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Complex frequency-domain vector potentials, point-source solutions and Green's functions in the theory of electromagnetic radiation from sources

In this chapter the complex frequency-domain vector potentials, point-source solutions and associated Green's tensors for the electromagnetic radiation from sources are introduced, as they are related to the complex frequency-domain coupled electromagnetic wave equations discussed in Section 24.4. The analysis runs parallel to the one in the time domain presented in Chapter 22.

25.1 Complex frequency-domain vector potentials in the theory of electromagnetic radiation from distributed sources

Along the same lines as in Section 22.1 the complex frequency-domain vector potentials in the theory of electromagnetic radiation from sources are introduced. The starting point for the discussion is the complex frequency-domain coupled electromagnetic wave equations (see Equations (24.4-1) and (24.4-2))

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p + \hat{\eta}_{k,r}\hat{E}_r = -\hat{J}_k^{\text{ext}} , \qquad (25.1\text{-}1)$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r + \hat{\zeta}_{j,p}\hat{H}_p = -\hat{K}_j^{\text{ext}} , \qquad (25.1\text{-}2)$$

where possible non-zero initial-value contributions have been incorporated in the volume source densities on the right-hand side.

Let, now, $\{\hat{E}_r,\hat{H}_p\} = \{\hat{E}_r^J,\hat{H}_p^J\}$ denote the causal electromagnetic field that is generated by the electric current source distribution $\hat{J}_k^{\text{ext}} = \hat{J}_k^{\text{ext}}(\boldsymbol{x},s)$ in the absence of a magnetic current source distribution, i.e. for $\hat{K}_j^{\text{ext}} = 0$. Then,

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p^J + \hat{\eta}_{k,r}\hat{E}_r^J = -\hat{J}_k^{\text{ext}} , \qquad (25.1\text{-}3)$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r^J + \hat{\zeta}_{j,p}\hat{H}_p^J = 0 . \qquad (25.1\text{-}4)$$

Taking advantage of the fact that the right-hand side of Equation (25.1-4) is zero, this equation is rewritten as

$$\hat{H}_p^J = -\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{E}_r^J \,, \tag{25.1-5}$$

where $\hat{\zeta}_{p,j}^{-1}$ is the tensor that is inverse to $\hat{\zeta}_{j,p}$ (see Equation (24.4-10)). Substitution of the expression for $\hat{H}_p^J$ in Equation (25.1-3) leads to the second-order vector differential equation

$$\varepsilon_{k,m,p}\partial_m(\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{E}_r^J) + \hat{\eta}_{k,r}\hat{E}_r^J = -\hat{J}_k^{\rm ext} \,. \tag{25.1-6}$$

By analogy with Equation (22.1-11) we now introduce the *complex frequency-domain electric current vector potential* $\hat{A}_r = \hat{A}_r(x,s)$ as the causal solution to the second-order vector differential equation

$$-\varepsilon_{k,m,p}\partial_m(s\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{A}_r) - s\hat{\eta}_{k,r}\hat{A}_r = -\hat{J}_k^{\rm ext} \,, \tag{25.1-7}$$

with the volume source density of electric current as the forcing term on the right-hand side. Comparison of Equations (25.1-6) and (25.1-7) leads to

$$\hat{E}_r^J = -s\hat{A}_r \,, \tag{25.1-8}$$

while Equation (25.1-5) subsequently leads to

$$\hat{H}_p^J = s\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{A}_r \,. \tag{25.1-9}$$

Let, next, $\{\hat{E}_r,\hat{H}_p\} = \{\hat{E}_r^K,\hat{H}_p^K\}$ denote the causal electromagnetic field that is generated by the magnetic current source distribution $\hat{K}_j^{\rm ext} = \hat{K}_j^{\rm ext}(x,s)$, in the absence of an electric current source distribution, i.e. for $\hat{J}_k^{\rm ext} = 0$. Then,

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p^K + \hat{\eta}_{k,r}\hat{E}_r^K = 0 \,, \tag{25.1-10}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r^K + \hat{\zeta}_{j,p}\hat{H}_p^K = -\hat{K}_j^{\rm ext} \,. \tag{25.1-11}$$

Taking advantage of the fact that the right-hand side of Equation (25.1-10) is zero, this equation is rewritten as

$$\hat{E}_r^K = \hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{H}_p^K \,, \tag{25.1-12}$$

where $\hat{\eta}_{r,k}^{-1}$ is the tensor that is inverse to $\hat{\eta}_{k,r}$ (see Equation (24.4-9)). Substitution of the expression for E_r^K in Equation (25.1-11) leads to the second-order vector differential equation

$$\varepsilon_{j,n,r}\partial_n(\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{H}_p^K) + \hat{\zeta}_{j,p}\hat{H}_p^K = -\hat{K}_j^{\rm ext} \,. \tag{25.1-13}$$

By analogy with Equation (22.1-20) we now introduce the *complex frequency-domain magnetic current vector potential* $\hat{F}_p = \hat{F}_p(x,s)$ as the causal solution to the second-order vector differential equation

$$-\varepsilon_{j,n,r}\partial_n(s\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{F}_p) - s\hat{\zeta}_{j,p}\hat{F}_p = -\hat{K}_j^{\rm ext} \,, \tag{25.1-14}$$

with the volume source density of magnetic current as the forcing term on the right-hand side. Comparison of Equations (25.1-13) and (25.1-14) leads to

$$\hat{H}_p^K = -s\hat{F}_p \,, \tag{25.1-15}$$

while Equation (25.1-12) subsequently leads to

$$\hat{E}_r^K = -s\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{F}_p \,. \tag{25.1-16}$$

Since the total electromagnetic field is the superposition of the two constituents, i.e.

$$\{\hat{E}_r,\hat{H}_p\} = \{\hat{E}_r^J + \hat{E}_r^K,\hat{H}_p^J + \hat{H}_p^K\}\ , \tag{25.1-17}$$

we end up with

$$\hat{E}_r = -s\hat{A}_r - s\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{F}_p\ , \tag{25.1-18}$$

$$\hat{H}_p = -s\hat{F}_p + s\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{A}_r\ , \tag{25.1-19}$$

as the complex frequency-domain counterparts of Equations (22.1-24) and (22.1-25).

Exercises

Exercise 25.1-1

Verify that Equations (25.1-18) and (25.1-19) satisfy Equations (25.1-1) and (25.1-2) provided that Equations (25.1-7) and (25.1-14) are satisfied.

25.2 Complex frequency-domain point-source solutions; complex frequency-domain Green's functions

Along the same lines as in Section 22.2 the complex frequency-domain point-source solutions and the complex frequency-domain Green's functions are introduced. To this end, the volume source density of electric current $\hat{J}_k^{\text{ext}} = \hat{J}_k^{\text{ext}}(\boldsymbol{x},s)$ and the volume source density of magnetic current $\hat{K}_j^{\text{ext}} = \hat{K}_j^{\text{ext}}(\boldsymbol{x},s)$ are written as a continuous superposition of point sources through the representations (see Equation (22.2-1) and (22.2-2))

$$\hat{J}_k^{\text{ext}}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \delta_{k,k'}\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{J}_{k'}^{\text{ext}}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{25.2-1}$$

and

$$\hat{K}_j^{\text{ext}}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \delta_{j,j'}\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{K}_{j'}^{\text{ext}}(\boldsymbol{x}',s)\,\mathrm{d}V\ , \tag{25.2-2}$$

where $\mathcal{D}^T$ is the spatial support of the distributed sources and where the sifting property of the Dirac delta distribution $\delta(\boldsymbol{x}-\boldsymbol{x}')$ operative at $\boldsymbol{x}'=\boldsymbol{x}$ has been used. Let, now, the tensor function of rank two $\hat{G}_{r,k'}^A = \hat{G}_{r,k'}^A(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order tensor differential equation (see Equation (25.1-7))

$$-\varepsilon_{k,m,p}\partial_m(s\hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\partial_n\hat{G}_{r,k'}^A) - s\hat{\eta}_{k,r}\hat{G}_{r,k'}^A = -\delta_{k,k'}\delta(\boldsymbol{x}-\boldsymbol{x}')\ , \tag{25.2-3}$$

and let the tensor function of rank two $\hat{G}_{p,j'}^F = \hat{G}_{p,j'}^F(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order tensor differential equation (see Equation (25.1-14))

$$-\varepsilon_{j,n,r}\partial_n(s\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\partial_m\hat{G}_{p,j'}^F) - s\hat{\zeta}_{j,p}\hat{G}_{p,j'}^F = -\delta_{j,j'}\delta(\boldsymbol{x}-\boldsymbol{x}')\ , \tag{25.2-4}$$

then Equations (25.1-7) and (25.1-14) are satisfied by

$$\hat{A}_r(x,s) = \int_{x' \in \mathcal{D}^T} \hat{G}^A_{r,k'}(x,x',s)\hat{J}^{\text{ext}}_{k'}(x',s)\,\mathrm{d}V \tag{25.2-5}$$

and

$$\hat{F}_p(x,s) = \int_{x' \in \mathcal{D}^T} \hat{G}^F_{p,j'}(x,x',s)\hat{K}^{\text{ext}}_{j'}(x',s)\,\mathrm{d}V\,, \tag{25.2-6}$$

respectively. The proof follows by observing that the differentiations in the left-hand sides of Equations (25.1-7) and (25.1-14) are with respect to x, whereas the integrations in the right-hand sides of Equations (25.2-5) and (25.2-6) are with respect to x'. The function $\hat{G}^A_{r,k'} = \hat{G}^A_{r,k'}(x,x',s)$ is the complex frequency-domain tensor Green's function associated with the electric-current vector potential $\hat{A}_r = \hat{A}_r(x,s)$; the function $\hat{G}^F_{p,j'} = \hat{G}^F_{p,j'}(x,x',s)$ is the complex frequency-domain tensor Green's function associated with the magnetic-current vector potential $\hat{F}_p = \hat{F}_p(x,s)$. The role that these Green's functions play in the solution of electromagnetic radiation problems will be more extensively discussed in Chapter 26.

Exercises

Exercise 25.2-1

Let $\hat{u} = \hat{u}(x,s)$ be the solution to the scalar Helmholtz equation

$$\partial_m\partial_m\hat{u} - (s^2/c^2)\hat{u} = -\hat{\rho}\,. \tag{25.2-7}$$

The spatial support of the sources with volume density $\hat{\rho} = \hat{\rho}(x,s)$ is $\mathcal{D}^T$. (a) Give the differential equation for the Green's function $\hat{G} = \hat{G}(x,x',s)$ and (b) express $\hat{u} = \hat{u}(x,s)$ as a superposition of point-source solutions.

Answers:

(a) $$\partial_m\partial_m\hat{G} - (s^2/c^2)\hat{G} = -\delta(x - x')\,, \tag{25.2-8}$$

and

(b) $$\hat{u}(x,s) = \int_{x' \in \mathcal{D}^T} \hat{G}(x,x',s)\hat{\rho}(x',s)\,\mathrm{d}V\,. \tag{25.2-9}$$