21.1-9}$$

Since

$$\varepsilon_0 E_k \partial_t E_k = \partial_t \left[\tfrac{1}{2}\varepsilon_0 E_k E_k\right], \tag{21.1-10}$$

and

$$\mu_0 H_j \partial_t H_j = \partial_t \left[\tfrac{1}{2}\mu_0 H_j H_j\right], \tag{21.1-11}$$

Equation (21.1-9) can be rewritten as

$$\partial_m(\varepsilon_{m,k,j} E_k H_j) + \partial_t \left[\tfrac{1}{2}\varepsilon_0 E_k E_k + \tfrac{1}{2}\mu_0 H_j H_j\right] + E_k J_k = 0\,, \tag{21.1-12}$$

which is the local form of the electromagnetic energy equation for a collection of electrically charged particles moving in vacuum.

To arrive at the physical interpretation of the different terms in this relation – note that at this point only the last term on the left-hand side has been physically interpreted – Equation (21.1-12) is integrated over a vacuum domain $\mathcal{D}$ that is bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$, where $\mathcal{S}_2$ completely surrounds $\mathcal{S}_1$ (Figure 21.1-2). Now, in view of Gauss' integral theorem, we have (note the orientations of the unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$)

$$\int_{x\in\mathcal{D}} \partial_m(\varepsilon_{m,k,j} E_k H_j)\,\mathrm{d}V = \int_{x\in\mathcal{S}_2} \varepsilon_{m,k,j}\nu_m E_k H_j\,\mathrm{d}A - \int_{x\in\mathcal{S}_1} \varepsilon_{m,k,j}\nu_m E_k H_j\,\mathrm{d}A\,. \tag{21.1-13}$$

Using Equation (21.1-13) in the integrated form of Equation (21.1-12), we arrive at

$$\int_{x\in\mathcal{S}_2} \varepsilon_{m,k,j}\nu_m E_k H_j\,\mathrm{d}A + \partial_t \int_{x\in\mathcal{D}} \left[\tfrac{1}{2}\varepsilon_0 E_k E_k + \tfrac{1}{2}\mu_0 H_j H_j\right]\mathrm{d}V + \int_{x\in\mathcal{D}} E_k J_k\,\mathrm{d}V$$

$$= \int_{x\in\mathcal{S}_1} \varepsilon_{m,k,j}\nu_m E_k H_j\,\mathrm{d}A\,. \tag{21.1-14}$$

Equation (21.1-14) is the global form of the electromagnetic energy equation for a collection of electrically charged particles moving in vacuum. It leads to the following physical interpretation of the different terms occurring in it.

The term

$$\dot{W} = \int_{x\in\mathcal{D}} E_k J_k\,\mathrm{d}V \tag{21.1-15}$$

is the *time rate at which mechanical work is done* by the electromagnetic field on the moving electrically charged particles. This mechanical work is used to change the kinetic energy of the particles. If (part of) this kinetic energy is irreversibly converted into heat, the term is (for that part) a loss term as far as the electromagnetic field is concerned.

The two quantities

$$W^{\mathrm{e}} = \int_{x\in\mathcal{D}} \tfrac{1}{2}\varepsilon_0 E_k E_k\,\mathrm{d}V \tag{21.1-16}$$

and

$$W^{\mathrm{m}} = \int_{x\in\mathcal{D}} \tfrac{1}{2}\mu_0 H_j H_j\,\mathrm{d}V \tag{21.1-17}$$

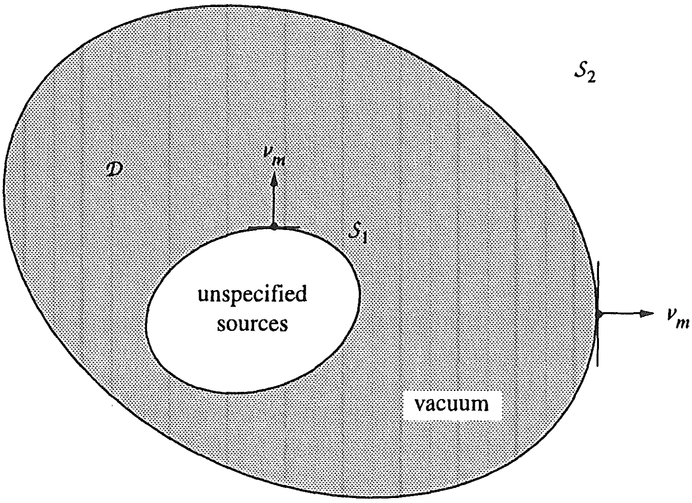

Figure 21.1-2 Vacuum domain $\mathcal{D}$ in between the closed surface S_1 that surrounds the unspecified sources of an electromagnetic field and the closed surface S_2 through which $\mathcal{D}$ interacts with its surroundings.

occur only as $\partial_t(W^{e} + W^{m})$ in Equation (21.1-14). Furthermore, $W^{e} \geqslant 0$ since $\frac{1}{2}\varepsilon_0 E_p E_p \geqslant 0$ and $W^{m} \geqslant 0$ since $\frac{1}{2}\mu_0 H_q H_q \geqslant 0$. On account of this ("work done can lead to a change in time of stored energy"), W^{e} is interpreted as the amount of *energy* that is *stored in the electric field* in $\mathcal{D}$, and W^{m} is interpreted as the amount of *energy* that is *stored in the magnetic field* in $\mathcal{D}$.

Finally,

$$P^{\text{out}} = \int_{x \in S_2} \varepsilon_{m,k,j} \nu_m E_k H_j \, \mathrm{d}A \tag{21.1-18}$$

is an integral over the surface S_2 through which $\mathcal{D}$ is in contact with its surroundings. Therefore, it seems natural to interpret this term as the *instantaneous power* that is, across S_2, *transferred from $\mathcal{D}$ toward its surroundings* (note the orientation of ν_m on S_2). Similarly,

$$P^{\text{in}} = \int_{x \in S_1} \varepsilon_{m,k,j} \nu_m E_k H_j \, \mathrm{d}A \tag{21.1-19}$$

can be interpreted as the *instantaneous power* that is, across S_1, *transferred from the interior* of S_1 (that can contain as yet unspecified sources of electromagnetic radiation) *toward $\mathcal{D}$* (note the orientation of ν_m on S_1).

Using Equations (21.1-15)-(21.1-19), Equation (21.1-14) can be rewritten as

$$\dot{W} + \partial_t(W^{e} + W^{m}) + P^{\text{out}} = P^{\text{in}} . \tag{21.1-20}$$

Equation (21.1-20) is denoted as the *global electromagnetic energy theorem* pertaining to the electromagnetic field in $\mathcal{D}$, and it expresses the conservation of energy in the system under consideration.

Now that Equation (21.1-14) has been physically interpreted, we return to Equation (21.1-9). In accordance with Equations (21.1-16) and (21.1-17) we introduce the *volume density of electric field energy*

$$w^{\mathrm{e}} = \tfrac{1}{2}\varepsilon_0 E_k E_k \tag{21.1-21}$$

and the *volume density of magnetic field energy*

$$w^{\mathrm{m}} = \tfrac{1}{2}\mu_0 H_j H_j \, . \tag{21.1-22}$$

Furthermore, we introduce, in view of Equations (21.1-18) and (21.1-19), the *area density of electromagnetic power flow*

$$S_m = \varepsilon_{m,k,j} E_k H_j \, ; \tag{21.1-23}$$

this vector is known as Poynting's vector. With the aid of Equation (21.1-6) and Equations (21.1-21)–(21.1-23), Equation (21.1-9) can be rewritten as

$$\partial_m S_m + \partial_t (w^{\mathrm{e}} + w^{\mathrm{m}}) + \dot{w} = 0 \, . \tag{21.1-24}$$

Equation (21.1-24) is known as *Poynting's theorem.* It is in accordance with the principle of the conservation energy and expresses the *local electromagnetic energy equation* for the system under consideration.

21.2 Energy theorem for the electromagnetic field in stationary matter

The starting point for our considerations on the transfer of electromagnetic energy in matter is the observation that the exchange of energy between some bounded piece of matter and the vacuum that surrounds it takes place across the boundary surface of the relevant piece of matter. Now, in vacuum, the area density of electromagnetic power flow that is associated with this energy transfer is given by Equation (21.1-23). Furthermore, on account of the boundary conditions (20.1-2) and (20.1-3), the normal component of Poynting's vector is continuous across any source-free interface (see Exercise 20.1-3), in particular across the one between vacuum and matter. Now, a necessary prerequisite for some vectorial quantity to be a candidate for the area density of power flow is that its normal component is continuous across any interface that is free from surface sources (passive interface). Otherwise, there would be a net gain or loss of energy in a domain of zero thickness, a possibility that would be in contradiction with the physical condition that outside (mathematical models of) surface sources storage of energy requires volume. Consequently, the Poynting vector

$$S_m = \varepsilon_{m,r,p} E_r H_p \tag{21.2-1}$$

can, also in matter, be taken as the *area density of electromagnetic power flow*.

Starting from Equation (21.2-1) we can, upon using the electromagnetic field equations in matter (see (18.3-11) and (18.3-12))

$$-\varepsilon_{k,m,p}\partial_m H_p + J_k + \partial_t D_k = -J_k^{\mathrm{ext}}, \tag{21.2-2}$$

$$\varepsilon_{j,n,r}\partial_n E_r + \partial_t B_j = -K_j^{\mathrm{ext}}, \tag{21.2-3}$$

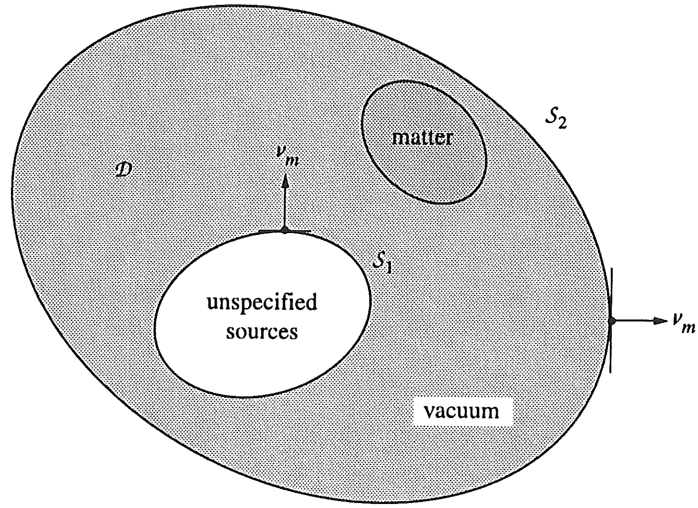

Figure 21.2-1 Vacuum domain $\mathcal{D}$ containing pieces of matter and specified sources in between the closed surface S_1 that surrounds the unspecified sources of an electromagnetic field and the closed surface S_2 through which $\mathcal{D}$ interacts with its surroundings.

derive the local form of the energy theorem. Proceeding as in Section 21.1, we multiply Equation (21.2-2) by E_k and Equation (21.2-3) by H_j, and add the results. In this way we obtain

$$\partial_m(\varepsilon_{m,r,p}E_rH_p) + E_kJ_k + E_k\partial_tD_k + H_j\partial_tB_j = -E_kJ_k^{\text{ext}} - H_jK_j^{\text{ext}}. \tag{21.2-4}$$

Equation (21.2-4) is the local form of the electromagnetic energy equation. To arrive at the physical interpretation of the different terms in this relation – note that at this point only the Poynting vector in the first term on the left-hand side has a physical interpretation – we integrate Equation (21.2-4) over the domain $\mathcal{D}$ that is bounded internally by the closed surface S_1 and externally by the closed surface S_2 where S_2 completely surrounds S_1 (Figure 21.2-1).

Using Equation (21.1-13), we obtain

$$\int_{x\in S_2} \varepsilon_{m,r,p}\nu_mE_rH_p\,\mathrm{d}A + \int_{x\in\mathcal{D}} (E_kJ_k + E_k\partial_tD_k + H_j\partial_tB_j)\,\mathrm{d}V$$

$$= \int_{x\in S_1} \varepsilon_{m,r,p}\nu_mE_rH_p\,\mathrm{d}A - \int_{x\in\mathcal{D}} (E_kJ_k^{\text{ext}} + H_jK_j^{\text{ext}})\,\mathrm{d}V\,. \tag{21.2-5}$$

As in Section 21.1, an equation of the kind of Equation (21.2-5) must express an energy conservation law. The first term on the left-hand side represents the instantaneous power

$$P^{\text{out}} = \int_{x\in S_2} \varepsilon_{m,k,j}\nu_mE_kH_j\,\mathrm{d}A \tag{21.2-6}$$

that flows across S_2 from $\mathcal{D}$ into its surroundings. The first term on the right-hand side represents the instantaneous power

$$P^{\text{in}} = \int_{x\in S_1} \varepsilon_{m,k,j}\nu_mE_kH_j\,\mathrm{d}A \tag{21.2-7}$$

that flows across $\mathcal{S}_1$, the interior of which may contain as yet unspecified sources of electromagnetic radiation, toward $\mathcal{D}$. The second term on the right-hand side represents the electromagnetic power

$$P^{\text{ext}} = -\int_{x \in \mathcal{D}} (E_k J_k^{\text{ext}} + H_j K_j^{\text{ext}})\,\text{d}V \tag{21.2-8}$$

that is generated by the specified sources in $\mathcal{D}$ whose action has been accounted for by the volume densities of exterior electric current and magnetic current. The minus sign in this term is typical for the fact that for the sources to deliver a positive power to the system, the volume current densities must be oriented opposite to the fields. As far as the second term on the left-hand side is concerned, we can without specifying the constitutive relations of the material present in $\mathcal{D}$ say no more than that it must represent the sum of the time rate $\dot{W}^{\text{h}}$ at which electromagnetic energy is irreversibly converted into heat and the time rate of change $\partial_t W^{\text{em}}$ of the electromagnetic energy W^{em} that is reversibly stored in $\mathcal{D}$. With this, Equation (21.2-5) can be written as

$$P^{\text{out}} + \dot{W}^{\text{h}} + \partial_t W^{\text{em}} = P^{\text{in}} + P^{\text{ext}}. \tag{21.2-9}$$

It is evident that, in interpreting Equation (21.2-5) as presented in Equation (21.2-9), the principle of conservation of energy has been used as a postulate.

Note: In view of the starting point Equation (21.2-1) of our considerations, and the fact that the continuity of the tangential parts of the electric and the magnetic field strenghts across an interface (see Section 20.1) has only been proved for time-invariant interfaces, our energy considerations in the present section hold for stationary matter only. The exchange of energy in moving pieces of matter is a subject of investigation in the theory of relativity and is beyond the scope of the present analysis.

Passive, dissipative, lossless or active media

From an energetic point of view, a material medium can be typified as passive, dissipative, lossless or active. These notions are elucidated by reconsidering the configuration of Figure 21.2-1. In this configuration the medium in $\mathcal{D}$ is denoted as *passive* if the net outflow of electromagnetic energy across the boundary surface of a domain is less than or equal to the energy delivered by the sources in that domain, i.e. if

$$\int_{t=-\infty}^{\infty} (P^{\text{out}} - P^{\text{in}})\,\text{d}t \leqslant \int_{t=-\infty}^{\infty} P^{\text{ext}}\,\text{d}t, \tag{21.2-10}$$

as *dissipative* if the net outflow of electromagnetic energy across the boundary surface of a domain is less than the energy delivered by the sources in that domain, i.e. if

$$\int_{t=-\infty}^{\infty} (P^{\text{out}} - P^{\text{in}})\,\text{d}t < \int_{t=-\infty}^{\infty} P^{\text{ext}}\,\text{d}t, \tag{21.2-11}$$

as *lossless* if the net outflow of electromagnetic energy across the boundary surface of a domain is equal to the energy delivered by the sources in that domain, i.e. if

$$\int_{t=-\infty}^{\infty} (P^{\text{out}} - P^{\text{in}})\,\mathrm{d}t = \int_{t=-\infty}^{\infty} P^{\text{ext}}\,\mathrm{d}t\,, \tag{21.2-12}$$

and as *active* if the net outflow of electromagnetic energy across the boundary surface of a domain exceeds the energy delivered by the sources in that domain, i.e. if

$$\int_{t=-\infty}^{\infty} (P^{\text{out}} - P^{\text{in}})\,\mathrm{d}t > \int_{t=-\infty}^{\infty} P^{\text{ext}}\,\mathrm{d}t\,. \tag{21.2-13}$$

From Equation (21.2-9) and Equations (21.2-10)–(21.2-13) it follows that for transient fields, for which

$$\int_{t=-\infty}^{\infty} \partial_t W^{\text{em}}\,\mathrm{d}t = \lim_{t\uparrow\infty} W^{\text{em}} - \lim_{t\downarrow-\infty} W^{\text{em}} = 0 - 0 = 0\,, \tag{21.2-14}$$

we equivalently have

$$\int_{t=-\infty}^{\infty} \dot{w}^{\text{h}}\,\mathrm{d}t \geqslant 0 \quad \text{for passive media,} \tag{21.2-15}$$

$$\int_{t=-\infty}^{\infty} \dot{w}^{\text{h}}\,\mathrm{d}t > 0 \quad \text{for dissipative media,} \tag{21.2-16}$$

$$\int_{t=-\infty}^{\infty} \dot{w}^{\text{h}}\,\mathrm{d}t = 0 \quad \text{for lossless media,} \tag{21.2-17}$$

$$\int_{t=-\infty}^{\infty} \dot{w}^{\text{h}}\,\mathrm{d}t < 0 \quad \text{for active media.} \tag{21.2-18}$$

Which one of these characteristics applies is, in practice, tested by employing *time-periodic fields*. Let T be the period in time of these fields and let

$$\langle ... \rangle_T = T^{-1} \int_{t=t_0}^{t_0+T} \ldots\,\mathrm{d}t \tag{21.2-19}$$

denote the time average over a single period. Then, we have, since

$$\langle \partial_t W^{\text{em}} \rangle_T = T^{-1} \int_{t=t_0}^{t_0+T} \partial_t W^{\text{em}}\,\mathrm{d}t = T^{-1}\left[W^{\text{em}}(t_0 + T) - W^{\text{em}}(t_0)\right] = 0\,, \tag{21.2-20}$$

for any t_0,

$$\langle P^{\text{out}} - P^{\text{in}} \rangle_T \leqslant \langle P^{\text{ext}} \rangle_T \quad \text{for passive media,} \tag{21.2-21}$$

$$\langle P^{\text{out}} - P^{\text{in}} \rangle_T < \langle P^{\text{ext}} \rangle_T \quad \text{for dissipative media,} \tag{21.2-22}$$

$$\langle P^{\text{out}} - P^{\text{in}} \rangle_T = \langle P^{\text{ext}} \rangle_T \quad \text{for lossless media,} \tag{21.2-23}$$

$$\langle P^{\text{out}} - P^{\text{in}} \rangle_T > \langle P^{\text{ext}} \rangle_T \quad \text{for active media,} \tag{21.2-24}$$

or, equivalently,

$$\langle \dot{w}^{\text{h}} \rangle_T \geqslant 0 \quad \text{for passive media,} \tag{21.2-25}$$

$$\langle \dot{w}^{\mathrm{h}} \rangle_T > 0 \quad \text{for dissipative media,} \tag{21.2-26}$$

$$\langle \dot{w}^{\mathrm{h}} \rangle_T = 0 \quad \text{for lossless media,} \tag{21.2-27}$$

$$\langle \dot{w}^{\mathrm{h}} \rangle_T < 0 \quad \text{for active media.} \tag{21.2-28}$$

Note: From the principle of conservation of energy it is clear that an electromagnetically active medium must have an energy resource in some other physical phenomenon. For example, the electromagnetic field can extract energy from an acoustic wave motion.

21.3 Energy theorem for the electromagnetic field in a medium with conductivity, permittivity and permeability

In this section the exchange of electromagnetic energy in a medium that is linear, time invariant, instantaneously reacting and locally reacting in its electromagnetic behaviour is considered. Its electromagnetic properties are characterised by a tensorial conductivity $\sigma_{k,r} = \sigma_{k,r}(\boldsymbol{x})$, a tensorial permittivity $\varepsilon_{k,r} = \varepsilon_{k,r}(\boldsymbol{x})$, and a tensorial permeability $\mu_{j,p} = \mu_{j,p}(\boldsymbol{x})$. Hence, the medium can be anisotropic and inhomogeneous. The pertaining constitutive relations are (see Section 19.2)

$$J_k = \sigma_{k,r} E_r \,, \tag{21.3-1}$$

$$D_k = \varepsilon_{k,r} E_r \,, \tag{21.3-2}$$

$$B_j = \mu_{j,p} H_p \,. \tag{21.3-3}$$

The permittivity and the permeability are assumed to be symmetric tensors, i.e. $\varepsilon_{k,r} = \varepsilon_{r,k}$ and $\mu_{j,p} = \mu_{p,j}$; no such assumption is made for the conductivity, i.e. in general, $\sigma_{k,r} \neq \sigma_{r,k}$. Then, we have

$$E_k J_k = \sigma_{k,r} E_k E_r \,, \tag{21.3-4}$$

$$E_k \partial_t D_k = \partial_t \left[\tfrac{1}{2} \varepsilon_{k,r} E_k E_r \right], \tag{21.3-5}$$

$$H_j \partial_t B_j = \partial_t \left[\tfrac{1}{2} \mu_{j,p} H_j H_p \right]. \tag{21.3-6}$$

Substitution of Equations (21.3-4)–(21.3-6) into Equation (21.2-4) leads to

$$\partial_m S_m + \dot{w}^{\mathrm{h}} + \partial_t (w^{\mathrm{e}} + w^{\mathrm{m}}) = \dot{w}^{\mathrm{ext}}, \tag{21.3-7}$$

in which

$$S_m = \varepsilon_{m,r,p} E_r H_p \tag{21.3-8}$$

is the *electromagnetic power flow density* (Poynting vector),

$$\dot{w}^{\mathrm{h}} = \sigma_{k,r} E_k E_r \tag{21.3-9}$$

is the *volume density of electromagnetic power* that is *irreversibly dissipated into heat,*

$$w^{\mathrm{e}} = \tfrac{1}{2} \varepsilon_{k,r} E_k E_r \tag{21.3-10}$$

is the *volume density of reversibly stored electric field energy*, and

$$w^{\mathrm{m}} = \tfrac{1}{2} \mu_{j,p} H_j H_p \tag{21.3-11}$$

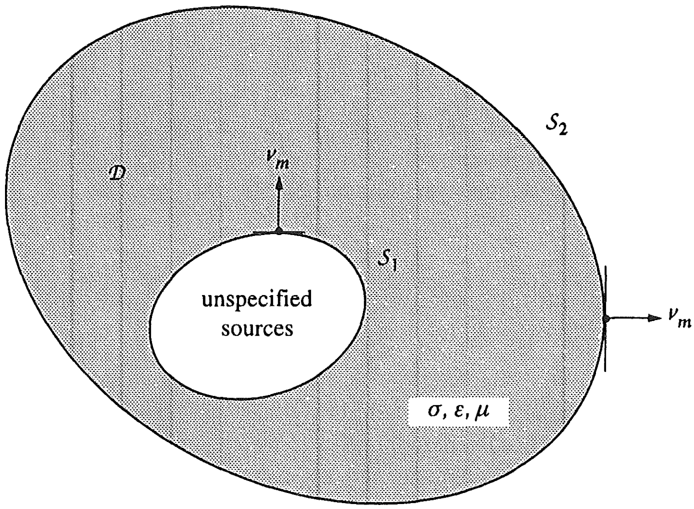

Figure 21.3-1 Domain $\mathcal{D}$ that contains a medium with conductivity, permittivity and permeability as its electromagnetic constitutive parameters and specified sources in between the closed surface S_1 that surrounds the unspecified sources of the electromagnetic field, and the closed surface S_2 across which $\mathcal{D}$ interacts with its surroundings.

is the *volume density of reversibly stored magnetic field energy*, and

$$\dot{w}^{\mathrm{ext}} = -E_k J_k^{\mathrm{ext}} - H_j K_j^{\mathrm{ext}}, \tag{21.3-12}$$

is the *volume density of power delivered by the sources.* Equation (21.3-7) is the *local electromagnetic energy relation* for the medium under consideration. The physical condition that $\dot{w}^{\mathrm{h}} > 0$, $w^{\mathrm{e}} > 0$ and $w^{\mathrm{m}} > 0$ for any non-vanishing electromagnetic field puts restrictions on the admissible values of the elements of $\sigma_{k,r}$, $\varepsilon_{k,r}$ and $\mu_{j,p}$. Tensors whose elements satisfy these restrictions are denoted as *positive definite.*

For the configuration of Figure 21.3-1 integration of Equation (21.3-7) over the domain $\mathcal{D}$ and application of Gauss' integral theorem to the term containing $\partial_m S_m$ lead to

$$\int_{x \in S_2} \nu_m S_m \, \mathrm{d}A + \int_{x \in \mathcal{D}} \dot{w}^{\mathrm{h}} \, \mathrm{d}V + \partial_t \int_{x \in \mathcal{D}} (w^{\mathrm{e}} + w^{\mathrm{m}}) \, \mathrm{d}V$$

$$= \int_{x \in S_1} \nu_m S_m \, \mathrm{d}A - \int_{x \in \mathcal{D}} (E_k J_k^{\mathrm{ext}} + H_j K_j^{\mathrm{ext}}) \, \mathrm{d}V, \tag{21.3-13}$$

in which

$$P^{\mathrm{out}} = \int_{x \in S_2} \nu_m S_m \, \mathrm{d}A \tag{21.3-14}$$

is the instantaneous *outward electromagnetic power flow across S_2 from $\mathcal{D}$ towards its surroundings,*

$$\dot{W}^{\mathrm{h}} = \int_{x \in \mathcal{D}} \dot{w}^{\mathrm{h}} \, \mathrm{d}V \tag{21.3-15}$$

is the *electromagnetic power irreversibly dissipated into heat* in $\mathcal{D}$,

$$W^{\mathrm{e}} = \int_{x \in \mathcal{D}} w^{\mathrm{e}} \, \mathrm{d}V \tag{21.3-16}$$

is the *electric field energy reversibly stored* in $\mathcal{D}$,

$$W^{\mathrm{m}} = \int_{x \in \mathcal{D}} w^{\mathrm{m}} \, \mathrm{d}V \tag{21.3-17}$$

is the *magnetic field energy reversibly stored* in $\mathcal{D}$,

$$P^{\mathrm{in}} = \int_{x \in \mathcal{S}_1} \nu_m S_m \, \mathrm{d}A \tag{21.3-18}$$

is the instantaneous *inward electromagnetic power flow across* $\mathcal{S}_1$ towards $\mathcal{D}$ from its surroundings, and

$$\dot{W}^{\mathrm{ext}} = \int_{x \in \mathcal{D}} \dot{w}^{\mathrm{ext}} \, \mathrm{d}V \tag{21.3-19}$$

is the *instantaneous electromagnetic power generated by the sources in* $\mathcal{D}$. Equation (21.3-13) is the *global electromagnetic energy relation* for the domain $\mathcal{D}$. With this, we conclude our energy considerations.

Exercises

Exercise 21.3-1

Show that $\partial_t\left[\frac{1}{2}\varepsilon_{k,r}E_kE_r\right] = E_k\varepsilon_{k,r}E_r$, provided that $\varepsilon_{k,r}$ is a symmetrical tensor (i.e. $\varepsilon_{k,r} = \varepsilon_{r,k}$).

Exercise 21.3-2

Show that $\partial_t\left[\frac{1}{2}\mu_{j,p}H_jH_p\right] = H_j\mu_{j,p}H_p$, provided that $\mu_{j,p}$ is a symmetrical tensor (i.e. $\mu_{j,p} = \mu_{p,j}$).

Exercise 21.3-3

Show that a necessary (but not sufficient) condition for the tensor $\sigma_{k,r}$ to be positive definite is $\sigma_{1,1} > 0$, $\sigma_{2,2} > 0$, $\sigma_{3,3} > 0$. (*Hint*: Take the condition that $\sigma_{k,r}E_kE_r > 0$ for any electric field, and consider the cases where the electric field has a single non-vanishing component only along each of the three axes of the chosen Cartesian reference frame.)

Exercise 21.3-4

Show that a necessary (but not sufficient) condition for the tensor $\varepsilon_{k,r}$ to be positive definite is $\varepsilon_{1,1} > 0$, $\varepsilon_{2,2} > 0$, $\varepsilon_{3,3} > 0$. (*Hint*: Take the condition that $\frac{1}{2}\varepsilon_{k,r}E_kE_r > 0$ for any electric field, and consider the cases where the electric field has a single non-vanishing component only along each of the three axes of the chosen Cartesian reference frame.)

Exercise 21.3-5

Show that a necessary (but not sufficient) condition for the tensor $\mu_{j,p}$ to be positive definite is $\mu_{1,1} > 0$, $\mu_{2,2} > 0$, $\mu_{3,3} > 0$. (*Hint*: Take the condition that $\frac{1}{2}\mu_{j,p}H_jH_p > 0$ for any magnetic field, and consider the cases where the magnetic field has a single non-vanishing component only along each of the three axes of the chosen Cartesian reference frame.)

Exercise 21.3-6

The electric field E_k of a uniformly charged ball with electric charge Q and radius a is radially symmetric around the centre O of the ball which is chosen as the origin of the coordinate system employed. Let $\boldsymbol{x}$ be the position vector from O to some point in space. Show (a) that the expression $E_k = Qx_k/4\pi\varepsilon_0 a^3$ when $0 \leqslant |\boldsymbol{x}| \leqslant a$ and $E_k = Qx_k/4\pi\varepsilon_0|\boldsymbol{x}|^3$ when $a \leqslant |\boldsymbol{x}| \leqslant \infty$ satisfies the field equations $\partial_k E_k = \rho/\varepsilon_0$ and $\varepsilon_{j,n,r}\partial_n E_r = 0$ for a static distribution of electric charge in vacuum with volume density of charge ρ. (In our case, $\rho = 3Q/4\pi a^3$) Determine: (b) the electric field energy $W^{e,1}$ stored in the domain occupied by the ball, (c) the electric field energy $W^{e,2}$ stored in the vacuum domain outside the ball, (d) the total electric field energy $W^e = W^{e,1} + W^{e,2}$ stored in the configuration. (e) Express $W^{e,1}$ and $W^{e,2}$ in fractions of W^e. (f) What happens to $W^{e,1}$, $W^{e,2}$ and W^e when Q remains fixed and $a \rightarrow 0$?

Answer:

(b) $W^{e,1} = Q/40\pi\varepsilon_0 a$;

(c) $W^{e,2} = Q^2/8\pi\varepsilon_0 a$;

(d) $W = 3Q^2/20\pi\varepsilon_0 a$;

(e) $W^{e,1} = W^e/6$, $W^{e,2} = 5W^e/6$;

(f) $W^{e,1} \rightarrow \infty$, $W^{e,2} \rightarrow \infty$ and $W^e \rightarrow \infty$ for $a \rightarrow 0$.

Exercise 21.3-7

The uniformly charged ball of Exercise 21.3-6 is a classical model for the electron (electric charge $-e$). By equating the electric field energy of the charge distribution to the relativistic mechanical rest energy $m_e c_0^2$ of the electron, we can determine the so-called *classical electron* *radius* r_e. Give (a) the expression for r_e, (b) substitute in this expression the values of e, μ_0 and m_e ($e = 1.602177 \times 10^{-19}$ C, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $m_e = 9.10938 \times 10^{-31}$ kg).

Answer: $r_e = 3e^2\mu_0/20\pi m_e$; $r_e = 1.690765 \times 10^{-15}$ m.

Exercise 21.3-8

Show that the energy theorem for a stationary flow of charged particles is given by $E_kJ_k + \partial_m(\varepsilon_{m,k,j}E_kH_j) = 0$, or $\dot{w} + \partial_mS_m = 0$.

Exercise 21.3-9

In the configuration of Figure 21.2-1 an electromagnetic field is present that has been generated by sources that were active in the time interval $-\infty < t < t_0$. At the instant $t = t_0$ the sources are switched off. Since the field is no longer sustained by sources, $W^{\text{em}} \to 0$ as $t \to \infty$. Use Equation (21.2-9) to show that

$$\int_{t=t_0}^{\infty} \dot{W}^{\text{h}}\,\mathrm{d}t + \int_{t=t_0}^{\infty} (P^{\text{out}} - P^{\text{in}})\,\mathrm{d}t = W^{\text{em}}(t_0)\,.$$

Exercise 21.3-10

Derive the local energy theorem for a medium with scalar conductivity $\sigma = \sigma(x)$, scalar permittivity $\varepsilon = \varepsilon(x)$ and scalar permeability $\mu = \mu(x)$.

Answer:

$$\partial_mS_m + \dot{w}^{\text{h}} + \partial_t(w^{\text{e}} + w^{\text{m}}) = \dot{w}^{\text{ext}},$$

in which

$$S_m = \varepsilon_{m,r,p}E_rH_p,\quad \dot{w}^{\text{h}} = \sigma E_kE_k,\quad w^{\text{e}} = \tfrac{1}{2}\varepsilon E_kE_k,\quad w^{\text{m}} = \tfrac{1}{2}\mu H_jH_j,\quad \dot{w}^{\text{ext}} = -(E_kJ_k^{\text{e}} + H_jK_j^{\text{e}}).$$

Exercise 21.3-11

(a) Show, for the medium considered in Exercise 21.3-10, that $\sigma \geqslant 0$ for a passive medium, $\sigma > 0$ for a dissipative medium, $\sigma = 0$ for a lossless medium, and $\sigma < 0$ for an active medium. (In a linear active medium the field values can grow beyond any limit; non-linear saturation effects limit, eventually, in practice the field values.) (b) What is the condition for $\frac{1}{2}\varepsilon E_kE_k > 0$ for any non-vanishing electric field? (c) What is the condition for $\frac{1}{2}\mu H_jH_j > 0$ for any non-vanishing magnetic field?

Answer: (b) $\varepsilon > 0$ (c) $\mu > 0$.

21.4 SI units of the quantities associated with the exchange of electromagnetic energy

The SI units of the quantities that are associated with the exchange of electromagnetic energy are listed in Table 21.4-1.

Table 21.4-1 Quantities related to the exchange of electromagnetic energy and their units in the International System of Units (SI)

Quantity		Unit	
Name	Symbol	Name	Symbol
Electric field energy	W^{e}	joule	J
Magnetic field energy	W^{m}	joule	J
Dissipated power	$\dot{W}^{h}$	watt	W
Power generated by external sources	$\dot{W}^{ext}$	watt	W
Electromagnetic power flow	P	watt	W
Volume density of electric field energy	w^{e}	joule/metre^3	J/m^3
Volume density of magnetic field energy	w^{m}	joule/metre^3	J/m^3
Volume density of dissipated power	$\dot{w}^{h}$	watt/metre^3	W/m^3
Volume density of power generated by external sources	$\dot{w}^{ext}$	watt/metre^3	W/m^3
Area density of electromagnetic power flow (Poynting vector)	S_m	watt/metre^2	W/m^2