22

Vector potentials, point-source solutions and Green's functions in the theory of electromagnetic radiation from sources

In this chapter the vector potentials for the electromagnetic field radiated by distributed sources are introduced. Next, the point-source solutions and the associated electromagnetic Green's tensors are introduced.

22.1 Vector potentials in the theory of electromagnetic radiation from distributed sources

The calculation of the electromagnetic field radiated by sources is, certainly in inhomogeneous and/or anisotropic media, a complicated affair. For this reason, it is standard practice to decompose, as far as possible, the problem of the determination of the total wave field into a number of subproblems, each of which either exhibits a particular feature or is easier to handle. In the present section, such a decomposition will be carried out for the electromagnetic radiation from sources in a linear, time-invariant, and locally reacting medium that may be arbitrarily inhomogeneous and/or anisotropic in its electromagnetic properties, where the type of source distribution (electric-current source, magnetic-current source) is the distinguishing feature. In the present section, an analysis in the time domain will be presented for an instantaneously reacting (i.e. lossless) medium; a complex frequency-domain analysis for a medium with relaxation will be presented in Chapter 25. The distinction as to the type of source will lead, in a natural fashion, to the introduction of the electromagnetic vector potentials.

Using the constitutive relations

$$D_k(x,t) = \varepsilon_{k,r}(x)E_r(x,t) \tag{22.1-1}$$

and

$$B_j(x,t) = \mu_{j,p}(x)H_p(x,t)\ , \tag{22.1-2}$$

Maxwell's equations become (see Equations (18.3-11) and (18.3-12))

$$-\varepsilon_{k,m,p}\partial_m H_p + \varepsilon_{k,r}\partial_t E_r = -J_k^{\text{ext}}, \tag{22.1-3}$$

$$\varepsilon_{j,n,r}\partial_n E_r + \mu_{j,p}\partial_t H_p = -K_j^{\text{ext}}. \tag{22.1-4}$$

Now let $\{E_r,H_p\} = \{E_r^J,H_p^J\}$ be the causal wave field that is generated by the electric-current source distribution $J_k^{\text{ext}} = J_k^{\text{ext}}(x,t)$ in the absence of a magnetic-current source distribution, i.e. for $K_j^{\text{ext}} = 0$. Then,

$$-\varepsilon_{k,m,p}\partial_m H_p^J + \varepsilon_{k,r}\partial_t E_r^J = -J_k^{\text{ext}}, \tag{22.1-5}$$

$$\varepsilon_{j,n,r}\partial_n E_r^J + \mu_{j,p}\partial_t H_p^J = 0\,. \tag{22.1-6}$$

Equation (22.1-5) is differentiated with respect to t to yield

$$-\varepsilon_{k,m,p}\partial_m\partial_t H_p^J + \varepsilon_{k,r}\partial_t^2 E_r^J = -\partial_t J_k^{\text{ext}}. \tag{22.1-7}$$

Taking advantage of the fact that the right-hand side of Equation (22.1-6) is zero, this equation is rewritten as

$$\partial_t H_p^J = -\mu_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n E_r^J\,, \tag{22.1-8}$$

where $\mu_{p,j}^{-1}$ is the tensor of rank two that is inverse to $\mu_{j,p}$, i.e.

$$\mu_{j,p}\mu_{p,j'}^{-1} = \delta_{j,j'}\,. \tag{22.1-9}$$

Substitution of the expression for $\partial_t H_p^J$ of Equation (22.1-8) in Equation (22.1-7) leads to the second-order vector differential equation

$$\varepsilon_{k,m,p}\partial_m(\mu_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n E_r^J) + \varepsilon_{k,r}\partial_t^2 E_r^J = -\partial_t J_k^{\text{ext}}. \tag{22.1-10}$$

Equation (22.1-10) induces us to introduce the *electric-current vector potential* $A_r = A_r(x,t)$ as the causal solution to the second-order vector differential equation (*electric vector potential wave equation*)

$$-\varepsilon_{k,m,p}\partial_m(\mu_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n A_r) - \varepsilon_{k,r}\partial_t^2 A_r = -J_k^{\text{ext}}, \tag{22.1-11}$$

with the volume source density of electric current as the forcing term on the right-hand side. Comparison of Equations (22.1-10) and (22.1-11) leads to

$$E_r^J = -\partial_t A_r\,, \tag{22.1-12}$$

while Equation (22.1-8) subsequently leads to

$$H_p^J = \mu_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n A_r\,, \tag{22.1-13}$$

in which causality has been used in performing the necessary integration with respect to time.

Let, next, $\{E_r,H_p\} = \{E_r^K,H_p^K\}$ be the causal wave field that is generated by the magnetic-current source distribution $K_j^{\text{ext}} = K_j^{\text{ext}}(x,t)$, in the absence of an electric-current source distribution, i.e. for $J_k^{\text{ext}} = 0$. Then,

$$-\varepsilon_{k,m,p}\partial_m H_p^K + \varepsilon_{k,r}\partial_t E_r^K = 0\,, \tag{22.1-14}$$

$$\varepsilon_{j,n,r}\partial_n E_r^K + \mu_{j,p}\partial_t H_p^K = -K_j^{\text{ext}}. \tag{22.1-15}$$

Taking advantage of the fact that the right-hand side of Equation (22.1-14) is zero, this equation is rewritten as

$$\partial_t E_r^K = \varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m H_p^K\,, \tag{22.1-16}$$

where $\varepsilon_{r,k}^{-1}$ is the tensor of rank two that is inverse to $\varepsilon_{k,r}$, i.e.

$$\varepsilon_{k,r}\varepsilon_{r,k'}^{-1} = \delta_{k,k'} . \tag{22.1-17}$$

Furthermore, Equation (22.1-15) is differentiated with respect to t to yield

$$\varepsilon_{j,n,r}\partial_n\partial_t E_r^K + \mu_{j,p}\partial_t^2 H_p^K = -\partial_t K_j^{\text{ext}}. \tag{22.1-18}$$

Substitution of the expression for $\partial_t E_r^K$ of Equation (22.1-16) in Equation (22.1-18) leads to the second-order vector differential equation

$$\varepsilon_{j,n,r}\partial_n(\varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m H_p^K) + \mu_{j,p}\partial_t^2 H_p^K = -\partial_t K_j^{\text{ext}}. \tag{22.1-19}$$

Equation (22.1-19) induces us to introduce the *magnetic-current vector potential* $F_p = F_p(\boldsymbol{x},t)$ as the causal solution to the second-order differential equation (*magnetic vector potential wave equation*)

$$-\varepsilon_{j,n,r}\partial_n(\varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m F_p) - \mu_{j,p}\partial_t^2 F_p = -K_j^{\text{ext}}, \tag{22.1-20}$$

with the volume source density of magnetic current as the forcing term on the right-hand side. Comparison of Equations (22.1-19) and (22.1-20) leads to

$$H_p^K = -\partial_t F_p , \tag{22.1-21}$$

while Equation (22.1-16) subsequently leads to

$$E_r^K = -\varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m F_p , \tag{22.1-22}$$

in which causality has been used in performing the necessary integration with respect to time.

Since the total wave field is the superposition of the two constituents, i.e.

$$\{E_r, H_p\} = \{E_r^J + E_r^K, H_p^J + H_p^K\} , \tag{22.1-23}$$

we end up with

$$E_r = -\partial_t A_r - \varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m F_p , \tag{22.1-24}$$

$$H_p = -\partial_t F_p + \mu_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n A_r , \tag{22.1-25}$$

which are the desired representations.

From the analysis it is clear that there exists some freedom in the choice of the vector potentials. In particular, this applies to the incorporation of differentiations or integrations with respect to time. In our procedure, care has been taken to arrive at final expressions in which each term contains only first-order differentiations, either with respect to time or with respect to the spatial coordinates, since this turns out to be the preferred structural distribution of derivatives for the study of wave phenomena.

Exercises

Exercise 22.1-1

Verify that Equations (22.1-24) and (22.1-25) satisfy Equations (22.1-3) and (22.1-4), provided that Equations (22.1-11) and (22.1-20) are satisfied.

22.2 Point-source solutions; Green's functions

In this section the decomposition of the total electromagnetic wave field into partial constituents is further carried through by considering each of the two separate volume source distributions (viz. the one of the electric-current type and the one of the magnetic-current type) as the continuous superposition of corresponding *point sources* in space–time. This is mathematically achieved by writing the volume source density of electric current $J_k^{\text{ext}} = J_k^{\text{ext}}(x,t)$ as

$$J_k^{\text{ext}}(x,t) = \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} \delta_{k,k'}\delta(x - x', t - t') J_{k'}^{\text{ext}}(x', t')\, \mathrm{d}V, \tag{22.2-1}$$

and the volume source density of magnetic current $K_j^{\text{ext}} = K_j^{\text{ext}}(x,t)$ as

$$K_j^{\text{ext}}(x,t) = \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} \delta_{j,j'}\delta(x - x', t - t') K_{j'}^{\text{ext}}(x', t')\, \mathrm{d}V, \tag{22.2-2}$$

where $\mathcal{D}^T$ is the spatial support of the distributed sources and where the sifting property of the Dirac delta distribution $\delta(x - x', t - t')$ operative at $x' = x$ and $t' = t$ has been used. Let, now, the tensor function of rank two $G^A_{r,k'} = G^A_{r,k'}(x,x',t,t')$ satisfy the second-order tensor differential equation (see Equation (22.1-11))

$$-\varepsilon_{k,m,p}\partial_m(\mu^{-1}_{p,j}\varepsilon_{j,n,r}\partial_n G^A_{r,k'}) - \varepsilon_{k,r}\partial_t^2 G^A_{r,k'} = -\delta_{k,k'}\delta(x - x', t - t'), \tag{22.2-3}$$

and let the tensor of rank two $G^F_{p,j'} = G^F_{p,j'}(x,x',t,t')$ satisfy the second-order tensor differential equation (see Equation (22.1-20))

$$-\varepsilon_{j,n,r}\partial_n(\varepsilon^{-1}_{r,k}\varepsilon_{k,m,p}\partial_m G^F_{p,j'}) - \mu_{j,p}\partial_t^2 G^F_{p,j'} = -\delta_{j,j'}\delta(x - x', t - t'), \tag{22.2-4}$$

then Equations (22.1-11) and (22.1-20) are satisfied by

$$A_r(x,t) = \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} G^A_{r,k'}(x,x',t,t') J_{k'}^{\text{ext}}(x', t')\, \mathrm{d}V \tag{22.2-5}$$

and

$$F_p(x,t) = \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} G^F_{p,j'}(x,x',t,t') K_{j'}^{\text{ext}}(x', t')\, \mathrm{d}V, \tag{22.2-6}$$

respectively. The proof follows by observing that the differentiations in the left-hand sides of Equations (22.1-11) and (22.1-20) are with respect to x and t, whereas the integrations in the right-hand sides of Equations (22.2-1) and (22.2-2) and (22.2-5) and (22.2-6), are with respect to x' and t'. The function $G^A_{r,k'} = G^A_{r,k'}(x,x',t,t')$ is denoted as the (tensor) *Green's function* associated with the *electric-current vector potential* $A_r = A_r(x,t)$; the function $G^F_{p,j'} = G^F_{p,j'}(x,x',t,t')$ is denoted as the (tensor) *Green's function* associated with the *magnetic-current vector potential* $F_p = F_p(x,t)$. In view of the time invariance of the media involved, both Green's functions depend on t and t' only via the difference $t - t'$. Furthermore, taking the Green's functions to be the causal solutions to Equations (22.2-3) and (22.2-4), respectively, the causality of $A_r = A_r(x,t)$ and $F_p = F_p(x,t)$ and, consequently, of the generated wave field is ensured.

The role that these Green's functions play in the solution of electromagnetic radiation problems will be more extensively discussed in Chapter 26.

Exercises

Exercise 22.2-1

Let $u = u(x,t)$ be the solution to the scalar wave equation

$$\partial_m \partial_m u - c^{-2} \partial_t^2 u = -\rho \qquad (22.2\text{-}7)$$

that is causally related to the action of the sources with volume density $\rho = \rho(x,t)$. The spatial support of the volume source density is $\mathcal{D}^T$. (a) Give the differential equation for the Green's function $G = G(x,x',t,t')$ and (b) express $u = u(x,t)$ as a superposition of point-source solutions.

Answer:

$$\text{(a)} \quad \partial_m \partial_m G - c^{-2} \partial_t^2 G = -\delta(x - x', t - t') \; ; \qquad (22.2\text{-}8)$$

and

$$\text{(b)} \quad u(x,t) = \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{x' \in \mathcal{D}^T} G(x,x',t,t') \rho(x',t') \, \mathrm{d}V \, . \qquad (22.2\text{-}9)$$