REPRESENTATION THEOREMS FOR THE DISPLACEMENT IN AN ELASTIC SOLID AND THEIR APPLICATION TO ELASTODYNAMIC DIFFRACTION THEORY

A. T. DE HOOP

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Chapter I

INTRODUCTION

1. GENERAL INTRODUCTION AND REVIEW OF THE LITERATURE

Recent developments in acoustic and electromagnetic diffraction theory show that the formulation of diffraction problems in terms of integral equations is a subject of growing importance (see Bouwkamp (13)). Therefore, it seems worth while to attempt a generalization of the relevant methods to the field of elastodynamic diffraction theory. Now it is a well-known fact that in a homogeneous, isotropic, elastic solid there are two velocities of propagation; the larger of the two is associated with the wave fronts of irrotational or compressional waves, the smaller of the two is associated with the wave fronts of equivoluminal or shear waves. In a medium of infinite extent the two types of waves can propagate independently; however, as soon as boundaries occur, an interaction between the two types of waves takes place. Therefore, the phenomena related to the diffraction of elastic waves are expected to be of a complicated nature.

One of the most important applications of the theory of elastic wave propagation is the field of seismology. This explains why the emphasis is not on the steady-state behaviour of a system but rather on its transient response to a source which starts to act at a certain instant. Also, most of the problems that have been investigated deal with the radiation from a source located in an elastic medium consisting of several layers with different elastic properties (model of the earth). In this respect we mention Lamb's (26) classical solution of the problem of the radiation from a line source or a point source located at the free surface bounding an elastic half-space. A recent book by Ewing, Jardetzky and Press (16) covers most of the work that has been done on this type of problems.

Another publication we want to mention is Cagniard's monograph (14) on the generalization of Lamb's problem to the case of a point source located in one of two coupled elastic half-spaces. In this monograph the author develops a general method of solving transient problems. The idea is roughly as follows. After having taken the Laplace transform with respect to time, the remaining boundary value problem is solved. The solution of this boundary value problem is then written in such a form that the transient problem under consideration can be solved more or less by inspection and not by evaluating a Mellin inversion integral. During

the whole procedure, the Laplace transform variable is real and positive.

To the opinion of the present author, it is slightly unelegant that Cagniard introduces, be it temporarily, a *complex* variable which, after some transformations, plays the role of the actual time. In the present thesis Cagniard's method is modified in such a way that the relevant variable is real all the way through. The method thus developed can be applied to all sorts of mixed initial and boundary value problems associated with the acoustic, electromagnetic or elastodynamic wave equation.

Coming to our subject proper, we observe that the first step towards the formulation of diffraction problems in terms of integral equations is a representation theorem for the displacement in an elastic solid similar to Kirchhoff's formula (19, 1, 42) in scalar wave propagation. Part of the thesis deals with the derivation of such a representation theorem. The special case of harmonic time dependence has been discussed by Kupradse and can be found in the German edition of his book (25).

With the aid of the representation theorem the problems concerning the diffraction by a perfectly rigid or a perfectly weak screen are reduced to the solution of certain (differential-)integral equations. Several problems dealing with the diffraction of a plane pulse by a half-plane are worked out in detail. In these examples, the Wiener-Hopf technique for solving certain integral equations plays an important role.

Special attention has been paid to the saltus-problem formulation of the diffraction by a screen of vanishing thickness. This investigation has been inspired by Kottler's theory of diffraction (23, 24) by a black screen.

The literature on the subject matter is scarce. Maue (32) solved the problem of the diffraction of a time-harmonic plane wave by a perfectly weak half-plane with the aid of the Wiener-Hopf technique. A recent paper by Knopoff (21) is of a more general character. In this paper the author derives a representation theorem for the acceleration vector. In applying this representation theorem to the Kirchhoff diffraction by an aperture in a plane screen, certain line integrals along the edge of the aperture are introduced. In Section 6 we show that the way in which this has been done is inconsistent with the proper saltus-problem formulation of the problem.

Several useful formulae in relation to the reflection and refraction of a plane elastic wave at the plane surface bounding two media with different elastic properties can be found in Kolsky's book (22) and in Schoch's review paper on acoustic diffraction theory (39).

The present thesis deals mainly with the analytical methods involved in solving the diffraction problems under consideration. The numerical evaluation of the results is still a project of considerable extent.

2. BASIC PARTIAL DIFFERENTIAL EQUATIONS IN ELASTIC WAVE PROPAGATION

We consider wave motions of small amplitude in a homogeneous, isotropic, elastic solid occupying the entire three-dimensional space. The displacement and the stress, which characterize the motion in this medium, satisfy the partial differential equations

$$\partial\tau_{ij}/\partial x_j - \rho(\partial^2 u_i/\partial t^2) = -f_i, \tag{2.1}$$

$$\tau_{ij} = c_{ij,pq}(\partial u_p/\partial x_q), \tag{2.2}$$

where

$$c_{ij,pq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{jp}\delta_{iq}). \tag{2.3}$$

The symbols in these equations have the following meaning:

u_i = displacement vector,
τ_{ij} = stress tensor,
f_i = density of body forces (per unit volume),
x_i = cartesian coordinates,
t = time,
ρ = density of the elastic medium,
λ, μ = Lamé constants of the elastic medium,
δ_{ij} = unit tensor: $\delta_{11}=\delta_{22}=\delta_{33}=1$, $\delta_{ij}=0$ if $i\neq j$.

If in an expression a lower case latin subscript occurs twice, the expression has to be summed over this subscript from 1 to 3.

Eq. (2.1) is Newton's equation of motion for an element of volume (27); eq. (2.2) is the stress-strain relation (28). Substitution of (2.3) in (2.2) shows the stress-strain relation written in full

$$\tau_{ij} = \lambda(\partial u_k/\partial x_k)\delta_{ij} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i). \tag{2.4}$$

In view of later applications we use the stress-strain relation in the form (2.3). The tensor $c_{ij,pq}$ satisfies a number of symmetry relations: $c_{ij,pq} = c_{ji,pq} = c_{ji,qp} = c_{ij,qp}$, $c_{ij,pq} = c_{pq,ij}$.

Elimination of τ_{ij} from (2.1) and (2.2) leads to the elastodynamic wave equation

$$c_{ij,pq}(\partial^2 u_p/\partial x_j\partial x_q) - \rho(\partial^2 u_i/\partial t^2) = -f_i. \tag{2.5}$$

The more familiar form of (2.5),

$$v_P^2 \text{ grad div } \vec{u} - v_S^2 \text{ curl curl } \vec{u} - \partial^2\vec{u}/\partial t^2 = -\vec{f}/\rho, \tag{2.6}$$

where $\vec{u} = (u_1, u_2, u_3)$, $\vec{f} = (f_1, f_2, f_3)$ and

$$v_P = \{(\lambda+2\mu)/\rho\}^{\frac{1}{2}}, \tag{2.7}$$

$$v_S = (\mu/\rho)^{\frac{1}{2}}, \tag{2.8}$$

shows the occurrence of two velocities of propagation (29): v_P is the velocity of propagation of compressional, irrotational or P-waves (for which curl $\vec{u} = \vec{0}$), v_S is the velocity of propagation of shear, equivoluminal or S-waves (for which div $\vec{u} = 0$). In an elastic medium of infinite extent the two types of waves propagate independently. At boundaries an interaction between the two types of waves takes place. This property makes elastodynamic boundary value problems of such a complicated nature.

We now proceed to give the form to which the equations reduce in two-dimensional problems. A problem is called two-dimensional if the geometrical configuration and all physical quantities involved are independent of one of the cartesian coordinates. Consequently, all derivatives with respect to that coordinate vanish. Let x_2 be this particular coordinate. The differential equations (2.1) and (2.2) show that the general two-dimensional wave motion in an elastic solid is the superposition of two separate systems of displacements and stresses. One system only contains u_2 and satisfies the equations *

$$\partial\tau_{2\beta}/\partial x_\beta - \rho(\partial^2 u_2/\partial t^2) = -f_2, \tag{2.9}$$

$$\tau_{2\beta} = \mu(\partial u_2/\partial x_\beta). \tag{2.10}$$

The other system only contains u_1 and u_3 and satisfies the equations

$$\partial\tau_{\alpha\beta}/\partial x_\beta - \rho(\partial^2 u_\alpha/\partial t^2) = -f_\alpha, \tag{2.11}$$

$$\tau_{\alpha\beta} = c_{\alpha\beta,\gamma\delta}(\partial u_\gamma/\partial x_\delta), \quad \tau_{22} = \lambda(\partial u_\gamma/\partial x_\gamma). \tag{2.12}$$

Elimination of the stress from (2.9) and (2.10) leads to the scalar wave equation

$$\mu(\partial^2 u_2/\partial x_\beta \partial x_\beta) - \rho(\partial^2 u_2/\partial t^2) = -f_2. \tag{2.13}$$

Eq. (2.13) indicates that a wave with displacement $(0, u_2, 0)$ is a pure shear wave; it is often called a SH-wave (horizontally polarized shear wave).

Elimination of the stress from (2.11) and (2.12) leads to the two-dimensional elastodynamic wave equation

$$c_{\alpha\beta,\gamma\delta}(\partial^2 u_\gamma/\partial x_\beta \partial x_\delta) - \rho(\partial^2 u_\alpha/\partial t^2) = -f_\alpha. \tag{2.14}$$

Eq. (2.14) indicates that a wave with displacement $(u_1, 0, u_3)$ con-

* Greek subscripts only run through the values 1 and 3. As before, latin subscripts run through the values 1, 2 and 3. If useful, the subscript 2 will be written explicitly.

sists of a compressional and a shear wave. The corresponding shear wave is often called a SV-wave (vertically polarized shear wave).

Chapter II

REPRESENTATION THEOREMS

3. DISPLACEMENT DUE TO A POINT FORCE VARYING IN TIME

The displacement $\vec{u} = \vec{u}(\xi_1, \xi_2, \xi_3, t)$ due to a force of magnitude $h(t)$, directed along the constant unit vector $\vec{a}$ and acting at the point $\xi_i = x_i$ $(i = 1, 2, 3)$ satisfies the inhomogeneous differential equation

$$v_P^2 \text{ grad div } \vec{u} - v_S^2 \text{ curl curl } \vec{u} - \partial^2\vec{u}/\partial t^2 =$$

$$= -(\vec{a}/\rho)\ \delta(\xi_1-x_1, \xi_2-x_2, \xi_3-x_3)\ h(t), \tag{3.1}$$

where $\delta(\xi_1-x_1, \xi_2-x_2, \xi_3-x_3)$ denotes the three-dimensional Dirac delta function. It is assumed that $h(t)$ is a continuous function of time, together with its first and second derivative. In the right-hand side of (3.1) we employ the identity

$$-\vec{a}\ \delta(\xi_1-x_1, \xi_2-x_2, \xi_3-x_3) = \text{grad div } (\vec{a}/4\pi r) - \text{curl curl } (\vec{a}/4\pi r), \tag{3.2}$$

where $r = \{(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2\}^{\frac{1}{2}} \gtrsim 0$. The displacement $\vec{u}$ is written in the form

$$\vec{u} = \text{grad div } \vec{A}_P - \text{curl curl } \vec{A}_S. \tag{3.3}$$

In order that the right-hand side of (3.3) is a solution of (3.1) it is sufficient that $\vec{A}_P$ and $\vec{A}_S$ satisfy the equations

$$v_P^2 \nabla^2\vec{A}_P - \partial^2\vec{A}_P/\partial t^2 = (\vec{a}/4\pi\rho r)\ h(t), \tag{3.4}$$

$$v_S^2 \nabla^2\vec{A}_S - \partial^2\vec{A}_S/\partial t^2 = (\vec{a}/4\pi\rho r)\ h(t), \tag{3.5}$$

where $\nabla^2 = \partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2 + \partial^2/\partial\xi_3^2$. With $\vec{A}_P = A_P\vec{a}$ and $\vec{A}_S = A_S\vec{a}$, eqs. (3.4) and (3.5) reduce to the inhomogeneous scalar wave equations

$$v_P^2 \nabla^2 A_P - \partial^2 A_P/\partial t^2 = h(t)/4\pi\rho r, \tag{3.6}$$

$$v_S^2 \nabla^2 A_S - \partial^2 A_S/\partial t^2 = h(t)/4\pi\rho r. \tag{3.7}$$

The solutions of (3.6) and (3.7) that are bounded at $r=0$ are

readily obtained as integrals similar to the advanced and retarded potentials in electromagnetic theory (see, e.g., Stratton (41)). The result can be written in the form

$$A_P(r,t) = \frac{1}{4\pi\rho}\left[\frac{1}{r}\int_0^\infty h(t\pm r/v_P \pm\nu)\nu d\nu - \frac{1}{r}\int_0^\infty h(t\pm\nu)\nu d\nu\right], \quad (3.8)$$

$$A_S(r,t) = \frac{1}{4\pi\rho}\left[\frac{1}{r}\int_0^\infty h(t\pm r/v_S \pm\nu)\nu d\nu - \frac{1}{r}\int_0^\infty h(t\pm\nu)\nu d\nu\right]. \quad (3.9)$$

The upper sign corresponds to a wave converging towards r=0, the lower sign corresponds to a wave diverging from r=0. The behaviour of h(t) at large values of $|t|$ is supposed to be such that the integrals in (3.8) and (3.9) exist. With the aid of (3.9) it can be shown that

$$\text{curl curl } \vec{A}_S = \text{grad div } \vec{A}_S - \frac{1}{4\pi\rho v_S^2}\frac{h(t\pm r/v_S)}{r}\vec{a}. \quad (3.10)$$

This result enables us to write the displacement in the form

$$\vec{u} = \text{grad div } (\vec{A}_P - \vec{A}_S) + \frac{1}{4\pi\rho v_S^2}\frac{h(t\pm r/v_S)}{r}\vec{a}. \quad (3.11)$$

Substitution of (3.8) and (3.9) in (3.11) gives, in subscript notation, the expression

$$u_i = \frac{1}{4\pi\rho}\left\{\frac{\partial^2}{\partial\xi_i\partial\xi_j}\left[\frac{1}{r}\int_0^\infty [h(t\pm r/v_P\pm\nu) - h(t\pm r/v_S\pm\nu)]\,\nu d\nu\right] + \right.$$
$$\left. + \frac{1}{v_S^2}\frac{h(t\pm r/v_S)}{r}\delta_{ij}\right\}a_j. \quad (3.12)$$

In the right-hand side of (3.12) only the lower sign is physically acceptable, since only this choice leads to waves diverging from the source. Explicit expressions for the components of the displacement under consideration are given by Love (30).

On the other hand, the solution of (3.1) corresponding to waves converging towards r=0 will play an important role in obtaining the three-dimensional representation theorem to be derived in Section 4.

4. THREE-DIMENSIONAL REPRESENTATION THEOREM

The object of the present section is to obtain a representation theorem similar to Kirchhoff's formula (19, 1, 42) in scalar wave propagation. As usual, this will be derived from Gauss' divergence theorem applied to a suitably chosen vector.

Let S be a sufficiently regular closed surface and let V be its interior. Further, we introduce the vectors u_i and w_i, which, together with their first and second derivatives, are continuous functions

of position and time. From Gauss' theorem, applied to the vector $w_i c_{ij,pq}(\partial u_p/\partial x_q)$, we obtain

$$\int_V c_{ij,pq} w_i(\partial^2 u_p/\partial x_j \partial x_q) dx_1 dx_2 dx_3 + \\ + \int_V c_{ij,pq}(\partial w_i/\partial x_j)(\partial u_p/\partial x_q) dx_1 dx_2 dx_3 = \\ = \int_S c_{ij,pq} w_i(\partial u_p/\partial x_q) n_j dS, \tag{4.1}$$

where n_i is the unit vector in the direction of the outward normal to S. Since $c_{ij,pq} = c_{pq,ij}$, an interchange of u_i and w_i, followed by subtraction of the resulting identity from (4.1), leads to

$$\int_V c_{ij,pq} \left[w_i(\partial^2 u_p/\partial x_j \partial x_q) - u_i(\partial^2 w_p/\partial x_j \partial x_q)\right] dx_1 dx_2 dx_3 = \\ = \int_S c_{ij,pq} \left[w_i(\partial u_p/\partial x_q) - u_i(\partial w_p/\partial x_q)\right] n_j dS. \tag{4.2}$$

Let (x_1, x_2, x_3) be any point of observation located inside S and denote the variables of integration in (4.2) by ξ_1, ξ_2, ξ_3. In (4.2), we take for u_i a solution of (2.5). Further, w_i is chosen as

$$w_i = \frac{1}{4\pi\rho}\left\{\frac{\partial^2}{\partial\xi_i \partial\xi_j}\left[\frac{1}{r}\int_0^\infty \left[h(t+r/v_P+\nu) - h(t+r/v_S+\nu)\right] \nu d\nu\right] + \\ + \frac{1}{v_S^2}\frac{h(t+r/v_S)}{r}\delta_{ij}\right\} a_j, \tag{4.3}$$

where a_i is a constant vector, $r = \{(\xi_1-x_1)^2+(\xi_2-x_2)^2+(\xi_3-x_3)^2\}^{\frac{1}{2}} \geq 0$ and h(t) is a continuous function of time, together with its first and second derivative. The behaviour of h(t) at large positive values of t is assumed to be such that the integrals in (4.3) exist. The vector w_i, given by (4.3), represents a wave motion converging towards r=0 and satisfies, as long as $r \neq 0$, the homogeneous elastodynamic wave equation (see Section 3)

$$c_{ij,pq}(\partial^2 w_p/\partial\xi_j \partial\xi_q) - \rho(\partial^2 w_i/\partial t^2) = 0. \tag{4.4}$$

In the neighbourhood of r=0 we have

$$w_i = \frac{h(t)}{4\pi\rho}\left\{\frac{1}{2}\left(\frac{1}{v_S^2} + \frac{1}{v_P^2}\right)\frac{1}{r}\delta_{ij} + \frac{1}{2}\left(\frac{1}{v_S^2} - \frac{1}{v_P^2}\right)\frac{(\xi_i - x_i)(\xi_j - x_j)}{r^3}\right\} a_j + \\ + O(1), \tag{4.5}$$

which indicates a behaviour of order $O(r^{-1})$ as $r \to 0$.

Since w_i is singular at r=0, eq. (4.2) cannot be applied to the entire domain inside S. To exclude the singularity, a sphere S_ε with radius $\varepsilon > 0$ is circumscribed around $\xi_i = x_i$ (i=1, 2, 3) and V is taken as the domain bounded externally by S and internally by S_ε. From (2.5), (4.4) and (4.2) we obtain

$$\int_V \left[w_i(\partial^2 u_i/\partial t^2) - u_i(\partial^2 w_i/\partial t^2)\right] \rho\, d\xi_1 d\xi_2 d\xi_3 - \int_V w_i f_i d\xi_1 d\xi_2 d\xi_3 = \\ = \int_{S+S_\varepsilon} c_{ij,pq}\left[w_i(\partial u_p/\partial\xi_q) - u_i(\partial w_p/\partial\xi_q)\right] n_j dS. \tag{4.6}$$

In the limit $\varepsilon \to 0$, the contribution of the surface integral over S_ε reduces to

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} c_{ij,pq} w_i (\partial u_p / \partial \xi_q) n_j dS = 0, \tag{4.7}$$

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} c_{ij,pq} u_i (\partial w_p / \partial \xi_q) n_j dS = a_i u_i(x_1, x_2, x_3, t)\, h(t). \tag{4.8}$$

With (4.7), (4.8) and the identity

$$w_i(\partial^2 u_i / \partial t^2) - u_i(\partial^2 w_i / \partial t^2) = (\partial / \partial t) \left[w_i(\partial u_i / \partial t) - u_i(\partial w_i / \partial t) \right], \tag{4.9}$$

integration of both sides of (4.6) over all values of t gives

$$\begin{aligned} &\int_{-\infty}^{\infty} a_i u_i(x_1, x_2, x_3, t)\, h(t) dt + \\ &+ \left[\int_V \left[w_i(\partial u_i / \partial t) - u_i(\partial w_i / \partial t) \right] \rho\, d\xi_1 d\xi_2 d\xi_3 \right] \Bigg|_{t=-\infty}^{t=\infty} = \\ &= \int_{-\infty}^{\infty} dt \int_V w_i f_i d\xi_1 d\xi_2 d\xi_3 + \\ &+ \int_{-\infty}^{\infty} dt \int_S c_{ij,pq} \left[w_i(\partial u_p / \partial \xi_q) - u_i(\partial w_p / \partial \xi_q) \right] n_j dS. \end{aligned} \tag{4.10}$$

Now, h(t) is chosen such that the second term on the left-hand side of (4.10) vanishes. Introduction of the tensor operator $\underline{\underline{G}}_{ij}$, defined for any quantity φ (scalar, vector, tensor) by

$$\begin{aligned} \underline{\underline{G}}_{ij}[\varphi] &= \underline{\underline{G}}_{ij}[\varphi(\xi_1, \xi_2, \xi_3, t)] = \\ &= \frac{1}{4\pi\rho} \left\{ \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{r} \int_0^{\infty} [\varphi(\xi_1, \xi_2, \xi_3, t - r/v_P - \nu) - \right.\right. \\ &\left.\left. - \varphi(\xi_1, \xi_2, \xi_3, t - r/v_S - \nu)]\, \nu d\nu \right] + \frac{1}{v_S^2} \frac{\varphi(\xi_1, \xi_2, \xi_3, t - r/v_S)}{r} \delta_{ij} \right\}, \end{aligned} \tag{4.11}$$

enables us to write (4.10) in the form

$$\begin{aligned} a_i \int_{-\infty}^{\infty} h(t) u_i(x_1, x_2, x_3, t) dt &= a_i \int_{-\infty}^{\infty} h(t) dt \int_V \underline{\underline{G}}_{ij} [f_j]\, d\xi_1 d\xi_2 d\xi_3 + \\ &+ a_i \int_{-\infty}^{\infty} h(t) dt \int_S c_{jk,pq} \{ \underline{\underline{G}}_{ij} [\partial u_p / \partial \xi_q] + \\ &+ (\partial / \partial x_q) \underline{\underline{G}}_{ip} [u_j] \}\, n_k dS, \end{aligned} \tag{4.12}$$

where the property $\partial w_i / \partial \xi_j = -\partial w_i / \partial x_j$ has been used. Since eq. (4.12) holds for any h(t) satisfying the proper conditions as regards continuity and behaviour at infinity and since a_i is an arbitrary constant vector, it follows that

$$u_i(x_1, x_2, x_3, t) = \int_V \underline{\underline{G}}_{ij} [f_j] d\xi_1 d\xi_2 d\xi_3 + \int_S c_{jk,pq} \underline{\underline{G}}_{ij} [\partial u_p/\partial \xi_q] n_k dS +$$
$$+ (\partial/\partial x_q) \int_S c_{jk,pq} \underline{\underline{G}}_{ip} [u_j] n_k dS. \tag{4.13}$$

Eq. (4.13) is valid for any point of observation inside S. For points of observation located outside S, the function w_i, given by (4.3), is regular inside S and hence, the sum of the three expressions on the right-hand side of (4.13) vanishes identically. The three terms on the right-hand side of (4.13) can be interpreted as follows. The first term represents the displacement due to the distribution of the body forces in V with density f_i. The second term represents the displacement due to a single layer distribution on S with density $c_{jk,pq}(\partial u_p/\partial \xi_q)n_k$. The third term represents the displacement due to a double layer distribution on S with density $u_j n_k$.

In subsequent applications we frequently deal with problems where the sources start to act at t=0, while the displacement is identically zero for negative values of t. In connection with these problems it will be useful to introduce the one-sided Laplace transform with respect to time. Let

$$U_i(x_1, x_2, x_3; s) = \int_0^\infty \exp(-st) u_i(x_1, x_2, x_3, t) dt, \tag{4.14}$$

where s is a real, positive number large enough to ensure the convergence of integrals of the type (4.14). If u_i and $\partial u_i/\partial t$ are continuous functions of time, $U_i(x_1, x_2, x_3; s)$ satisfies the equation

$$c_{ij,pq} (\partial^2 U_p/\partial x_j \partial x_q) - \rho s^2 U_i = -F_i, \tag{4.15}$$

where $F_i = F_i(x_1, x_2, x_3; s)$ denotes the one-sided Laplace transform of $f_i(x_1, x_2, x_3, t)$.

The representation theorem for $U_i(x_1, x_2, x_3; s)$ is obtained by multiplying through in eq. (4.13) by exp(-st) and integrating over all positive values of t. The result is

$$U_i(x_1, x_2, x_3; s) = \int_V G_{ij} F_j d\xi_1 d\xi_2 d\xi_3 +$$
$$+ \int_S c_{jk,pq} G_{ij} (\partial U_p/\partial \xi_q) n_k dS + (\partial/\partial x_q) \int_S c_{jk,pq} G_{ip} U_j n_k dS, \tag{4.16}$$

in which

$$G_{ij}(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3; s) =$$
$$= \frac{1}{4\pi\rho} \left\{ \frac{1}{s^2} \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{\exp(-sr/v_P)}{r} - \frac{\exp(-sr/v_S)}{r} \right] + \right.$$
$$\left. + \frac{1}{v_S^2} \frac{\exp(-sr/v_S)}{r} \delta_{ij} \right\} \tag{4.17}$$

is the Green's function, for an infinite medium, associated with the differential equation (4.15).

5. TWO-DIMENSIONAL REPRESENTATION THEOREMS

The two-dimensional representation theorems associated with the wave equations (2.13) and (2.14) respectively are obtained as follows. Let C be a simple closed curve in the x_1, x_3-plane and let D be its interior. In the three-dimensional representation theorem (4.13) we take for S the closed surface consisting of the plane portions $\xi_2 = x_2 - L$, $\xi_2 = x_2 + L$, $(x_1, x_3) \in D$, together with the cylindrical part $-L \leqq \xi_2 - x_2 \leqq L$, $(x_1, x_3) \in C$, where $L > 0$. Due to the behaviour of $\underline{\underline{G}}_{ij}[\varphi]$ at large values of $|\xi_2 - x_2|$, the contribution of $\xi_2 = x_2 - L$ and $\xi_2 = x_2 + L$, $(x_1, x_3) \in D$, to the surface integrals vanishes in the limit $L \to \infty$. In this way we obtain from (4.13)

$$u_i(x_1, x_3, t) = \int_D \underline{\underline{\Gamma}}_{ij} [f_j] \, d\xi_1 d\xi_3 +$$
$$+ \int_C c_{j\varkappa,p\delta} \underline{\underline{\Gamma}}_{ij} [\partial u_p / \partial \xi_\delta] \, n_\varkappa ds + (\partial / \partial x_\delta) \int_C c_{j\varkappa,p\delta} \underline{\underline{\Gamma}}_{ip} [u_j] \, n_\varkappa ds, \tag{5.1}$$

where the operator $\underline{\underline{\Gamma}}_{ij}[\varphi]$, for any quantity $\varphi(\xi_1, \xi_3, t)$ independent of ξ_2, is defined by

$$\underline{\underline{\Gamma}}_{ij} [\varphi(\xi_1, \xi_3, t)] = \int_{-\infty}^{\infty} \underline{\underline{G}}_{ij} [\varphi(\xi_1, \xi_3, t)] \, d\xi_2. \tag{5.2}$$

It is assumed that the behaviour of $\varphi(\xi_1, \xi_3, t)$ at large negative values of t is such that the integral in (5.2) exists.

From the definition (4.11) of $\underline{\underline{G}}_{ij}[\varphi]$ we see that $\underline{\underline{\Gamma}}_{2\beta}[\varphi(\xi_1, \xi_3, t)] = 0$. Introduction of the variable of integration

$$\tau = t - \frac{1}{v_S} \{ (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 \}^{\frac{1}{2}} \tag{5.3}$$

in the expression for $\underline{\underline{\Gamma}}_{22}[\varphi]$ gives the result

$$\underline{\underline{\Gamma}}_{22} [\varphi(\xi_1, \xi_3, t)] = \frac{1}{2\pi\rho v_S^2} \int_{-\infty}^{t - r/v_S} \frac{\varphi(\xi_1, \xi_3, \tau)}{\{(t-\tau)^2 - r^2/v_S^2\}^{\frac{1}{2}}} \, d\tau, \tag{5.4}$$

where now $r = \{ (\xi_1 - x_1)^2 + (\xi_3 - x_3)^2 \}^{\frac{1}{2}} \geqq 0$. Similarly, we obtain

$$\underline{\underline{\Gamma}}_{\alpha\beta} [\varphi(\xi_1, \xi_3, t)] =$$
$$= \frac{1}{2\pi\rho} \left\{ \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left[\int_{-\infty}^{t - r/v_P} \frac{d\tau}{\{(t-\tau)^2 - r^2/v_P^2\}^{\frac{1}{2}}} \int_0^\infty \varphi(\xi_1, \xi_3, \tau - \nu) \nu d\nu - \right.\right.$$
$$\left. - \int_{-\infty}^{t - r/v_S} \frac{d\tau}{\{(t-\tau)^2 - r^2/v_S^2\}^{\frac{1}{2}}} \int_0^\infty \varphi(\xi_1, \xi_3, \tau - \nu) \nu d\nu \right] +$$

$$+ \left[\frac{1}{v_s^2} \int_{-\infty}^{t-r/v_s} \frac{\varphi(\xi_1, \xi_3, \tau)}{\{(t-\tau)^2 - r^2/v_s^2\}^{\frac{1}{2}}} \, d\tau \right] \delta_{\alpha\beta} \Big\} . \tag{5.5}$$

With these results we obtain from (5.1) the two-dimensional representation theorems

$$u_2(x_1, x_3, t) = \int_D \underline{\underline{\Gamma}}_{22} [f_2] \, d\xi_1 d\xi_3 +$$
$$+ \int_C \mu \, \underline{\underline{\Gamma}}_{22} [\partial u_2/\partial \xi_\varkappa] \, n_\varkappa ds + (\partial/\partial x_\varkappa) \int_C \mu \, \underline{\underline{\Gamma}}_{22} [u_2] \, n_\varkappa ds \tag{5.6}$$

and

$$u_\alpha(x_1, x_3, t) = \int_D \underline{\underline{\Gamma}}_{\alpha\beta} [f_\beta] \, d\xi_1 d\xi_3 +$$
$$+ \int_C c_{\beta\varkappa,\gamma\delta} \, \underline{\underline{\Gamma}}_{\alpha\beta} [\partial u_\gamma/\partial \xi_\delta] \, n_\varkappa ds + (\partial/\partial x_\delta) \int_C c_{\beta\varkappa,\gamma\delta} \, \underline{\underline{\Gamma}}_{\alpha\gamma} [u_\beta] \, n_\varkappa ds \tag{5.7}$$

Eq. (5.6) is nothing but Volterra's solution (45, 2) of the two-dimensional scalar wave equation. This result, of course, would be expected from eq. (2.13).

The Laplace transform $U_2(x_1, x_3; s)$ of $u_2(x_1, x_3, t)$ satisfies the differential equation

$$\mu(\partial^2 U_2/\partial x_\beta \, \partial x_\beta) - \rho s^2 U_2 = -F_2. \tag{5.8}$$

The representation theorem for $U_2(x_1, x_3; s)$ is obtained by multiplying through in eq. (5.6) by exp(-st) and integrating over all positive values of t. Since (46)

$$\int_{r/v}^{\infty} \exp(-st) \, (t^2 - r^2/v^2)^{-\frac{1}{2}} dt = K_0(sr/v), \quad (s>0), \tag{5.9}$$

where K_0 denotes the modified Bessel function of the second kind and order zero, the result is

$$U_2(x_1, x_3; s) = \int_D \Gamma_{22} F_2 \, d\xi_1 d\xi_3 +$$
$$+ \int_C \mu \, \Gamma_{22} (\partial U_2/\partial \xi_\varkappa) n_\varkappa ds + (\partial/\partial x_\varkappa) \int_C \mu \, \Gamma_{22} U_2 n_\varkappa ds, \tag{5.10}$$

where

$$\Gamma_{22}(x_1, x_3; \xi_1, \xi_3; s) = \frac{1}{2\pi\rho v_s^2} K_0(sr/v_s). \tag{5.11}$$

Similarly, $U_\alpha(x_1, x_3; s)$ satisfies the differential equation

$$c_{\alpha\beta,\gamma\delta}(\partial^2 U_\gamma/\partial x_\beta \partial x_\delta) - \rho s^2 U_\alpha = -F_\alpha. \tag{5.12}$$

Multiplying through in eq. (5.7) by exp(-st) and integrating over all positive values of t, we obtain the representation theorem for $U_\alpha(x_1, x_3; s)$:

$$U_\alpha(x_1, x_3; s) = \int_D \Gamma_{\alpha\beta} F_\beta d\xi_1 d\xi_3 +$$
$$+ \int_C c_{\beta\varkappa,\gamma\delta} \Gamma_{\alpha\beta} (\partial U_\gamma / \partial \xi_\delta) n_\varkappa ds + (\partial / \partial x_\delta) \int_C c_{\beta\varkappa,\gamma\delta} \Gamma_{\alpha\gamma} U_\beta n_\varkappa ds, \tag{5.13}$$

where

$$\Gamma_{\alpha\beta}(x_1, x_3; \xi_1, \xi_3; s) = \frac{1}{2\pi\rho} \left\{ \frac{1}{s^2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left[K_o(sr/v_P) - K_o(sr/v_S) \right] + \right.$$
$$\left. + \frac{1}{v_S^2} K_o(sr/v_S) \delta_{\alpha\beta} \right\}. \tag{5.14}$$

Since (5.9) can be rewritten as

$$\int_{-\infty}^{\infty} \frac{\exp[-(s/v)(\xi^2+r^2)^{\frac{1}{2}}]}{(\xi^2+r^2)^{\frac{1}{2}}} d\xi = 2 K_o(sr/v), \quad (s>0), \tag{5.15}$$

eqs. (5.11) and (5.14) are in accordance with (4.17).

Chapter III

DIFFRACTION OF ELASTIC WAVES BY A SCREEN OF VANISHING THICKNESS

6. GENERAL REMARKS ON DIFFRACTION OF ELASTIC WAVES

Consider the scattering or diffraction of an arbitrary incident wave by a "screen" Σ of finite extent and vanishing thickness. In the elastic solid, Σ is a two-dimensional region across which the displacement and the stress may be discontinuous. The shape, dimensions and location of Σ are assumed to be independent of time. Although the displacement and its first spatial derivatives are, in general, discontinuous across Σ, we still assume that at an arbitrary distance from Σ the displacement and its first and second derivatives are continuous and that Newton's equation of motion (2.1) and the stress-strain relation (2.2) are satisfied. This condition limits the number of quantities, the jumps of which can be prescribed arbitrarily. It is easy to verify that, e.g., the three components of the displacement and the three components of the traction (i.e. the force per unit area) may jump across Σ by arbitrary amounts. This is an important fact, since these are the quantities that appear in the representation theorem (4.13).

Let the incident wave u_i^i hit the screen at $t=t_o$. When $t \geq t_o$, due to the presence of the screen, a scattered wave u_i^s is generated; when $t < t_o$, $u_i^s \equiv 0$ everywhere in space. In subsequent calculations the effect of body forces will be neglected. Both the incident and the scattered wave then satisfy the homogeneous elastodynamic wave equation

$$c_{ij,pq}(\partial^2 u_p/\partial x_j \partial x_q) - \rho(\partial^2 u_i/\partial t^2) = 0. \tag{6.1}$$

With the aid of the representation theorem (4.13) the displacement u_i^s will now be expressed in terms of the jumps across Σ in the displacement and the traction. Let n_i^+ and n_i^- denote the unit vectors in the direction of the normal to Σ^+ and Σ^- respectively; Σ^+ is one face of Σ and Σ^- is the other face. The positive sense of n_i^+ and n_i^- is taken towards Σ; hence, $n_i^+ = -n_i^-$. The jump in the displacement is denoted by

$$[u_i]_-^+ = u_i^+ - u_i^- . \tag{6.2}$$

A similar notation will be used to denote the jumps in the first derivatives of u_i. In order that the representation theorem can be

applied, we have to construct a closed surface on which the normal is defined everywhere. To this aim we introduce a toroid-like surface S_ε consisting of the points at a distance ε from the edge of the screen (Fig. 1). Further, let S_R be a sphere arcund the point of observation (x_1, x_2, x_3), the radius R of which is chosen such that Σ lies entirely within S_R. Application of the representation theorem (4. 13) to the domain bounded externally by S_R and internally by Σ^+, Σ^- and S_ε gives

$$\begin{aligned} u_i^s(x_1, x_2, x_3, t) = & \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ij} \left[\partial u_p^s / \partial \xi_q\right]_-^+ n_k^+ \, dS + \\ & + (\partial / \partial x_q) \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ip} \left[u_j^s\right]_-^+ n_k^+ \, dS + \\ & + \int_{S_R + S_\varepsilon} c_{jk,pq} \underline{\underline{G}}_{ij} \left[\partial u_p^s / \partial \xi_q\right] n_k \, dS + \\ & + (\partial / \partial x_q) \int_{S_R + S_\varepsilon} c_{jk,pq} \underline{\underline{G}}_{ip} \left[u_j^s\right] n_k dS. \end{aligned} \tag{6. 3}$$

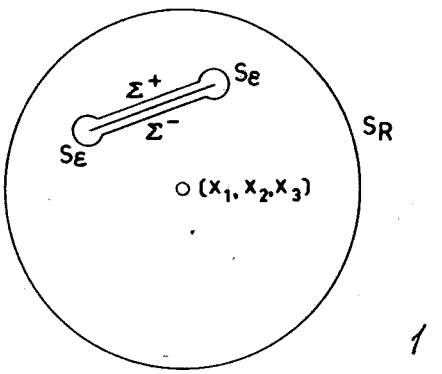

Fig. 1. Domain to which the representation theorem is applied.

By virtue of the initial condition $u_i^s \equiv 0$, when $t < t_0$, together with the finiteness of the velocities of propagation, the contribution of S_R to the surface integrals vanishes for sufficiently large values of R. Further, it is assumed that the quantities $[u_i^s]_-^+$ and $c_{jk,pq}[\partial u_p^s / \partial \xi_q]_-^+$ are such that the integrals over S_ε vanish in the limit $\varepsilon \to 0$. Moreover, since u_i^i and its derivatives are continuous across Σ we have, when $u_i = u_i^i + u_i^s$ is the total wave motion, $[u_i^s]_-^+ = [u_i]_-^+$ and $[\partial u_p^s / \partial \xi_q]_-^+ = [\partial u_p / \partial \xi_q]_-^+$. The expression (6. 3) for u_i^s then reduces to

$$\begin{aligned} u_i^s(x_1, x_2, x_3, t) = & \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ij} \left[\partial u_p / \partial \xi_q\right]_-^+ n_k^+ dS + \\ & + (\partial / \partial x_q) \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ip} \left[u_j\right]_-^+ n_k^+ \, dS. \end{aligned} \tag{6. 4}$$

The first term on the right-hand side of (6.4) is the displacement due to a single layer distribution on Σ. The second term is

the displacement due to a double layer distribution on Σ. It can be shown that the term due to the single layer distribution leads to a displacement which is continuous across Σ, but gives a traction which jumps across Σ by the assumed amount. On the other hand, the term due to the double layer distribution leads to a displacement which jumps across Σ by the assumed amount, but gives a traction which is continuous across Σ. The proofs run along the same lines as those in potential theory (see Kellogg (18)) and are given in Kupradse (25) in the case of harmonic time dependence.

The corresponding two-dimensional results follow from the representation theorems (5.6) and (5.7). A method similar to the one given above leads to the following expressions for the scattered wave

$$u_2^s(x_1, x_3, t) = \int_\Sigma \mu \underline{\underline{\Gamma}}_{22} [\partial u_2/\partial \xi_\varkappa]_-^+ n_\varkappa^+ ds + \\ + (\partial/\partial x_\varkappa) \int_\Sigma \mu \underline{\underline{\Gamma}}_{22} [u_2]_-^+ n_\varkappa^+ ds, \tag{6.5}$$

$$u_\alpha^s(x_1, x_3, t) = \int_\Sigma c_{\beta\varkappa,\gamma\delta} \underline{\underline{\Gamma}}_{\alpha\beta} [\partial u_\gamma/\partial \xi_\delta]_-^+ n_\varkappa^+ ds + \\ + (\partial/\partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta} \underline{\underline{\Gamma}}_{\alpha\gamma} [u_\beta]_-^+ n_\varkappa^+ ds, \tag{6.6}$$

where Σ now denotes the intersection of the screen with the plane x_2 = constant.

For convenience, we also list the corresponding results for the Laplace transform of the scattered wave. They are obtained in the usual way by multiplying through in the relevant equation by exp(-st) and integrating over all positive values of t.

For three-dimensional diffraction problems we obtain in this way

$$U_i^s(x_1, x_2, x_3; s) = \int_\Sigma c_{jk,pq} G_{ij} [\partial U_p/\partial \xi_q]_-^+ n_k^+ dS + \\ + (\partial/\partial x_q) \int_\Sigma c_{jk,pq} G_{ip} [U_j]_-^+ n_k^+ dS. \tag{6.7}$$

For two-dimensional diffraction problems we have

$$U_2^s(x_1, x_3; s) = \int_\Sigma \mu \Gamma_{22} [\partial U_2/\partial \xi_\varkappa]_-^+ n_\varkappa^+ ds + \\ + (\partial/\partial x_\varkappa) \int_\Sigma \mu \Gamma_{22} [U_2]_-^+ n_\varkappa^+ ds \tag{6.8}$$

and

$$U_\alpha^s(x_1, x_3; s) = \int_\Sigma c_{\beta\varkappa,\gamma\delta} \Gamma_{\alpha\beta} [\partial U_\gamma/\partial \xi_\delta]_-^+ n_\varkappa^+ ds + \\ + (\partial/\partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta} \Gamma_{\alpha\gamma} [U_\beta]_-^+ n_\varkappa^+ ds. \tag{6.9}$$

In (6.8) and (6.9), Σ denotes the intersection of the screen with the plane x_2 = constant.

We now include some remarks on the analogous problems in electromagnetic diffraction theory. For an extensive investigation

of electromagnetic representation theorems the reader is referred to Bouwkamp's review paper (7) and to the relevant chapter in Baker and Copson (3). Consider the diffraction of an electromagnetic wave by a screen of vanishing thickness. In general, all three components of the electric field and all three components of the magnetic field will be discontinuous across the screen. However, since Maxwell's equations have to be satisfied at an arbitrary distance from the screen, the amounts by which the six aforementioned quantities jump cannot be prescribed arbitrarily. It is easy to verify that if, e.g., the amounts by which the tangential components of the electric and the magnetic field jump are prescribed, the jumps in the normal components follow by virtue of Maxwell's equations. This implies that a representation theorem, in which only the tangential components of the electric and the magnetic field occur, is a suitable one. Such a representation theorem is known (Bouwkamp (8)). Physically, the surface distribution of the jumps in the tangential components of the electric and the magnetic field are equivalent to a surface distribution of magnetic and electric currents respectively. The Green's function occurring in this representation theorem is not of a point source type but of a dipole type; this ensures that the divergences of the fields thus generated vanish identically. Some authors (Heins and Silver (17)), however, prefer the use of a different representation theorem, in which the Green's function is of a point source type. Such a representation theorem is known, too, but here also the normal components of the electric and the magnetic field occur (Bouwkamp (9)). When the latter type of representation theorem is applied to the diffraction by a screen of vanishing thickness, the jumps in the normal components have to be in accordance with the prescribed jumps in the tangential components of the field quantities. Moreover, it turns out that, in order to get the same scattered field as the one determined from the surface distribution of magnetic and electric currents, certain line integrals along the edge of the diffracting screen have to be added (Bouwkamp (10)). The physical explanation of this is as follows. The surface distributions of the jumps in the normal components of the electric and the magnetic field are equivalent to a surface distribution of electric and magnetic charges respectively. By virtue of the equation of continuity (for both electric and magnetic currents and charges) the charge distributions follow from the assumed current distributions. Furthermore, the sudden termination of a current at the edge of the screen leads to a line charge along the edge of the screen (Stratton (43)). These line charges give rise to the line integrals mentioned earlier.

In elastodynamic diffraction theory the situation is different. Since there is no restriction upon the source distributions (single layer and double layer) occurring in the representation theorem for the displacement, analogous to the equation of continuity in electromagnetic theory, no additional line integrals are to be expected. In this respect we mention a recent paper by Knopoff (21). In this paper, the author derives a three-dimensional representa-

tion theorem for the acceleration vector, in which the divergence of $\vec{u}$, the tangential components of curl $\vec{u}$, the normal component of $\vec{u}$ and the tangential components of $\vec{u}$ occur. It is easy to verify that the jumps in these six quantities can be prescribed arbitrarily. Nevertheless, in applying this representation theorem to the diffraction of elastic waves by a screen of vanishing thickness, the author introduces certain line integrals along the edge of the screen. The way in which this has been done is inconsistent with the proper saltus problem formulation of the problem.

7. DIFFRACTION OF ELASTIC WAVES AS A BOUNDARY VALUE PROBLEM

When the physical properties of the diffracting screen Σ are given in terms of boundary values of the different quantities on Σ, two cases are of primary interest: (a) Σ is perfectly rigid (i.e. Σ is a domain of vanishing displacement), (b) Σ is perfectly weak (i.e. Σ is a domain of vanishing traction).

The scattered wave u_i^s arising from the diffraction of an incident wave u_i^i by a perfectly rigid screen is subject to the following conditions:

(i) u_i^s is a solution of the elastodynamic wave equation (6.1);
(ii) $u_i^s = -u_i^i$ on Σ^+ and Σ^-;
(iii) $u_i^s \equiv 0$ everywhere in space when $t<t_o$;
(iv) the kinetic and the potential energy density are integrable everywhere in space.

The scattered wave u_i^s arising from the diffraction of an incident wave u_i^i by a perfectly weak screen is subject to the following conditions:

(i) u_i^s is a solution of the elastodynamic wave equation (6.1);
(ii) $c_{ij,pq}n_j(\partial u_p^s/\partial x_q) = -c_{ij,pq}n_j(\partial u_p^i/\partial x_q)$ on Σ^+ and Σ^-;
(iii) $u_i^s \equiv 0$ everywhere in space when $t<t_o$;
(iv) the kinetic and the potential energy density are integrable everywhere in space.

It will now be shown that in both cases the scattered wave u_i^s is uniquely determined by the conditions (i) - (iv). Let u_i be the difference of two possible solutions. In (4.1) we take $w_i = \partial u_i/\partial t$. Since u_i satisfies the homogeneous elastodynamic wave equation (6.1) we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V (\partial u_i/\partial t)(\partial u_i/\partial t)\,\rho\, dx_1dx_2dx_3 +$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\int_V c_{ij,pq}(\partial u_i/\partial x_j)(\partial u_p/\partial x_q)dx_1dx_2dx_3 =$$

$$=\int_{\Sigma^+ + \Sigma^- + S_\varepsilon + S_R} c_{ij,pq}(\partial u_i/\partial t)(\partial u_p/\partial x_q)n_j dS, \qquad (7.1)$$

where S_ε is the toroid-like surface introduced in Section 6 and

S_R is a sphere of radius R around the origin. The radius of S_R is chosen such that Σ lies entirely within S_R. In (7.1), V is the domain bounded externally by S_R and internally by Σ^+, Σ^- and S_ε. The first term on the left-hand side is the time derivative of the kinetic energy; the second term on the left-hand side is the time derivative of the potential energy. By virtue of condition (ii) the surface integral over Σ^+ and Σ^- vanishes. By virtue of condition (iii), together with the finiteness of the velocities of propagation, the surface integral over S_R vanishes for sufficiently large values of R. By virtue of condition (iv) the surface integral over S_ε vanishes in the limit $\varepsilon \to 0$. Consequently, eq. (7.1) requires that the sum of the kinetic and the potential energy is a constant, independent of time, at all instants $t > t_o$. By virtue of the initial condition and the continuity of u_i and its first derivatives, this constant has the value zero. Since, further, the potential and the kinetic energy density are non-negative functions of position and time this means that u_i is a constant, independent of position and time. Since u_i vanishes at $t = t_o$, we have $u_i \equiv 0$. Hence, the uniqueness has been proved (see also Love (31)). It may be remarked that condition (iv) is necessary to ensure the existence of the integrals on the left-hand side of (7.1), especially in the neighbourhood of the edge of Σ.

To obtain the solution of the boundary value problems stated above, there are principally two different methods. The first method assumes that the technique of the separation of variables can be applied, thus reducing the problem to solving ordinary differential equations. The separation constants are then determined from the initial and boundary conditions. This method can only be applied in a limited number of geometrical configurations. The second method reduces the problem to solving certain (differential-)integral equations. This method has no restriction concerning the geometry of the diffraction problem. The way in which these (differential-)integral equations are obtained will be briefly outlined below.

In the case of diffraction by a perfectly rigid screen we obtain from (6.4) the expression

$$u_i^s(x_1, x_2, x_3, t) = \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ij} \left[\partial u_p/\partial \xi_q\right]_-^+ n_k^+ \, dS. \tag{7.2}$$

The boundary condition then leads, for points located on Σ, to the (pure) integral equation

$$\int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ij} \left[\partial u_p/\partial \xi_q\right]_-^+ n_k^+ \, dS = -u_i^i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Sigma. \tag{7.3}$$

For the analogous two-dimensional diffraction problems we have, from (6.5),

$$u_2^s(x_1, x_3, t) = \int_\Sigma \mu \, \underline{\underline{\Gamma}}_{22} \left[\partial u_2/\partial \xi_\varkappa\right]_-^+ n_\varkappa^+ ds \tag{7.4}$$

and, from (6.6),

$$u_\alpha^s(x_1, x_3, t) = \int_\Sigma c_{\beta\varkappa,\gamma\delta}\underline{\underline{\Gamma}}_{\alpha\beta}\,[\partial u_\gamma/\partial\xi_\delta]_-^+\, n_\varkappa^+ ds, \tag{7.5}$$

with the resulting (pure) integral equations

$$\int_\Sigma \mu\,\underline{\underline{\Gamma}}_{22}[\partial u_2/\partial\xi_\varkappa]_-^+\, n_\varkappa^+ ds = -u_2^i(x_1, x_3, t), \qquad (x_1, x_3)\epsilon\Sigma, \tag{7.6}$$

and

$$\int_\Sigma c_{\beta\varkappa,\gamma\delta}\underline{\underline{\Gamma}}_{\alpha\beta}\,[\partial u_\gamma/\partial\xi_\delta]_-^+\, n_\varkappa^+ ds = -u_\alpha^i(x_1, x_3, t), \qquad (x_1, x_3)\epsilon\Sigma\,. \tag{7.7}$$

In terms of the corresponding Laplace transforms with respect to time we have, from (6.7),

$$U_i^s(x_1, x_2, x_3; s) = \int_\Sigma c_{jk,pq}G_{ij}\,[\partial U_p/\partial\xi_q]_-^+\, n_k^+ dS, \tag{7.8}$$

which leads to the integral equation

$$\int_\Sigma c_{jk,pq}G_{ij}\,[\partial U_p/\partial\xi_q]_-^+\, n_k^+ dS = -U_i^i(x_1, x_2, x_3; s), \qquad (x_1, x_2, x_3)\epsilon\Sigma. \tag{7.9}$$

For the analogous two-dimensional diffraction problems we have, from (6.8),

$$U_2^s(x_1, x_3; s) = \int_\Sigma \mu\,\Gamma_{22}[\partial U_2/\partial\xi_\varkappa]_-^+\, n_\varkappa^+ ds \tag{7.10}$$

and, from (6.9),

$$U_\alpha^s(x_1, x_3; s) = \int_\Sigma c_{\beta\varkappa,\gamma\delta}\Gamma_{\alpha\beta}\,[\partial U_\gamma/\partial\xi_\delta]_-^+\, n_\varkappa^+ ds, \tag{7.11}$$

which lead to the integral equations

$$\int_\Sigma \mu\,\Gamma_{22}\,[\partial U_2/\partial\xi_\varkappa]_-^+\, n_\varkappa^+ ds = -U_2^i(x_1, x_3; s), \quad (x_1, x_3)\epsilon\Sigma, \tag{7.12}$$

and

$$\int_\Sigma c_{\beta\varkappa,\gamma\delta}\Gamma_{\alpha\beta}\,[\partial U_\gamma/\partial\xi_\delta]_-^+\, n_\varkappa^+ ds = -U_\alpha^i(x_1, x_3; s), \qquad (x_1, x_3)\epsilon\Sigma\,. \tag{7.13}$$

In the case of diffraction by a perfectly weak screen we obtain from (6.4) the expression

$$u_i^s(x_1, x_2, x_3, t) = (\partial/\partial x_q)\int_\Sigma c_{jk,pq}\underline{\underline{G}}_{ip}\,[u_j]_-^+\, n_k^+ dS. \tag{7.14}$$

The boundary condition then leads, for points located on Σ, to the differential-integral equation

$$c_{rs,ih}\, n_r^+ (\partial^2/\partial x_h \partial x_q) \int_\Sigma c_{jk,pq} \underline{\underline{G}}_{ip} [u_j]_-^+ n_k^+ dS =$$
$$= -c_{rs,ih}\, n_r^+ (\partial u_i{}^i/\partial x_h), \qquad (x_1, x_2, x_3) \epsilon \Sigma. \tag{7.15}$$

For the analogous two-dimensional diffraction problems we have, from (6.5),

$$u_2^s(x_1, x_3, t) = (\partial/\partial x_\varkappa) \int_\Sigma \mu\, \underline{\underline{\Gamma}}_{22} [u_2]_-^+ n_\varkappa^+ ds \tag{7.16}$$

and, from (6.6),

$$u_\alpha^s(x_1, x_3, t) = (\partial/\partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta} \underline{\underline{\Gamma}}_{\alpha\gamma} [u_\beta]_-^+ n_\varkappa^+ ds, \tag{7.17}$$

with the resulting differential-integral equations

$$\mu n_\lambda^+ (\partial^2/\partial x_\lambda \partial x_\varkappa) \int_\Sigma \mu\, \underline{\underline{\Gamma}}_{22} [u_2]_-^+ n_\varkappa^+ ds = -\mu n_\lambda^+ (\partial u_2{}^i/\partial x_\lambda),$$
$$(x_1, x_3) \epsilon \Sigma, \tag{7.18}$$

and

$$c_{\lambda\mu,\alpha\nu}\, n_\lambda^+ (\partial^2/\partial x_\nu \partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta}\, \underline{\underline{\Gamma}}_{\alpha\gamma} [u_\beta]_-^+ n_\varkappa^+ ds =$$
$$= -c_{\lambda\mu,\alpha\nu} n_\lambda^+ (\partial u_\alpha{}^i/\partial x_\nu), \qquad (x_1, x_3) \epsilon \Sigma. \tag{7.19}$$

In terms of the corresponding Laplace transforms with respect to time we have, from (6.7),

$$U_i^s(x_1, x_2, x_3; s) = (\partial/\partial x_q) \int_\Sigma c_{jk,pq} G_{ip} [U_j]_-^+ n_k^+ dS, \tag{7.20}$$

which leads to the differential-integral equation

$$c_{rs,ih}\, n_r^+ (\partial^2/\partial x_h \partial x_q) \int_\Sigma c_{jk,pq} G_{ip} [U_j]_-^+ n_k^+ dS =$$
$$= -c_{rs,ih}\, n_r^+ (\partial U_i{}^i/\partial x_h), \qquad (x_1, x_2, x_3) \epsilon \Sigma. \tag{7.21}$$

For the analogous two-dimensional diffraction problems we have, from (6.8),

$$U_2^s(x_1, x_3; s) = (\partial/\partial x_\varkappa) \int_\Sigma \mu\, \Gamma_{22} [U_2]_-^+ n_\varkappa^+ ds \tag{7.22}$$

and, from (6.9),

$$U_\alpha^s(x_1, x_3; s) = (\partial/\partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta}\, \Gamma_{\alpha\gamma} [U_\beta]_-^+ n_\varkappa^+ ds, \tag{7.23}$$

which expressions lead to the differential-integral equations

$$\mu\, n_\lambda^+ (\partial^2/\partial x_\lambda \partial x_\varkappa) \int_\Sigma \mu\, \Gamma_{22} [U_2]_-^+ n_\varkappa^+ ds = -\mu\, n_\lambda^+ (\partial U_2{}^i/\partial x_\lambda),$$
$$(x_1, x_3) \epsilon \Sigma, \tag{7.24}$$

and

$$c_{\lambda\mu,\alpha\nu}\, n_\lambda^+ (\partial^2/\partial x_\nu \partial x_\delta) \int_\Sigma c_{\beta\varkappa,\gamma\delta}\, \Gamma_{\alpha\gamma} [U_\beta]_-^+ n_\varkappa^+ \, ds =$$
$$= -c_{\lambda\mu,\alpha\nu}\, n_\lambda^+ (\partial U_\alpha{}^i/\partial x_\nu), \qquad (x_1, x_3) \in \Sigma . \tag{7.25}$$

The theory outlined in the present section will be applied to a few problems concerning the diffraction by a half-plane. When the relevant problem is formulated in terms of the Laplace transforms, the (differential-)integral equations are of the Wiener-Hopf type and hence, can be solved with the aid of the Wiener-Hopf technique (6, 12).

8. DIFFRACTION OF ELASTIC WAVES AS A SALTUS PROBLEM

In the optical theory of diffraction by a black screen of vanishing thickness the following assumptions concerning the wave function (due to Kirchhoff (20, 4)) are often made: on the illuminated part of the screen (in the sense of geometrical optics) the wave function and its normal derivative are equal to their corresponding values as if the screen were absent; on the dark part of the screen the wave function and its normal derivative vanish. Substitution of these assumed values in Kirchhoff's formula (19, 1) gives the well-known "Kirchhoff approximation" *. It can be shown that the wave function thus obtained does not reproduce the assumed values at the screen (Poincaré (35), Bouwkamp (11), Baker and Copson (5)) and hence, is not a solution of the diffraction problem stated as a boundary value problem. In fact, the values of the wave function and of its normal derivative cannot be prescribed simultaneously on a closed surface (the corresponding three-dimensional hypersurface in four-dimensional x_1, x_2, x_3, t - space has a space-like orientation, see M. Riesz (36)).

Kottler (23) has pointed out that if, in applying Kirchhoff's formula, it is assumed that the wave function and its normal derivative jump across the screen by given amounts, the assumed discontinuities are exactly reproduced. Consequently, from Kirchhoff's assumptions a rigourous solution of a saltus problem is obtained rather than an approximate solution of a boundary value problem. The physical properties of the screen are now specified in terms of the jumps of the wave function and of its normal derivative across the screen. If these jumps are numerically equal to the corresponding values of the incident wave at the screen, the screen is called perfectly absorbing or "black".

An analogous method will now be developed in elastodynamic diffraction theory. Let Σ be a screen of vanishing thickness. The physical properties of Σ are now specified in terms of the amounts by which the displacement and the traction jump across Σ. This implies that the densities of the single layer distribution and the double layer distribution on Σ are known functions of position and

* Usually the Kirchhoff approximation is given in the case of harmonic time dependence.

time. The scattered wave is then directly given by eqs. (6.4), (6.5) and (6.6). Similarly, the Laplace transform of the scattered wave is given by eqs. (6.7), (6.8) and (6.9).

The solution of the saltus problem is unique when the following conditions are satisfied:

(i) u_i^s is a solution of the elastodynamic wave equation;

(ii) the quantities $c_{jk,pq}[\partial u_p/\partial \xi_q]_-^+ n_k^+$ and $[u_i]_-^+$ are known, integrable, functions of position on Σ and time;

(iii) $u_i^s \equiv 0$ everywhere in space, when $t<t_o$;

(iv) when S_ε denotes the toroid-like surface consisting of the points at a distance ε from the edge of Σ,

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} c_{jk,pq}\underline{\underline{G}}_{ij} [\partial u_p^s/\partial \xi_q] n_k dS = 0$$

and

$$\lim_{\varepsilon\to 0} (\partial/\partial x_q) \int_{S_\varepsilon} c_{jk,pq}\underline{\underline{G}}_{ip} [u_j^s] n_k dS = 0.$$

For the proof we observe that the difference of two possible solutions satisfies all the requirements that were needed in the derivation of eq. (6.4). Since this difference is continuous across Σ, the resulting scattered wave vanishes identically.

Chapter IV

DIFFRACTION OF SH-WAVES BY A HALF-PLANE

9. DIFFRACTION OF A PLANE SH-PULSE BY A PERFECTLY RIGID HALF-PLANE

Let x, y, z denote right-handed cartesian coordinates in three-dimensional space. Consider the two-dimensional problem of the diffraction of a plane SH-pulse by a perfectly rigid half-plane coinciding with $z=0$, $0<x<\infty$ (Fig. 2). The incident wave $\vec{u}^i = (0, u_y^i, 0)$ is given by

$$u_y^i(x, z, t) = f[t-(x/v_S)\cos\theta_S-(z/v_S)\sin\theta_S], \tag{9.1}$$

where θ_S is the angle of incidence and $f(t) = 0$ when $t<0$. We restrict the angle of incidence to $0\leqslant\theta_S\leqslant\pi/2$; the scattered wave $\vec{u}^s = (0, u_y^s, 0)$ then satisfies, everywhere in space, the condition $\vec{u}^s = \vec{0}$ when $t<0$.

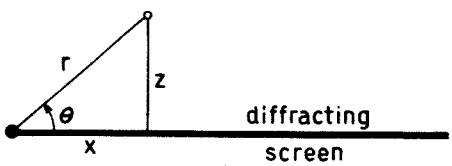

Fig. 2. Cartesian and polar coordinates used in the diffraction by a half-plane.

The Laplace transform of the scattered wave is given by

$$U_y^s(x, z; s) = \int_0^\infty \exp(-st)u_y^s(x, z, t)dt, \tag{9.2}$$

where s is a *real* positive number, large enough to ensure the convergence of integrals of the type (9.2). The Laplace transform of the incident wave is given by

$$U_y^i(x, z; s) = F(s)\exp\left[-(s/v_S)(x\cos\theta_S+z\sin\theta_S)\right], \tag{9.3}$$

where

$$F(s) = \int_0^\infty \exp(-st)f(t)dt. \tag{9.4}$$

Similarly, T_{yz} denotes the Laplace transform of τ_{yz}.

From (7.10) we obtain the following expression for the Laplace transform of the scattered wave

$$U_y^s(x,z;s) = -\int_0^\infty \Gamma_{yy}[T_{yz}]_-^+ d\xi, \tag{9.5}$$

where $[T_{yz}]_-^+ = T_{yz}(\xi,+0;s) - T_{yz}(\xi,-0;s)$. According to (5.11) we have

$$\Gamma_{yy} = \frac{1}{2\pi\mu} K_o(sR/v_S), \tag{9.6}$$

where $R = \{(x-\xi)^2+z^2\}^{\frac{1}{2}} \geqslant 0$.

It is anticipated that the diffraction problem will be solved with the aid of two-sided Laplace transforms with respect to x. Let

$$\int_0^\infty \exp(-sp\xi)\,[T_{yz}]_-^+ d\xi = F(s)A(p), \quad (-(1/v_S)\cos\theta_S < \mathrm{Re}\, p). \tag{9.7}$$

In view of subsequent calculations the transform variable has been chosen as sp rather than p; since s is a real and positive number this amounts to a change of scale in the complex p-plane. As will be seen from the solution of the problem, A(p) does not depend on s. The indicated domain of regularity of A(p) is determined from the asymptotic relation $[T_{yz}]_-^+ \sim O[\exp\{-(s\xi/v_S)\cos\theta_S\}]$ as $\xi\to\infty$. This relation follows from the physical assumption that the scattered wave predicted from the geometrical solution of the diffraction problem is predominant. Further, it can be shown (Watson (47)) that

$$\frac{1}{\pi}\int_{-\infty}^\infty \exp(-spx)K_o[(s/v_S)(x^2+z^2)^{\frac{1}{2}}]dx = \frac{\exp(-s\gamma_S|z|)}{s\gamma_S},$$

$$(-1/v_S < \mathrm{Re}\, p < 1/v_S), \tag{9.8}$$

where $\gamma_S = \gamma_S(p) = (1/v_S^2 - p^2)^{\frac{1}{2}}$. The sign of the square root has to be chosen such that $\mathrm{Re}\,\gamma_S \geqslant 0$ in the indicated strip of convergence. In view of subsequent calculations we choose $\mathrm{Re}\,\gamma_S \geqslant 0$ everywhere in the p-plane. This implies that branch cuts are introduced at $\mathrm{Im}\, p=0$, $1/v_S < |\mathrm{Re}\, p| < \infty$. Eq. (9.5) is multiplied through by $\exp(-spx)$ and integrated over all x. Application of the convolution theorem to the right-hand side gives

$$\int_{-\infty}^\infty \exp(-spx)\, U_y^s(x,z;s)dx = -\frac{F(s)}{2\mu s}\frac{\exp(-s\gamma_S|z|)}{\gamma_S}A(p). \tag{9.9}$$

In the limit z=0 we obtain

$$\int_{-\infty}^\infty \exp(-spx)U_y^s(x,0;s)\,dx = -\frac{F(s)}{2\mu s}\frac{A(p)}{\gamma_S}. \tag{9.10}$$

By virtue of the boundary condition, $U_y^s(x,0;s) = -U_y^i(x,0;s)$ when $0<x<\infty$, we have

$$\int_0^\infty \exp(-spx)U_y^s(x,0;s)dx = -\frac{F(s)}{s(p-p_o)}, \quad (-(1/v_S)\cos\theta_S < \mathrm{Re}\, p), \tag{9.11}$$

where $p_o = -(1/v_S)\cos\theta_S$. Further, let

$$\int_{-\infty}^{0} \exp(-spx)U_y^s(x, 0;s)dx = -\frac{F(s)}{s}B(p), \ (\mathrm{Re}\ p<1/v_S), \tag{9.12}$$

where the domain of regularity has been determined from the asymptotic relation $U_y^s(x, 0;s) \sim O\left[(-x)^{-\frac{1}{2}}\exp(sx/v_S)\right]$ as $x\to-\infty$. This relation follows from (9.5) by substituting in the right-hand side the asymptotic expansion of K_o. Again, the factor in front of $B(p)$ has been chosen such that $B(p)$ does not depend on s.
Eq. (9.10) reduces to

$$B(p) + \frac{1}{p-p_o} = \frac{1}{2\mu}\frac{A(p)}{\gamma_S(p)}, \ (-(1/v_S)\cos\theta_S < \mathrm{Re}\ p < 1/v_S). \tag{9.13}$$

Eq. (9.13) holds in the indicated strip of regularity common to all transforms involved. The kernel function $\gamma_S(p)$ is now written in the form (6, 12)

$$\gamma_S(p) = \gamma_S^+(p)\,\gamma_S^-(p), \tag{9.14}$$

where $\gamma_S^+(p)$ and its reciprocal are regular in the right half-plane $-1/v_S < \mathrm{Re}\ p$ and $\gamma_S^-(p)$ and its reciprocal are regular in the left half-plane $\mathrm{Re}\ p<1/v_S$. By inspection we see that this is accomplished by writing

$$\gamma_S^+(p) = (1/v_S + p)^{\frac{1}{2}}, \quad \gamma_S^-(p) = (1/v_S - p)^{\frac{1}{2}}. \tag{9.15}$$

Eq. (9.13) is now rewritten as

$$\gamma_S^-(p)\,B(p) + \frac{1}{p-p_o}\left\{\gamma_S^-(p) - \gamma_S^-(p_o)\right\} = \frac{1}{2\mu}\frac{A(p)}{\gamma_S^+(p)} - \frac{\gamma_S^-(p_o)}{p-p_o}. \tag{9.16}$$

The left-hand side of (9.16) is regular in the left half-plane $\mathrm{Re}\ p<1/v_S$; the right-hand side is regular in the right half-plane $-(1/v_S)\cos\theta_S<\mathrm{Re}\ p$. Eq. (9.16), valid in the common strip, implies that either side of (9.16) is the analytic continuation of the other side. Therefore, both sides represent one and the same entire function. Since $A(p)$ and $B(p)$ are bounded in $-(1/v_S)\cos\theta_S<\mathrm{Re}\ p$ and $\mathrm{Re}\ p<1/v_S$ respectively, this entire function is at most $O(p^{\frac{1}{2}})$ as $|p|\to\infty$. An extension of Liouville's theorem (44) shows that this is a constant. The behaviour of the right-hand side as $|p|\to\infty$ shows that this constant has the value zero. Consequently,

$$A(p) = 2\mu\frac{\gamma_S^+(p)\,\gamma_S^-(p_o)}{p-p_o}. \tag{9.17}$$

From (9.9) we deduce that the scattered wave can be written as the following Mellin inversion integral

$$U_y^s(x, z;s) = -\frac{F(s)}{4\pi\mu i}\int_{c-i\infty}^{c+i\infty}\exp(spx - s\gamma_S|z|)\frac{A(p)}{\gamma_S(p)}dp, \tag{9.18}$$

where the path of integration, Re p=c, is restricted to the strip $-(1/v_S)\cos\theta_S < c < 1/v_S$. The singularities of the integrand are: a simple pole at $p=p_o$ and branch points at $p=\pm 1/v_S$. Besides, the behaviour of the integrand as $|p|\to\infty$ shows that the conditions for the application of Jordan's lemma (48) are satisfied.

The next step towards the solution of the transient problem is to transform the integral on the right-hand side in such a way that it can be recognized as the Laplace transform of a certain function of time (Cagniard (14), Pekeris (33, 34)) *. Let $r=(x^2+z^2)^{\frac{1}{2}}$ and $\theta=\arctan(z/x)$ be polar coordinates in the plane y=constant ($0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$). By virtue of the symmetry property $U_y^s(x,z;s) = U_y^s(x,-z;s)$, it is sufficient to investigate the region $z \geqslant 0$ (or $0 \leqslant \theta \leqslant \pi$) only. The path of integration is modified such that

$$px - \gamma_S(p)\, z = -t, \tag{9.19}$$

where t, the new variable of integration, is real and positive. Solving for p we find

$$p = -(t/r)\cos\theta \pm i(t^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\sin\theta, \tag{9.20}$$

where the positive square root is taken. When $r/v_S \leqslant t < \infty$, eq. (9.20) represents a hyperbola whose point of intersection with the real axis always lies between the branch points $p = -1/v_S$ and $p = 1/v_S$ (Fig. 3). Therefore, no difficulties arise in connection with the branch cuts. On the other hand, the contribution of the pole $p = p_o$ has to be taken into account separately for values of θ

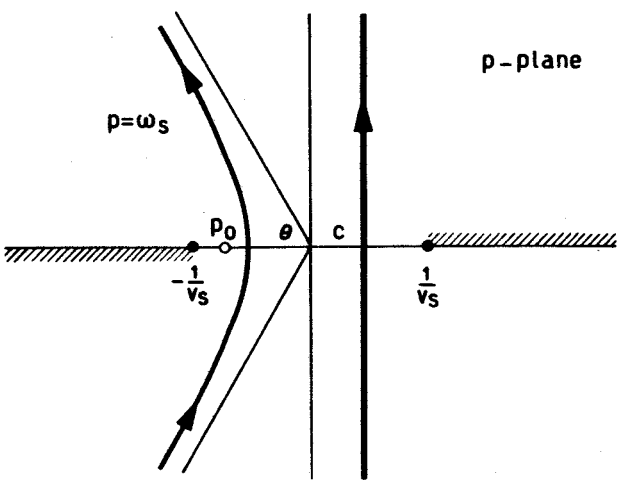

Fig. 3. Paths of integration for diffraction of a plane SH-wave by a half-plane.

* For another modification of the technique, see Sauter (37, 38).

in the region $0 \leqslant \theta < \theta_S$. It is easily verified that this contribution gives the scattered wave that would be predicted from the geometrical solution of the diffraction problem. The integral along the hyperbola is introduced as the *diffracted* wave $U_y^d(r, \theta; s)$. Since the modified path of integration is symmetric with respect to the real axis and since s and t are both real, the diffracted wave can be written in the form

$$U_y^d(r, \theta; s) = -\frac{F(s)}{2\pi\mu} \int_{r/v_S}^{\infty} \exp(-st) \operatorname{Im} \left\{ \frac{A(\omega_S)}{\gamma_S(\omega_S)} \frac{\partial \omega_S}{\partial t} \right\} dt, \qquad (9.21)$$

where

$$\omega_S = \omega_S(r, \theta, t) = -(t/r)\cos\theta + i(t^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\sin\theta. \qquad (9.22)$$

A further simplification is obtained by making use of the relation

$$\frac{1}{\gamma_S(\omega_S)} \frac{\partial \omega_S}{\partial t} = i(t^2 - r^2/v_S^2)^{-\frac{1}{2}}. \qquad (9.23)$$

Eq. (9.21) then reduces to

$$U_y^d(r, \theta; s) = -\frac{F(s)}{2\pi\mu} \int_{r/v_S}^{\infty} \exp(-st)(t^2 - r^2/v_S^2)^{-\frac{1}{2}} \operatorname{Re} \{A(\omega_S)\}\, dt. \qquad (9.24)$$

The right-hand side of (9.24) indicates that the diffracted wave is influenced by both the wave shape of the incident wave and the geometry of the diffraction problem. In order to separate the two effects, (9.24) is written in the form

$$U_y^d(r, \theta; s) = F(s)\, \Phi_y^{(S)}(r, \theta; s), \qquad (9.25)$$

where

$$\Phi_y^{(S)}(r, \theta; s) = \\ = -\frac{1}{2\pi\mu} \int_{r/v_S}^{\infty} \exp(-st)\, (t^2 - r^2/v_S^2)^{-\frac{1}{2}} \operatorname{Re} \{A(\omega_S)\}\, dt. \qquad (9.26)$$

The function $\varphi_y^{(S)}(r, \theta, t)$ of which $\Phi_y^{(S)}(r, \theta; s)$ is the Laplace transform satisfies the integral equation

$$\Phi_y^{(S)}(r, \theta; s) = \int_0^{\infty} \exp(-st)\varphi_y^{(S)}(r, \theta, t)dt, \qquad (9.27)$$

where $\Phi_y^{(S)}(r, \theta; s)$ is given by (9.26). By inspection we obtain the solution

$$\varphi_y^{(S)}(r, \theta, t) = -\frac{1}{2\pi\mu}(t^2 - r^2/v_S^2)^{-\frac{1}{2}} \operatorname{Re} \{A(\omega_S)\}\, H(t - r/v_S), \\ (0 \leqslant \theta \leqslant \pi), \qquad (9.28)$$

where H(t) denotes Heaviside's unit step function: H(t) = 0 when $t < 0$, H(t) = 1 when $t > 0$. Since the right-hand side satisfies the conditions for the application of Lerch's theorem (Doetsch (15)), the solution is unique. The diffracted wave is then given by the composition product

$$u_y^d(r,\theta,t) = \left\{ \int_{r/v_S}^{t} f(t-\tau)\varphi_y^{(S)}(r,\theta,\tau)d\tau \right\} H(t-r/v_S), \quad (0 \leqslant \theta \leqslant \pi). \tag{9.29}$$

This result shows that the diffracted wave is a cylindrical wave originating at the edge of the diffracting half-plane and whose wave front travels with the velocity v_S.

The geometrical solution of the diffraction problem (contribution from the pole $p=p_o$) is given by

$$u_y^{geom}(r,\theta,t) = \begin{cases} 0, & (0 \leqslant \theta < \theta_S), \\ f[t-(r/v_S)\cos(\theta-\theta_S)], & (\theta_S < \theta < 2\pi-\theta_S), \\ f[t-(r/v_S)\cos(\theta-\theta_S)] - f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_S < \theta \leqslant 2\pi). \end{cases} \tag{9.30}$$

The total wave motion is obtained as the superposition of the diffracted wave and the geometrical solution given in (9.30). The special values $\theta = \theta_S$ and $\theta = 2\pi - \theta_S$ have to be investigated individually. In order to get an expression which is valid at all values of θ, the definition of the geometrical solution is generalized to

$$u_y^{geom}(r,\theta,t) = \frac{1}{2}\left\{ u_y^{geom}(r,\theta-0,t) + u_y^{geom}(r,\theta+0,t) \right\}, \tag{9.31}$$

where the terms on the right-hand side are given by (9.30). In addition, the expression for the diffracted wave is generalized to

$$u_y^d(r,\theta,t) = \left\{ \lim_{\varepsilon\to 0} \int_{r/v_S+\varepsilon}^{t} f(t-\tau)\varphi_y^{(S)}(r,\theta,\tau)d\tau \right\} H(t-r/v_S), \quad (0 \leqslant \theta \leqslant \pi). \tag{9.32}$$

When $\pi \leqslant \theta \leqslant 2\pi$, the diffracted wave is obtained from the symmetry relation $u_y^d(r,\theta,t) = u_y^d(r,2\pi-\theta,t)$. For all values of θ, the total wave motion is then given by

$$u_y(r,\theta,t) = u_y^{geom}(r,\theta,t) + u_y^d(r,\theta,t). \tag{9.33}$$

The corresponding wave fronts are shown in Fig. 4.

Carrying out, in the right-hand side of (9.28), the algebraic operations, we obtain

$$\varphi_y^{(S)}(r,\theta,t) =$$

$$= \frac{v_S}{2^{\frac{1}{2}}\pi r}\left\{\frac{\sin\frac{1}{2}(\theta_S-\theta)}{v_S t/r-\cos(\theta_S-\theta)} - \frac{\sin\frac{1}{2}(\theta_S+\theta)}{v_S t/r-\cos(\theta_S+\theta)}\right\}\frac{H(t-r/v_S)}{(v_S t/r-1)^{\frac{1}{2}}},$$

$$(0\leqslant\theta\leqslant 2\pi). \quad (9.34)$$

This expression also follows from the results obtained by Sommerfeld (40), who discussed the problem of scalar diffraction by a half-plane with the aid of multi-valued wave functions.

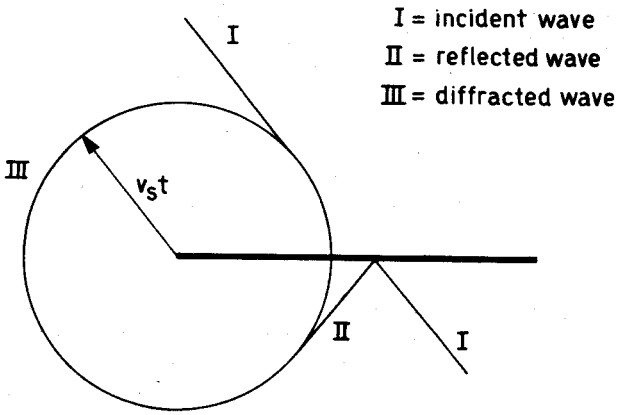

Fig. 4. Wave fronts for diffraction of a plane SH-wave by a half-plane.

10. DIFFRACTION OF A PLANE SH-PULSE BY A PERFECTLY WEAK HALF-PLANE

Consider the diffraction of a plane SH-pulse by a perfectly weak half-plane coinciding with $z=0$, $0<x<\infty$. From (7.22) we obtain the following expression for the Laplace transform of the scattered wave

$$U_y^s(x,z;s) = -\frac{\partial}{\partial z}\int_0^\infty \mu\,\Gamma_{yy}[U_y]_-^+\,d\xi, \quad (10.1)$$

where $[U_y]_-^+ = U_y(\xi,+0;s)-U_y(\xi,-0;s)$ and

$$\Gamma_{yy} = \frac{1}{2\pi\mu}K_0\left[(s/v_S)\{(x-\xi)^2+z^2\}^{\frac{1}{2}}\right]. \quad (10.2)$$

Since the analysis in the present section runs parallel to the one given in Section 9, we confine our attention to the essential steps. Again the two-sided Laplace transforms with respect to x are introduced. Let

$$\int_0^\infty \exp(-sp\xi)[U_y]_-^+ d\xi = -\frac{F(s)}{s} B(p), (-(1/v_S)\cos\theta_S < \mathrm{Re}\ p). \quad (10.3)$$

From (10.1), (10.3) and (9.8) we obtain

$$\int_{-\infty}^\infty \exp(-spx) U_y^s(x,z;s)dx = \mp \frac{F(s)}{2s} \exp(-s\gamma_S|z|)B(p), \quad (10.4)$$

where the upper sign applies when $z>0$ and the lower sign when $z<0$. Hence,

$$\int_{-\infty}^\infty \exp(-spx) T_{yz}^s(x,z;s)dx = \frac{\mu}{2} F(s)\exp(-s\gamma_S|z|)\gamma_S(p)B(p). \quad (10.5)$$

In the limit $z=0$ we obtain from the last equation

$$\int_{-\infty}^\infty \exp(-spx) T_{yz}^s(x,0;s)dx = \frac{\mu}{2} F(s)\gamma_S(p)B(p). \quad (10.6)$$

By virtue of the boundary condition we have $T_{yz}^s(x,0;s) = -T_{yz}^i(x,0;s)$ when $0<x<\infty$. Consequently,

$$\int_0^\infty \exp(-spx) T_{yz}^s(x,0;s)dx = \frac{\mu}{v_S} \frac{F(s)\sin\theta_S}{p-p_o}, \ (-(1/v_S)\cos\theta_S < \mathrm{Re}\ p), \quad (10.7)$$

where $p_o = -(1/v_S)\cos\theta_S$. Further, let

$$\int_{-\infty}^0 \exp(-spx) T_{yz}^s(x,0;s)dx = F(s)A(p), \qquad (\mathrm{Re}\ p < 1/v_S). \quad (10.8)$$

Eq. (10.6) then reduces to

$$A(p) + \frac{\mu}{v_S} \frac{\sin\theta_S}{p-p_o} = \frac{\mu}{2}\gamma_S(p)B(p), \ (-(1/v_S)\cos\theta_S < \mathrm{Re}\ p < 1/v_S). \quad (10.9)$$

Eq. (10.9) holds in the indicated strip of regularity common to all transforms involved. With the factorization of $\gamma_S(p)$, given in (9.14), eq. (10.9) is rewritten as

$$\frac{A(p)}{\gamma_S^-(p)} + \frac{\mu}{v_S} \frac{\sin\theta_S}{p-p_o}\left(\frac{1}{\gamma_S^-(p)} - \frac{1}{\gamma_S^-(p_o)}\right) = \frac{\mu}{2}\gamma_S^+(p)B(p) - \frac{\mu}{v_S}\frac{\sin\theta_S}{(p-p_o)\gamma_S^-(p_o)}. \quad (10.10)$$

Application of the usual reasoning leads to the solution

$$B(p) = \frac{2}{v_S} \frac{\sin\theta_S}{(p-p_o)\gamma_S^+(p)\gamma_S^-(p_o)}. \quad (10.11)$$

From (10.4) we deduce that the scattered wave can be written as the following Mellin inversion integral

$$U_y^s(x,z;s) = \mp \frac{F(s)}{4\pi i} \cdot \int_{c-i\infty}^{c+i\infty} \exp(spx - s\gamma_S|z|)B(p)dp, \quad (10.12)$$

where the path of integration, Re p=c, is restricted to the strip $-(1/v_S)\cos\theta_S < c < 1/v_S$. In the same way as in Section 9 the path of integration is changed into the hyperbola, given by (9.20). The integral along the hyperbola is introduced as the diffracted wave and can, when $0 \leqslant \theta \leqslant \pi$, be written in the form

$$U_y^d(r,\theta;s) = -\frac{F(s)}{2\pi}\int_{r/v_S}^{\infty} \exp(-st)(t^2-r^2/v_S^2)^{-\frac{1}{2}}\mathrm{Re}\{\gamma_S(\omega_S)B(\omega_S)\}\,dt, \tag{10.13}$$

in which $\omega_S=\omega_S(r,\theta,t)$ is given by (9.22). The right-hand side of (10.13) is of the general form

$$U_y^d(r,\theta;s) = F(s)\,\Psi_y^{(S)}(r,\theta;s), \tag{10.14}$$

where

$$\Psi_y^{(S)}(r,\theta;s) = -\frac{1}{2\pi}\int_{r/v_S}^{\infty} \exp(-st)(t^2-r^2/v_S^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_S(\omega_S)B(\omega_S)\}\,dt \tag{10.15}$$

represents the effect of the geometry of the diffraction problem. The function $\psi_y^{(S)}(r,\theta,t)$ of which $\Psi_y^{(S)}(r,\theta;s)$ is the Laplace transform satisfies the integral equation

$$\Psi_y^{(S)}(r,\theta;s) = \int_0^{\infty}\exp(-st)\psi_y^{(S)}(r,\theta,t)\,dt, \tag{10.16}$$

where $\Psi_y^{(S)}(r,\theta;s)$ is given by (10.15). By inspection we obtain the solution

$$\psi_y^{(S)}(r,\theta,t) = -\frac{1}{2\pi}(t^2-r^2/v_S^2)^{-\frac{1}{2}}\mathrm{Re}\{\gamma_S(\omega_S)B(\omega_S)\}H(t-r/v_S), \quad (0\leqslant\theta\leqslant\pi). \tag{10.17}$$

The diffracted wave is then given by the composition product

$$u_y^d(r,\theta,t) = \left\{\int_{r/v_S}^{t} f(t-\tau)\psi_y^{(S)}(r,\theta,\tau)\,d\tau\right\} H(t-r/v_S), \quad (0\leqslant\theta\leqslant\pi). \tag{10.18}$$

In this case, too, the diffracted wave is a cylindrical wave originating at the edge of the diffracting half-plane and whose wave front travels with the velocity v_S.

The geometrical solution of the diffraction problem (contribution from the pole $p=p_o$) is in this case given by

$$u_y^{geom}(r,\theta,t) =$$

$$= \begin{cases} 0, & (0\leqslant\theta<\theta_S), \\ f[t-(r/v_S)\cos(\theta-\theta_S)], & (\theta_S<\theta<2\pi-\theta_S), \\ f[t-(r/v_S)\cos(\theta-\theta_S)] + f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_S<\theta\leqslant 2\pi). \end{cases} \tag{10.19}$$

In order to get an expression which is also valid at $\theta=\theta_S$ and $\theta=2\pi-\theta_S$, the definition of the geometrical solution is generalized to

$$u_y^{geom}(r,\theta,t) = \tfrac{1}{2}\{u_y^{geom}(r,\theta-0,t) + u_y^{geom}(r,\theta+0,t)\}, \tag{10.20}$$

where the terms on the right-hand side are given by (10.19). In addition, the expression for the diffracted wave is generalized to

$$u_y^d(r,\theta,t) = \left\{\lim_{\varepsilon\to 0}\int_{r/v_S+\varepsilon}^{t} f(t-\tau)\psi_y^{(S)}(r,\theta,\tau)d\tau\right\}H(t-r/v_S), \quad (0\leqslant\theta\leqslant\pi). \tag{10.21}$$

When $\pi\leqslant\theta\leqslant 2\pi$, the diffracted wave is obtained from the symmetry relation $u_y^d(r,\theta,t)=-u_y^d(r,2\pi-\theta,t)$. For all values of θ, the total wave motion is then given by

$$u_y(r,\theta,t) = u_y^{geom}(r,\theta,t) + u_y^d(r,\theta,t). \tag{10.22}$$

The corresponding wave fronts are shown in Fig. 4, Section 9.

Carrying out, in the right-hand side of (10.17), the algebraic operations, we obtain

$$\psi_y^{(S)}(r,\theta,t) = \\ = \frac{v_S}{2^{\frac{1}{2}}\pi r}\left\{\frac{\sin\frac{1}{2}(\theta_S-\theta)}{v_S t/r-\cos(\theta_S-\theta)} + \frac{\sin\frac{1}{2}(\theta_S+\theta)}{v_S t/r-\cos(\theta_S+\theta)}\right\}\frac{H(t-r/v_S)}{(v_S t/r-1)^{\frac{1}{2}}}, \quad (0\leqslant\theta\leqslant 2\pi). \tag{10.23}$$

This expression, too, follows from the results obtained by Sommerfeld (40).

11. DIFFRACTION OF A PLANE SH-PULSE BY A HALF-PLANE AS A SALTUS PROBLEM

When an elastodynamic diffraction problem is stated as a saltus problem, we prescribe the amounts by which the displacement and the traction jump across the screen. In Section 8 we have seen that these amounts can be prescribed arbitrarily as long as they are integrable functions of position on the screen. Further, it is clear that either the jumps themselves can be prescribed as a function of time or their Laplace transforms as a function of the transform variable s. In order to give a uniform presentation, we prescribe the Laplace transforms of the jumps. In this case, the transient solution is obtained in exactly the same way as in Section 9 and Section 10, namely by a modification of Cagniard's method.

Although the jumps can be prescribed more or less arbitrarily, only a few examples are of practical interest. It has often been attempted to consider the solution of certain saltus problems as "approximate" solutions of certain boundary value problems. However, in what sense this would be an approximation is not quite

clear without further explanation. The examples we intend to give show that, when the jumps are prescribed as if the geometrical solution of the diffraction problem were the exact one, the geometrical part of the solution is reproduced; in addition, there appears a diffracted wave which is continuous across the screen.

We now proceed to give some examples in which the incident wave is the plane SH-pulse

$$u_y^i(x,z,t) = f[t-(x/v_S)\cos\theta_S - (z/v_S)\sin\theta_S], \tag{11.1}$$

where the angle of incidence θ_S is restricted to values $0 \leqslant \theta_S \leqslant \pi/2$ and $f(t)=0$ when $t<0$. The Laplace transform of the incident wave is then given by

$$U_y^i(x,z;s) = F(s)\exp[-(s/v_S)(x\cos\theta_S + z\sin\theta_S)], \tag{11.2}$$

where

$$F(s) = \int_0^\infty \exp(-st)f(t)dt. \tag{11.3}$$

In the first place the jumps are prescribed in accordance with the geometrical solution of the diffraction by a perfectly rigid half-plane, viz.

$$[U_y]_-^+ = 0, \tag{11.4}$$

$$[T_{yz}]_-^+ = 2\,s\,F(s)(\mu/v_S)\sin\theta_S \exp[-(s/v_S)\xi\cos\theta_S]. \tag{11.5}$$

Consequently, we have

$$\int_0^\infty \exp(-sp\xi)[T_{yz}]_-^+ d\xi = F(s)A(p), \quad (-(1/v_S)\cos\theta_S < \mathrm{Re}\ p), \tag{11.6}$$

where

$$A(p) = \frac{(2\mu/v_S)\sin\theta_S}{p-p_o}, \tag{11.7}$$

in which $p_o = -(1/v_S)\cos\theta_S$. From (9.9) it follows that the scattered wave is given by the Mellin inversion integral

$$^{(I)}U_y^s(x,z;s) = -\frac{F(s)}{4\pi\mu i}\int_{c-i\infty}^{c+i\infty} \exp(spx - s\gamma_S|z|)\frac{A(p)}{\gamma_S(p)}dp. \tag{11.8}$$

The path of integration, Re $p=c$, is restricted to the strip $-(1/v_S)\cos\theta_S < c < 1/v_S$. In exactly the same way as outlined in Section 9 we arrive at the expression for the diffracted wave

$$^{(I)}u_y^d(r,\theta,t) = \left\{\lim_{\varepsilon\to 0}\int_{r/v_S+\varepsilon}^t f(t-\tau)\,^{(I)}\varphi_y^{(S)}(r,\theta,\tau)d\tau\right\}H(t-r/v_S), \tag{11.9}$$

in which

$$^{(I)}\varphi_y^{(S)}(r,\theta,t) = -\frac{1}{2\pi\mu}(t^2-r^2/v_S^2)^{-\frac{1}{2}}\,\mathrm{Re}\,\{A(\omega_S)\}H(t-r/v_S),$$
$$(0\leqslant\theta\leqslant\pi), \qquad (11.10)$$

and $\omega_S=\omega_S(r,\theta,t)$ is given by (9.22). In the region $\pi\leqslant\theta\leqslant 2\pi$, the diffracted wave follows from the symmetry relation $^{(I)}u_y^d(r,\theta,t) = {}^{(I)}u_y^d(r,2\pi-\theta,t)$. The geometrical solution is given by

$$^{(I)}u_y^{\mathrm{geom}}(r,\theta,t) = \tfrac{1}{2}\left\{^{(I)}u_y^{\mathrm{geom}}(r,\theta-0,t) + {}^{(I)}u_y^{\mathrm{geom}}(r,\theta+0,t)\right\}, \qquad (11.11)$$

where the terms on the right-hand side follow from

$$^{(I)}u_y^{\mathrm{geom}}(r,\theta,t) =$$
$$=\begin{cases} 0, & (0\leqslant\theta<\theta_S),\\ f[t-(r/v_S)\cos(\theta-\theta_S)], & (\theta_S<\theta<2\pi-\theta_S),\\ f[t-(r/v_S)\cos(\theta-\theta_S)] - f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_S<\theta\leqslant 2\pi). \end{cases} \qquad (11.12)$$

The total wave motion is then the superposition of the geometrical solution and the diffracted wave. The corresponding wave fronts are shown in Fig. 4, Section 9.

Carrying out, in the right-hand side of (11.10), the algebraic operations we find

$$^{(I)}\varphi_y^{(S)}(r,\theta,t) =$$
$$= \frac{v_S}{2\pi r}\left\{\frac{\sin(\theta_S-\theta)}{v_S t/r-\cos(\theta_S-\theta)} + \frac{\sin(\theta_S+\theta)}{v_S t/r-\cos(\theta_S+\theta)}\right\}\frac{H(t-r/v_S)}{(v_S^2t^2/r^2-1)^{\frac{1}{2}}},$$
$$(0\leqslant\theta\leqslant 2\pi). \qquad (11.13)$$

In the second place the jumps are prescribed in accordance with the geometrical solution of the diffraction by a perfectly weak half-plane, viz.

$$[U_y]_-^+ = -2\,F(s)\exp[-(s/v_S)\xi\cos\theta_S], \qquad (11.14)$$

$$[T_{yz}]_-^+ = 0. \qquad (11.15)$$

Consequently, we have

$$\int_0^\infty \exp(-sp\xi)[U_y]_-^+\,d\xi = -\frac{F(s)}{s}B(p), \quad (-(1/v_S)\cos\theta_S<\mathrm{Re}\,p), \qquad (11.16)$$

where

$$B(p) = \frac{2}{p-p_o}. \qquad (11.17)$$

From (10.4) it follows that the scattered wave is given by the Mellin inversion integral

$$^{(II)}U_y^s(x,z;s) = \mp \frac{F(s)}{4\pi i}\int_{c-i\infty}^{c+i\infty} \exp(spx - s\gamma_s|z|)B(p)dp, \qquad (11.18)$$

in which the path of integration is restricted to the strip $-(1/v_s)\cos\theta_s < c < 1/v_s$. In exactly the same way as outlined in Section 10 we arrive at the expression for the diffracted wave

$$^{(II)}u_y^d(r,\theta,t) = \left\{\lim_{\varepsilon\to 0}\int_{r/v_s+\varepsilon}^{t} f(t-\tau)\,^{(II)}\psi_y^{(S)}(r,\theta,\tau)d\tau\right\}H(t-r/v_s), \qquad (11.19)$$

in which

$$^{(II)}\psi_y^{(S)}(r,\theta,t) = -\frac{1}{2\pi}(t^2-r^2/v_s^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_s(\omega_s)B(\omega_s)\}\,H(t-r/v_s), \quad (0\leqslant\theta\leqslant\pi). \qquad (11.20)$$

In the region $\pi\leqslant\theta\leqslant 2\pi$, the diffracted wave follows from the symmetry relation $^{(II)}u_y^d(r,\theta,t) = -\,^{(II)}u_y^d(r,2\pi-\theta,t)$. The geometrical solution is given by

$$^{(II)}u_y^{geom}(r,\theta,t) = \tfrac{1}{2}\left\{^{(II)}u_y^{geom}(r,\theta-0,t) + {}^{(II)}u_y^{geom}(r,\theta+0,t)\right\}, \qquad (11.21)$$

where the terms on the right-hand side follow from

$$^{(II)}u_y^{geom}(r,\theta,t) = \begin{cases} 0, & (0\leqslant\theta<\theta_s), \\ f[t-(r/v_s)\cos(\theta-\theta_s)], & (\theta_s<\theta<2\pi-\theta_s), \\ f[t-(r/v_s)\cos(\theta-\theta_s)] + f[t-(r/v_s)\cos(\theta+\theta_s)], & (2\pi-\theta_s<\theta\leqslant 2\pi). \end{cases} \qquad (11.22)$$

The total wave motion is then the superposition of the geometrical solution and the diffracted wave. The corresponding wave fronts are shown in Fig. 4, Section 9.

Carrying out, in the right-hand side of (11.20), the algebraic operations, we find

$$^{(II)}\psi_y^{(S)}(r,\theta,t) = \frac{v_s}{2\pi r}\left\{\frac{\sin(\theta_s-\theta)}{v_s t/r-\cos(\theta_s-\theta)} - \frac{\sin(\theta_s+\theta)}{v_s t/r-\cos(\theta_s+\theta)}\right\}\frac{H(t-r/v_s)}{(v_s^2t^2/r^2-1)^{\frac{1}{2}}}, \quad (0\leqslant\theta\leqslant 2\pi). \qquad (11.23)$$

Finally, we consider the saltus problem where the jumps in the displacement and the traction are numerically equal to the corresponding values of the incident wave (Kirchhoff's assumptions), viz.

$$[U_y]_-^+ = -F(s)\exp[-(s/v_s)\xi\cos\theta_s], \qquad (11.24)$$

$$[T_{yz}]_-^+ = s\,F(s)(\mu/v_S)\sin\theta_S \exp[-(s/v_S)\xi\cos\theta_S]. \qquad (11.25)$$

Comparison of (11.24) and (11.25) with (11.4), (11.5), (11.14) and (11.15) shows that the total wave motion in this case is given by

$$^{(K)}u_y(r,\theta,t) = \tfrac{1}{2}\left\{^{(I)}u_y(r,\theta,t) + {}^{(II)}u_y(r,\theta,t)\right\}. \qquad (11.26)$$

This leads to a geometrical solution which is given by

$$^{(K)}u_y^{\,geom}(r,\theta,t) = \tfrac{1}{2}\left\{^{(K)}u_y^{\,geom}(r,\theta-0,t) + {}^{(K)}u_y^{\,geom}(r,\theta+0,t)\right\}, \qquad (11.27)$$

where the terms on the right-hand side follow from

$$^{(K)}u_y^{\,geom}(r,\theta,t) = \begin{cases} 0, & (0\leqslant\theta<\theta_S), \\ f[t-(r/v_S)\cos(\theta-\theta_S)], & (\theta_S<\theta\leqslant 2\pi). \end{cases} \qquad (11.28)$$

Further, we have from (11.13) and (11.23) the result

$$\tfrac{1}{2}\left\{^{(I)}\varphi_y^{(S)}(r,\theta,t) + {}^{(II)}\psi_y^{(S)}(r,\theta,t)\right\} =$$

$$= \frac{v_S}{2\pi r}\left\{\frac{\sin(\theta_S-\theta)}{v_S t/r-\cos(\theta_S-\theta)}\right\}\frac{H(t-r/v_S)}{(v_S^2t^2/r^2-1)^{\frac{1}{2}}}, \quad (0\leqslant\theta\leqslant 2\pi). \qquad (11.29)$$

The corresponding wave fronts are shown in Fig. 5.

The structure of the geometrical solution $^{(K)}u_y^{\,geom}(r,\theta,t)$ explains why the assumptions (11.24) and (11.25) are supposed to solve the problem of the diffraction by a perfectly absorbing screen (in optical terms a "black" screen).

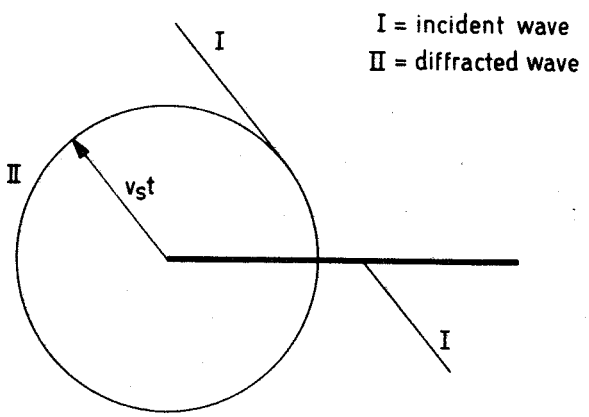

Fig. 5. Wave fronts for Kirchhoff diffraction of a plane SH-wave by a half-plane.

Chapter V

DIFFRACTION OF P-WAVES BY A HALF-PLANE

12. DIFFRACTION OF A PLANE P-PULSE BY A PERFECTLY RIGID HALF-PLANE

The present section deals with the two-dimensional problem of the diffraction of a plane compressional pulse by a perfectly rigid half-plane coinciding with $z=0$, $0<x<\infty$. The incident wave is represented by

$$u_x^i(x,z,t) = \cos\theta_P\, f[t-(x/v_P)\cos\theta_P - (z/v_P)\sin\theta_P], \tag{12.1}$$

$$u_z^i(x,z,t) = \sin\theta_P\, f[t-(x/v_P)\cos\theta_P - (z/v_P)\sin\theta_P], \tag{12.2}$$

where θ_P is the angle of incidence $(0\leqslant\theta_P\leqslant\pi/2)$ and $f(t)=0$ when $t<0$. For the corresponding Laplace transforms with respect to time we obtain

$$U_x^i(x,z;s) = F(s)\cos\theta_P \exp[-(s/v_P)(x\cos\theta_P + z\sin\theta_P)], \tag{12.3}$$

$$U_z^i(x,z;s) = F(s)\sin\theta_P \exp[-(s/v_P)(x\cos\theta_P + z\sin\theta_P)], \tag{12.4}$$

where

$$F(s) = \int_0^\infty \exp(-st)f(t)dt. \tag{12.5}$$

Similarly, U_x, U_z, T_{xz} and T_{zz} denote the Laplace transforms of u_x, u_z, τ_{xz} and τ_{zz} respectively. Eq. (7.11) leads to the following expressions for the Laplace transforms of the components of the scattered wave

$$U_x^s(x,z;s) = -\int_0^\infty \Gamma_{xx}[T_{xz}]_-^+ d\xi - \int_0^\infty \Gamma_{xz}[T_{zz}]_-^+ d\xi, \tag{12.6}$$

$$U_z^s(x,z;s) = -\int_0^\infty \Gamma_{zx}[T_{xz}]_-^+ d\xi - \int_0^\infty \Gamma_{zz}[T_{zz}]_-^+ d\xi, \tag{12.7}$$

in which $[T_{xz}]_-^+ = T_{xz}(\xi,+0;s) - T_{xz}(\xi,-0;s)$, $[T_{zz}]_-^+ = T_{zz}(\xi,+0;s) - T_{zz}(\xi,-0;s)$ and

$$\Gamma_{xx} = \frac{1}{2\pi\rho s^2}\left\{\frac{\partial^2}{\partial x^2}K_0(sR/v_P) + \frac{\partial^2}{\partial z^2}K_0(sR/v_S)\right\}, \tag{12.8}$$

$$\Gamma_{xz} = \Gamma_{zx} = \frac{1}{2\pi\rho s^2}\left\{\frac{\partial^2}{\partial x\partial z}K_o(sR/v_P) - \frac{\partial^2}{\partial x\partial z}K_o(sR/v_S)\right\}, \tag{12.9}$$

$$\Gamma_{zz} = \frac{1}{2\pi\rho s^2}\left\{\frac{\partial^2}{\partial z^2}K_o(sR/v_P) + \frac{\partial^2}{\partial x^2}K_o(sR/v_S)\right\}, \tag{12.10}$$

with $R = [(x-\xi)^2+z^2]^{\frac{1}{2}} \geqslant 0$. The right-hand sides of (12.8), (12.9) and (12.10) follow from (5.14) together with the equation

$$(\partial^2/\partial x^2+\partial^2/\partial z^2-s^2/v_S^2)K_o(sR/v_S) = 0. \tag{12.11}$$

In the same way as in the preceding sections we introduce the two-sided Laplace transforms with respect to x. Let

$$\int_0^\infty \exp(-sp\xi)[T_{xz}]_-^+ d\xi = F(s)A(p), \quad (-(1/v_P)\cos\theta_P < \mathrm{Re}\, p), \tag{12.12}$$

$$\int_0^\infty \exp(-sp\xi)[T_{zz}]_-^+ d\xi = F(s)B(p), \quad (-(1/v_P)\cos\theta_P < \mathrm{Re}\, p). \tag{12.13}$$

The indicated domain of regularity of A(p) and B(p) follows from the assumption that the geometrical solution of the diffraction problem determines the asymptotic expansion of $[T_{xz}]_-^+$ and $[T_{zz}]_-^+$ as $\xi \to \infty$, which means $[T_{xz}]_-^+ \sim O[\exp\{-(s/v_P)\xi\cos\theta_P\}]$ and $[T_{zz}]_-^+ \sim O[\exp\{-(s/v_P)\xi\cos\theta_P\}]$. Transformation of the right-hand sides of (12.6) and (12.7) gives, with the aid of (9.8) and the convolution theorem, the result

$$\int_{-\infty}^\infty \exp(-spx)U_x^s(x,z;s)dx =$$

$$= -\frac{F(s)}{2\rho s}\left[\left\{p^2A(p) \mp p\gamma_P B(p)\right\}\frac{\exp(-s\gamma_P|z|)}{\gamma_P} + \right.$$

$$\left. + \left\{\gamma_S^2 A(p) \pm p\gamma_S B(p)\right\}\frac{\exp(-s\gamma_S|z|)}{\gamma_S}\right], \tag{12.14}$$

$$\int_{-\infty}^\infty \exp(-spx)U_z^s(x,z;s)dx =$$

$$= -\frac{F(s)}{2\rho s}\left[\left\{\mp p\gamma_P A(p) + \gamma_P^2 B(p)\right\}\frac{\exp(-s\gamma_P|z|)}{\gamma_P} + \right.$$

$$\left. + \left\{\pm p\gamma_S A(p) + p^2 B(p)\right\}\frac{\exp(-s\gamma_S|z|)}{\gamma_S}\right], \tag{12.15}$$

where

$$\gamma_P = \gamma_P(p) = (1/v_P^2 - p^2)^{\frac{1}{2}} \tag{12.16}$$

and

$$\gamma_S = \gamma_S(p) = (1/v_S^2 - p^2)^{\frac{1}{2}}. \tag{12.17}$$

The sign of the square roots has to be taken such that $\mathrm{Re}\,\gamma_P \geqslant 0$ and $\mathrm{Re}\,\gamma_S \geqslant 0$. The upper sign in (12.14) and (12.15) applies when $z>0$ and the lower sign when $z<0$. In the limit $z=0$ these equations reduce to

$$\int_{-\infty}^{\infty} \exp(-spx)U_x^s(x,0;s)dx = -\frac{F(s)}{2\rho s}(p^2+\gamma_P\gamma_S)\frac{A(p)}{\gamma_P}, \qquad (12.18)$$

$$\int_{-\infty}^{\infty} \exp(-spx)U_z^s(x,0;s)dx = -\frac{F(s)}{2\rho s}(p^2+\gamma_P\gamma_S)\frac{B(p)}{\gamma_S}. \qquad (12.19)$$

By virtue of the boundary conditions we have $U_x^s(x,0;s) = -U_x^i(x,0;s)$ and $U_z^s(x,0;s) = -U_z^i(x,0;s)$ when $0<x<\infty$; hence

$$\int_0^{\infty} \exp(-spx)U_x^s(x,0;s)dx = -\frac{F(s)\cos\theta_P}{s(p-p_o)},$$

$$(-(1/v_P)\cos\theta_P < \mathrm{Re}\,p), \qquad (12.20)$$

$$\int_0^{\infty} \exp(-spx)U_z^s(x,0;s)dx = -\frac{F(s)\sin\theta_P}{s(p-p_o)},$$

$$(-(1/v_P)\cos\theta_P < \mathrm{Re}\,p), \qquad (12.21)$$

where $p_o = -(1/v_P)\cos\theta_P$. Further, we introduce the functions

$$\int_{-\infty}^{0} \exp(-spx)U_x^s(x,0;s)\,dx = -\frac{F(s)}{s}C(p), \quad (\mathrm{Re}\,p<1/v_P), \qquad (12.22)$$

$$\int_{-\infty}^{0} \exp(-spx)U_z^s(x,0;s)dx = -\frac{F(s)}{s}D(p), \quad (\mathrm{Re}\,p<1/v_P). \qquad (12.23)$$

The indicated domain of regularity follows from the asymptotic behaviour of the right-hand sides of (12.6) and (12.7) as $x\to-\infty$ and $z=0$. Substitution of (12.20) - (12.23) in (12.18) and (12.19), followed by division by the common factor $F(s)/s$, gives

$$C(p) + \frac{\cos\theta_P}{p-p_o} = \frac{1}{4\rho}\left(\frac{1}{v_S^2}+\frac{1}{v_P^2}\right)K(p)\frac{A(p)}{\gamma_P(p)},$$

$$(-(1/v_P)\cos\theta_P < \mathrm{Re}\,p < 1/v_P), \qquad (12.24)$$

$$D(p) + \frac{\sin\theta_P}{p-p_o} = \frac{1}{4\rho}\left(\frac{1}{v_S^2}+\frac{1}{v_P^2}\right)K(p)\frac{B(p)}{\gamma_S(p)},$$

$$(-(1/v_P)\cos\theta_P < \mathrm{Re}\,p < 1/v_P), \qquad (12.25)$$

where

$$K(p) = \frac{2}{v_S^{-2}+v_P^{-2}}\left[p^2+\gamma_P(p)\gamma_S(p)\right]. \qquad (12.26)$$

The only singularities of K(p) are branch points at $p = \pm 1/v_P$ and $p = \pm 1/v_S$. Its behaviour at infinity is found to be

$$K(p) = 1 + O(p^{-2}) \text{ as } |p| \to \infty. \tag{12.27}$$

Eqs. (12.24) and (12.25) hold in the indicated strip of regularity common to all transforms involved.

In order to apply the Wiener-Hopf technique, K(p) is written in the form

$$K(p) = K^+(p)K^-(p), \tag{12.28}$$

where $K^+(p)$ and its reciprocal are regular in the right half-plane $-1/v_P < \text{Re } p$ and $K^-(p)$ and its reciprocal are regular in the left half-plane $\text{Re } p < 1/v_P$. Furthermore, we make this factorization unique by requiring

$$K^+(p) = 1 + O(p^{-1}) \text{as } |p| \to \infty \tag{12.29}$$

and

$$K^-(p) = 1 + O(p^{-1}) \text{ as } |p| \to \infty . \tag{12.30}$$

Explicit expressions for $K^+(p)$ and $K^-(p)$ are derived in Section 13. Similarly, we write

$$\gamma_P(p) = \gamma_P{}^+(p)\,\gamma_P{}^-(p), \tag{12.31}$$

where

$$\gamma_P{}^+(p) = (1/v_P + p)^{\frac{1}{2}}, \tag{12.32}$$

$$\gamma_P{}^-(p) = (1/v_P - p)^{\frac{1}{2}}. \tag{12.33}$$

It is clear that $\gamma_P{}^+(p)$ and its reciprocal are regular in the right half-plane $-1/v_P < \text{Re } p$ and that $\gamma_P{}^-(p)$ and its reciprocal are regular in the left half-plane $\text{Re } p < 1/v_P$. A similar factorization holds for $\gamma_S(p)$; it is obtained by replacing, in the relevant expressions, v_P by v_S.

Eqs. (12.24) and (12.25) are now rewritten as

$$\begin{aligned} &\frac{C(p)\,\gamma_P{}^-(p)}{K^-(p)} + \frac{\cos\theta_P}{p-p_o}\left(\frac{\gamma_P{}^-(p)}{K^-(p)} - \frac{\gamma_P{}^-(p_o)}{K^-(p_o)}\right) = \\ &= \frac{1}{4\rho}\left(\frac{1}{v_S{}^2} + \frac{1}{v_P{}^2}\right)\frac{K^+(p)}{\gamma_P{}^+(p)}\,A(p) - \frac{\gamma_P{}^-(p_o)\cos\theta_P}{(p-p_o)K^-(p_o)}, \end{aligned} \tag{12.34}$$

$$\begin{aligned} &\frac{D(p)\,\gamma_S{}^-(p)}{K^-(p)} + \frac{\sin\theta_P}{p-p_o}\left(\frac{\gamma_S{}^-(p)}{K^-(p)} - \frac{\gamma_S{}^-(p_o)}{K^-(p_o)}\right) = \\ &= \frac{1}{4\rho}\left(\frac{1}{v_S{}^2} + \frac{1}{v_P{}^2}\right)\frac{K^+(p)}{\gamma_S{}^+(p)}\,B(p) - \frac{\gamma_S{}^-(p_o)\sin\theta_P}{(p-p_o)K^-(p_o)}. \end{aligned} \tag{12.35}$$

The usual reasoning leads to the solution

$$A(p) = \frac{4\rho}{v_S^{-2}+v_P^{-2}} \frac{\gamma_P^{+}(p)\gamma_P^{-}(p_o)\cos\theta_P}{(p-p_o)K^{+}(p)K^{-}(p_o)}, \tag{12.36}$$

$$B(p) = \frac{4\rho}{v_S^{-2}+v_P^{-2}} \frac{\gamma_S^{+}(p)\gamma_S^{-}(p_o)\sin\theta_P}{(p-p_o)K^{+}(p)K^{-}(p_o)}. \tag{12.37}$$

Now that expressions for A(p) and B(p) have been obtained, we turn our attention to the determination of the transient solution of the problem. From (12.14) and (12.15) we conclude that $U_x^s(x,z;s)$ and $U_z^s(x,z;s)$ can be written as the following Mellin inversion integrals

$$U_x^s(x,z;s) = -\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty}\exp(spx)\,.$$
$$\cdot\left[\left\{p^2A(p) \mp p\,\gamma_P(p)B(p)\right\}\frac{\exp[-s\gamma_P(p)|z|]}{\gamma_P(p)} + \right.$$
$$\left. + \left\{\gamma_S^2(p)A(p) \pm p\gamma_S(p)B(p)\right\}\frac{\exp[-s\gamma_S(p)|z|]}{\gamma_S(p)}\right]dp, \tag{12.38}$$

$$U_z^s(x,z;s) = -\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty}\exp(spx)\,.$$
$$\cdot\left[\left\{\mp p\,\gamma_P(p)A(p) + \gamma_P^2(p)B(p)\right\}\frac{\exp[-s\gamma_P(p)|z|]}{\gamma_P(p)} + \right.$$
$$\left. + \left\{\pm p\gamma_S(p)A(p) + p^2B(p)\right\}\frac{\exp[-s\gamma_S(p)|z|]}{\gamma_S(p)}\right]dp, \tag{12.39}$$

where the path of integration, Re p=c, is located in the strip $-(1/v_P)\cos\theta_P < c < 1/v_P$. The integrands at the right-hand sides of (12.38) and (12.39) are singular at the simple pole $p=p_o$ and at the branch points $p=\pm 1/v_P$ and $p=\pm 1/v_S$. It has been mentioned earlier in this section that the square roots defining γ_P and γ_S have to be chosen such that $\mathrm{Re}\,\gamma_P \geqslant 0$ and $\mathrm{Re}\,\gamma_S \geqslant 0$ on the path of integration. With a view to subsequent deformations of the path of integration, we impose this condition on γ_P and γ_S everywhere in the p-plane. This implies that the integrands are made single-valued by introducing branch cuts at Im p=0, $1/v_P < |\mathrm{Re}\,p| < \infty$ and at Im p=0, $1/v_S < |\mathrm{Re}\,p| < \infty$.

The right-hand sides of (12.38) and (12.39) show that the scattered wave consists of a compressional wave and a shear wave, both having a part which is symmetrical * with respect to z=0

* A vector $(u_x, 0, u_z)$ is called symmetrical with respect to z=0 when $u_x(-z)=u_x(z)$ and $u_z(-z)=-u_z(z)$; it is called antisymmetrical when $u_x(-z)=-u_x(z)$ and $u_z(-z)=u_z(z)$.

and a part which is antisymmetrical with respect to $z=0$. Due to the complexity of the expressions involved, the different terms will be discussed separately. In each term the path of integration will be modified in such a way that the resulting expression can be recognized as the Laplace transform of a certain function of time.

In the first place we consider the compressional wave which is symmetrical with respect to $z=0$. Its components are given by

$$^{(I)}U_x^s(x,z;s) = -\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty}\exp[spx-s\gamma_P(p)|z|]\,p^2A(p)\,\frac{dp}{\gamma_P(p)}, \tag{12.40}$$

$$^{(I)}U_z^s(x,z;s) = \pm\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty}\exp[spx-s\gamma_P(p)|z|]\,p\gamma_P(p)A(p)\,\frac{dp}{\gamma_P(p)}. \tag{12.41}$$

We introduce the polar coordinates $r=(x^2+z^2)^{\frac{1}{2}}$ and $\theta=\arctan(z/x)$, $(0\leqslant r<\infty,\ 0\leqslant\theta\leqslant 2\pi)$, and confine our attention to the region $z\geqslant 0$ or $0\leqslant\theta\leqslant\pi$. The path of integration is transformed into the hyperbola

$$p = -(t/r)\cos\theta \pm i(t^2/r^2-1/v_P^2)^{\frac{1}{2}}\sin\theta, \tag{12.42}$$

with $r/v_P\leqslant t<\infty$. The contribution from additional circular arcs at infinity vanishes by virtue of Jordan's lemma (48). Since the point of intersection of the hyperbola (12.42) with the real axis always lies between $p=-1/v_P$ and $p=1/v_P$, no difficulties arise in connection with the branch cuts. In changing the path of integration we may pass the pole $p=p_o$. The contribution from the latter will be taken into account later on; the integral along the hyperbola is introduced as the *diffracted* wave. Introduction of the function

$$\omega_P = \omega_P(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2-1/v_P^2)^{\frac{1}{2}}\sin\theta, \quad (0\leqslant\theta\leqslant\pi), \tag{12.43}$$

where the square root is taken to be positive, enables us to write the diffracted wave in the form

$$^{(I)}U_x^d(r,\theta;s) = -\frac{F(s)}{2\pi\rho}\int_{r/v_P}^{\infty}\exp(-st)(t^2-r^2/v_P^2)^{-\frac{1}{2}}\ \mathrm{Re}\,\{\omega_P^2A(\omega_P)\}\,dt, \tag{12.44}$$

$$^{(I)}U_z^d(r,\theta;s) = \frac{F(s)}{2\pi\rho}\int_{r/v_P}^{\infty}\exp(-st)(t^2-r^2/v_P^2)^{-\frac{1}{2}}\ \mathrm{Re}\{\omega_P\gamma_P(\omega_P)A(\omega_P)\}\,dt, \tag{12.45}$$

where we have used the relation

$$\gamma_P(\omega_P) = (t/r)\sin\theta + i(t^2/r^2 - 1/v_P^2)^{\frac{1}{2}}\cos\theta, \quad (0 \leqslant \theta \leqslant \pi). \tag{12.46}$$

The expressions are simplified by introducing the polar components of the displacement. The result is

$$^{(I)}U_r^d(r,\theta;s) =$$

$$= \frac{F(s)}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)(1-r^2/v_P^2t^2)^{-\frac{1}{2}} \ \mathrm{Re}\{\omega_P A(\omega_P)\}\, dt, \tag{12.47}$$

$$^{(I)}U_\theta^d(r,\theta;s) = -\frac{F(s)}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)\ \mathrm{Im}\{\omega_P A(\omega_P)\}\, dt. \tag{12.48}$$

The right-hand sides of these expressions are of the form

$$^{(I)}U_r^d(r,\theta;s) = F(s)\,\Phi_r^{(P)}(r,\theta;s), \tag{12.49}$$

$$^{(I)}U_\theta^d(r,\theta;s) = F(s)\,\Phi_\theta^{(P)}(r,\theta;s), \tag{12.50}$$

where

$$\Phi_r^{(P)}(r,\theta;s) =$$

$$= \frac{1}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)(1-r^2/v_P^2t^2)^{-\frac{1}{2}} \ \mathrm{Re}\{\omega_P A(\omega_P)\}\, dt, \tag{12.51}$$

$$\Phi_\theta^{(P)}(r,\theta;s) = -\frac{1}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)\ \mathrm{Im}\{\omega_P A(\omega_P)\}\, dt. \tag{12.52}$$

The functions $\varphi_r^{(P)}(r,\theta,t)$ and $\varphi_\theta^{(P)}(r,\theta,t)$ of which $\Phi_r^{(P)}(r,\theta;s)$ and $\Phi_\theta^{(P)}(r,\theta;s)$ are the respective Laplace transforms are readily obtained as

$$\varphi_r^{(P)}(r,\theta,t) = \frac{1}{2\pi\rho r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\omega_P A(\omega_P)\}H(t-r/v_P), \tag{12.53}$$

$$\varphi_\theta^{(P)}(r,\theta,t) = -\frac{1}{2\pi\rho r}\mathrm{Im}\{\omega_P A(\omega_P)\}H(t-r/v_P). \tag{12.54}$$

The polar components of the diffracted compressional wave which is symmetrical with respect to $z=0$ are then given by the composition products

$$^{(I)}u_r^d(r,\theta,t) = \left\{\int_{r/v_P}^{t} f(t-\tau)\varphi_r^{(P)}(r,\theta,\tau)d\tau\right\}H(t-r/v_P), \tag{12.55}$$

$$^{(I)}u_\theta^d(r,\theta,t) = \left\{\int_{r/v_P}^{t} f(t-\tau)\varphi_\theta^{(P)}(r,\theta,\tau)d\tau\right\}H(t-r/v_P), \tag{12.56}$$

in which $0 \leqslant \theta \leqslant \pi$.

In the second place we consider the scattered compressional wave which is antisymmetrical with respect to z=0. The Laplace transforms of its components are given by

$$^{(II)}U_x^s(x,z;s) = \pm \frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_P(p)|z|]p\, B(p)dp, \qquad (12.57)$$

$$^{(II)}U_z^s(x,z;s) = -\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_P(p)|z|]\gamma_P(p)B(p)dp. \qquad (12.58)$$

In a way similar to the one outlined above, the diffracted wave is introduced. The transformation of the path of integration enables us to write the Laplace transforms of its polar components in the form

$$^{(II)}U_r^d(r,\theta;s) = F(s)\,\Psi_r^{(P)}(r,\theta;s), \qquad (12.59)$$

$$^{(II)}U_\theta^d(r,\theta;s) = F(s)\,\Psi_\theta^{(P)}(r,\theta;s), \qquad (12.60)$$

where

$$\Psi_r^{(P)}(r,\theta;s) =$$

$$= -\frac{1}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_P(\omega_P)B(\omega_P)\}dt, \qquad (12.61)$$

$$\Psi_\theta^{(P)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{r/v_P}^{\infty} \exp(-st)\,\mathrm{Im}\{\gamma_P(\omega_P)B(\omega_P)\}dt, \qquad (12.62)$$

with $0 \leqslant \theta \leqslant \pi$. The functions $\psi_r^{(P)}(r,\theta,t)$ and $\psi_\theta^{(P)}(r,\theta,t)$ of which $\Psi_r^{(P)}(r,\theta;s)$ and $\Psi_\theta^{(P)}(r,\theta;s)$ are the respective Laplace transforms are readily obtained as

$$\psi_r^{(P)}(r,\theta,t) =$$

$$= -\frac{1}{2\pi\rho r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_P(\omega_P)B(\omega_P)\}H(t-r/v_P), \qquad (12.63)$$

$$\psi_\theta^{(P)}(r,\theta,t) = \frac{1}{2\pi\rho r}\,\mathrm{Im}\{\gamma_P(\omega_P)B(\omega_P)\}H(t-r/v_P). \qquad (12.64)$$

The polar components of the diffracted compressional wave which is antisymmetrical with respect to z=0 are then given by the composition products

$$^{(II)}u_r^d(r,\theta,t) = \left\{\int_{r/v_P}^{t} f(t-\tau)\psi_r^{(P)}(r,\theta,\tau)d\tau\right\} H(t-r/v_P), \qquad (12.65)$$

$$^{(II)}u_\theta^d(r,\theta,t) = \left\{\int_{r/v_P}^{t} f(t-\tau)\psi_\theta^{(P)}(r,\theta,\tau)d\tau\right\} H(t-r/v_P), \qquad (12.66)$$

in which $0 \leqslant \theta \leqslant \pi$.

We now proceed to give the analogous results for the scattered shear wave which is symmetrical with respect to $z=0$. Eqs. (12.38) and (12.39) give the following expressions for its components

$$^{(III)}U_x^s(x,z;s) = -\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_S(p)|z|]\ \gamma_S(p)A(p)dp, \tag{12.67}$$

$$^{(III)}U_z^s(x,z;s) = \mp\frac{F(s)}{4\pi\rho i}\int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_S(p)|z|]\ p\,A(p)\,dp. \tag{12.68}$$

In the first instance the path of integration is transformed into the hyperbola

$$p = -(t/r)\cos\theta \pm i(t^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\sin\theta, \tag{12.69}$$

with $r/v_S \leqslant t < \infty$ and $0 \leqslant \theta \leqslant \pi$. The contribution from additional circular arcs at infinity vanishes by virtue of Jordan's lemma. The point of intersection of the hyperbola (12.69) with the real axis is located at $p = -(1/v_S)\cos\theta$. Only in the region $0 \leqslant |\cos\theta| < v_S/v_P$, this point lies between $p = -1/v_P$ and $p = 1/v_P$. In this region we are free to cross the real axis. When $v_S/v_P < \cos\theta \leqslant 1$, however, the point of intersection lies to the left of $p = -1/v_P$. Since $p = -1/v_P$ is a branch point of the integrands, we are not allowed to cross the corresponding branch cut Im $p=0$, $-\infty < \mathrm{Re}\ p \leqslant -1/v_P$. Accordingly, the integral along the hyperbola (12.69) has to be supplemented by an integral around the branch cut from $p = -(1/v_S)\cos\theta - i\delta$ to $p = -(1/v_S)\cos\theta + i\delta$, where $\delta \to 0$ $(\delta > 0)$. In order to identify this branch cut integral as the Laplace transform of a certain function of time, the path of integration is taken as

$$p = -(t/r)\cos\theta + (1/v_S^2 - t^2/r^2)^{\frac{1}{2}}\sin\theta \pm i\delta,\ (\delta \to 0), \tag{12.70}$$

where $t_{PS} < t < r/v_S$ and

$$t_{PS} = (r/v_P)\cos\theta + (r/v_S)(1 - v_S^2/v_P^2)^{\frac{1}{2}}\sin\theta, \tag{12.71}$$

plus an integral along a circle with radius ε around $p = -1/v_P$ (Fig. 6). It is easily verified that the integral along the circle vanishes in the limit $\varepsilon \to 0$. An analogous situation does not arise in the region $-1 \leqslant \cos\theta < -v_S/v_P$. Since A(p) and B(p) are regular in the right half-plane $-(1/v_P)\cos\theta_P < \mathrm{Re}\ p$, the point $p = 1/v_P$ is not a branch point of the integrands of the shear waves and we may freely cross the real axis as long as the point of intersection lies to the left of $p = 1/v_S$, which is always the case.

Introduction of the functions

$$\omega_S = \omega_S(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\sin\theta \tag{12.72}$$

and

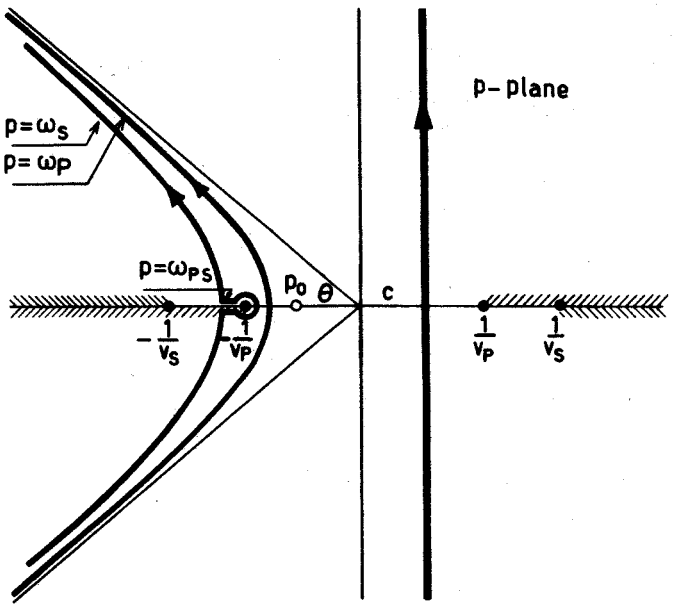

Fig. 6. Paths of integration for diffraction of a plane P-wave by a perfectly rigid half-plane.

$$\omega_{PS} = \omega_{PS}(r, \theta, t) = -(t/r)\cos\theta + (1/v_S^2 - t^2/r^2)^{\frac{1}{2}}\sin\theta + i\delta, \quad (\delta \to 0), \tag{12.73}$$

enables us to write the diffracted wave as

$$\begin{aligned} {}^{(III)}U_x^d(r, \theta; s) &= \\ &= \frac{F(s)}{2\pi\rho}\int_{t_{PS}}^{r/v_S} \exp(-st)(r^2/v_S^2 - t^2)^{-\frac{1}{2}}\, \mathrm{Im}\{\gamma_S^2(\omega_{PS})A(\omega_{PS})\}\, dt - \\ &- \frac{F(s)}{2\pi\rho}\int_{r/v_S}^{\infty} \exp(-st)(t^2 - r^2/v_S^2)^{-\frac{1}{2}}\, \mathrm{Re}\{\gamma_S^2(\omega_S)A(\omega_S)\}\, dt, \end{aligned} \tag{12.74}$$

$$\begin{aligned} {}^{(III)}U_z^d(r, \theta; s) &= \\ &= \frac{F(s)}{2\pi\rho}\int_{t_{PS}}^{r/v_S} \exp(-st)(r^2/v_S^2 - t^2)^{-\frac{1}{2}}\, \mathrm{Im}\{\omega_{PS}\gamma_S(\omega_{PS})A(\omega_{PS})\}\, dt - \\ &- \frac{F(s)}{2\pi\rho}\int_{r/v_S}^{\infty} \exp(-st)(t^2 - r^2/v_S^2)^{-\frac{1}{2}}\, \mathrm{Re}\{\omega_S\gamma_S(\omega_S)A(\omega_S)\}\, dt, \end{aligned} \tag{12.75}$$

where we have used the relations

$$\gamma_S(\omega_{PS}) = (t/r)\sin\theta + (1/v_S^2 - t^2/r^2)^{\frac{1}{2}}\cos\theta \tag{12.76}$$

and

$$\gamma_S(\omega_S) = (t/r)\sin\theta + i(t^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\cos\theta, \tag{12.77}$$

with $0 \leqslant \theta \leqslant \pi$. It must be observed that the first term on the right-hand sides of (12.74) and (12.75) is only present in the region $0 \leqslant \theta < \arccos(v_S/v_P)$. The polar components of the diffracted wave under consideration can be written in the form

$$^{(III)}U_r^d(r,\theta;s) = F(s)\,\Phi_r^{(PS)}(r,\theta;s) + F(s)\,\Phi_r^{(S)}(r,\theta;s), \tag{12.78}$$

$$^{(III)}U_\theta^d(r,\theta;s) = F(s)\,\Phi_\theta^{(PS)}(r,\theta;s) + F(s)\,\Phi_\theta^{(S)}(r,\theta;s), \tag{12.79}$$

where

$$\Phi_r^{(PS)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{t_{PS}}^{r/v_S} \exp(-st)\,\mathrm{Im}\{\gamma_S(\omega_{PS})A(\omega_{PS})\}\,dt, \tag{12.80}$$

$$\Phi_\theta^{(PS)}(r,\theta;s) = -\frac{1}{2\pi\rho r}\int_{t_{PS}}^{r/v_S} \exp(-st)(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\,\mathrm{Im}\{\gamma_S(\omega_{PS})A(\omega_{PS})\}\,dt \tag{12.81}$$

and

$$\Phi_r^{(S)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{r/v_S}^{\infty} \exp(-st)\,\mathrm{Im}\{\gamma_S(\omega_S)A(\omega_S)\}\,dt, \tag{12.82}$$

$$\Phi_\theta^{(S)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{r/v_S}^{\infty} \exp(-st)(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_S(\omega_S)A(\omega_S)\}\,dt. \tag{12.83}$$

The functions $\varphi_r^{(PS)}(r,\theta,t)$, $\varphi_\theta^{(PS)}(r,\theta,t)$, $\varphi_r^{(S)}(r,\theta,t)$ and $\varphi_\theta^{(S)}(r,\theta,t)$ of which $\Phi_r^{(PS)}(r,\theta;s)$, $\Phi_\theta^{(PS)}(r,\theta;s)$, $\Phi_r^{(S)}(r,\theta;s)$ and $\Phi_\theta^{(S)}(r,\theta;s)$ are the respective Laplace transforms are obtained as

$$\varphi_r^{(PS)}(r,\theta,t) = \frac{1}{2\pi\rho r}\,\mathrm{Im}\{\gamma_S(\omega_{PS})A(\omega_{PS})\}\,[H(t-t_{PS}) - H(t-r/v_S)], \tag{12.84}$$

$$\varphi_\theta^{(PS)}(r,\theta,t) =$$

$$= -\frac{1}{2\pi\rho r}(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\,\mathrm{Im}\{\gamma_S(\omega_{PS})A(\omega_{PS})\}\,[H(t-t_{PS}) - H(t-r/v_S)] \tag{12.85}$$

and

$$\varphi_r^{(S)}(r,\theta,t) = \frac{1}{2\pi\rho r}\,\mathrm{Im}\{\gamma_S(\omega_S)A(\omega_S)\}\,H(t-r/v_S), \tag{12.86}$$

$$\varphi_\theta^{(S)}(r,\theta,t) = \frac{1}{2\pi\rho r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\,\mathrm{Re}\{\gamma_S(\omega_S)A(\omega_S)\}\,H(t-r/v_S). \tag{12.87}$$

The polar components of the diffracted shear wave which is symmetrical with respect to z=0 are then given by the composition products

$$
\begin{aligned}
&{}^{(III)}u_r^{\,d}(r,\theta,t) = \\
&= \left\{ \int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)\varphi_r^{(PS)}(r,\theta,\tau)d\tau \right\} H(t-t_{PS}) + \\
&+ \left\{ \int_{r/v_S}^{t} f(t-\tau)\varphi_r^{(S)}(r,\theta,\tau)d\tau \right\} H(t-r/v_S),
\end{aligned}
\tag{12.88}
$$

$$
\begin{aligned}
{}^{(III)}u_\theta^{\,d}(r,\theta,t) &= \left\{ \int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)\varphi_\theta^{(PS)}(r,\theta,\tau)d\tau \right\} H(t-t_{PS}) + \\
&+ \left\{ \int_{r/v_S}^{t} f(t-\tau)\varphi_\theta^{(S)}(r,\theta,\tau)d\tau \right\} H(t-r/v_S),
\end{aligned}
\tag{12.89}
$$

in which $0 \leqslant \theta \leqslant \pi$ and where the first term on the right-hand sides of (12.88) and (12.89) is only present in the region $0 \leqslant \theta < \arccos(v_S/v_P)$.

Finally, we consider the scattered shear wave which is antisymmetrical with respect to z=0. According to (12.38) and (12.39), the Laplace transforms of its components are given by

$$
{}^{(IV)}U_x^{\,s}(x,z;s) = \mp \frac{F(s)}{4\pi\rho i} \int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_S(p)|z|] \quad p\gamma_S(p)B(p)\frac{dp}{\gamma_S(p)},
\tag{12.90}
$$

$$
{}^{(IV)}U_z^{\,s}(x,z;s) = -\frac{F(s)}{4\pi\rho i} \int_{c-i\infty}^{c+i\infty} \exp[spx - s\gamma_S(p)|z|] \quad p^2B(p)\frac{dp}{\gamma_S(p)}.
\tag{12.91}
$$

In exactly the same way as outlined above, the diffracted wave is introduced. The transformation of the path of integration enables us to write the Laplace transforms of the polar components of the corresponding diffracted wave in the form

$$
{}^{(IV)}U_r^{\,d}(r,\theta;s) = F(s)\,\Psi_r^{(PS)}(r,\theta;s) + F(s)\,\Psi_r^{(S)}(r,\theta;s),
\tag{12.92}
$$

$$
{}^{(IV)}U_\theta^{\,d}(r,\theta;s) = F(s)\,\Psi_\theta^{(PS)}(r,\theta;s) + F(s)\,\Psi_\theta^{(S)}(r,\theta;s),
\tag{12.93}
$$

where

$$
\Psi_r^{(PS)}(r,\theta;s) = \frac{1}{2\pi\rho r} \int_{t_{PS}}^{r/v_S} \exp(-st)\,\mathrm{Im}\{\omega_{PS}B(\omega_{PS})\}\,dt,
\tag{12.94}
$$

$$
\begin{aligned}
&\Psi_\theta^{(PS)}(r,\theta;s) = \\
&= -\frac{1}{2\pi\rho r} \int_{t_{PS}}^{r/v_S} \exp(-st)(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\,\mathrm{Im}\{\omega_{PS}B(\omega_{PS})\}\,dt
\end{aligned}
\tag{12.95}
$$

and

$$\Psi_r^{(S)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{r/v_S}^{\infty} \exp(-st)\,\mathrm{Im}\{\omega_S B(\omega_S)\}\,dt, \tag{12.96}$$

$$\Psi_\theta^{(S)}(r,\theta;s) = \frac{1}{2\pi\rho r}\int_{r/v_S}^{\infty} \exp(-st)(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\ \mathrm{Re}\{\omega_S B(\omega_S)\}\,dt, \tag{12.97}$$

with $0\leqslant\theta\leqslant\pi$. The functions $\psi_r^{(PS)}$, $\psi_\theta^{(PS)}$, $\psi_r^{(S)}$ and $\psi_\theta^{(S)}$ of which $\Psi_r^{(PS)}$, $\Psi_\theta^{(PS)}$, $\Psi_r^{(S)}$ and $\Psi_\theta^{(S)}$ are the respective Laplace transforms are obtained as

$$\psi_r^{(PS)}(r,\theta,t) = \frac{1}{2\pi\rho r}\,\mathrm{Im}\{\omega_{PS}B(\omega_{PS})\}\,[H(t-t_{PS})-H(t-r/v_S)] \tag{12.98}$$

$$\psi_\theta^{(PS)}(r,\theta,t) =$$

$$= -\frac{1}{2\pi\rho r}(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\,\mathrm{Im}\{\omega_{PS}B(\omega_{PS})\}\ [H(t-t_{PS})-H(t-r/v_S)] \tag{12.99}$$

and

$$\psi_r^{(S)}(r,\theta,t) = \frac{1}{2\pi\rho r}\,\mathrm{Im}\{\omega_S B(\omega_S)\}H(t-r/v_S), \tag{12.100}$$

$$\psi_\theta^{(S)}(r,\theta,t) = \frac{1}{2\pi\rho r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\mathrm{Re}\{\omega_S B(\omega_S)\}H(t-r/v_S). \tag{12.101}$$

The polar components of the diffracted shear wave which is antisymmetrical with respect to z=0 are then given by the composition products

$$^{(IV)}u_r^d(r,\theta,t) =$$

$$= \left\{\int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)\psi_r^{(PS)}(r,\theta,\tau)d\tau\right\} H(t-t_{PS}) +$$

$$+ \left\{\int_{r/v_S}^{t} f(t-\tau)\psi_r^{(S)}(r,\theta,\tau)d\tau\right\} H(t-r/v_S), \tag{12.102}$$

$$^{(IV)}u_\theta^d(r,\theta,t) =$$

$$= \left\{\int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)\psi_\theta^{(PS)}(r,\theta,\tau)d\tau\right\} H(t-t_{PS}) +$$

$$+ \left\{\int_{r/v_S}^{t} f(t-\tau)\psi_\theta^{(S)}(r,\theta,\tau)d\tau\right\} H(t-r/v_S), \tag{12.103}$$

in which $0\leqslant\theta\leqslant\pi$ and where the first term on the right-hand sides of (12.102) and (12.103) is only present in the region $0\leqslant\theta<\arccos(v_S/v_P)$.

Now that the diffracted waves have been discussed, we investigate the contribution from the pole $p=p_o$. It can be shown that the incident wave plus this contribution gives the geometrical solution of the diffraction problem. Introduction of the angle θ_S which is related to the angle of incidence θ_P through Snell's law

$$(1/v_P)\cos\theta_P = (1/v_S)\cos\theta_S, \quad (\arccos(v_S/v_P) \leqslant \theta_S \leqslant \pi/2), \tag{12.104}$$

enables us to write the result in the form

$$u_x^{geom}(r,\theta,t) = \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta < 2\pi-\theta_S), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PS}\sin\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_S < \theta < 2\pi-\theta_P), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PP}\cos\theta_P f[t-(r/v_P)\cos(\theta+\theta_P)] + \\ \quad + R_{PS}\sin\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_P < \theta \leqslant 2\pi), \end{cases} \tag{12.105}$$

$$u_z^{geom}(r,\theta,t) = \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta < 2\pi-\theta_S), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PS}\cos\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_S < \theta < 2\pi-\theta_P), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] - R_{PP}\sin\theta_P\, f[t-(r/v_P)\cos(\theta+\theta_P)] + \\ \quad + R_{PS}\cos\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_P < \theta \leqslant 2\pi), \end{cases} \tag{12.106}$$

where

$$R_{PP} = -\frac{\cos(\theta_S+\theta_P)}{\cos(\theta_S-\theta_P)} \tag{12.107}$$

and

$$R_{PS} = -\frac{\sin 2\theta_P}{\cos(\theta_S-\theta_P)} \tag{12.108}$$

are the amplitudes of the reflected P- and S-wave respectively, when a plane P-wave is incident upon a perfectly rigid plane boundary.

The values $\theta=\theta_P$, $\theta=2\pi-\theta_S$ and $\theta=2\pi-\theta_P$ require a special in-

vestigation. At these values the geometrical solution is taken as one half of the sum of the limiting values at either side of the ray under consideration. Further, the diffracted waves are taken as the values which are obtained by first substituting the relevant value of θ and afterwards approaching the lower limits of integration from above. With the expressions thus generalized, the total wave motion is everywhere the superposition of the geometrical solution and the diffracted waves.

From the results it is clear that in the first place the diffracted waves consist of a cylindrical compressional wave and a cylindrical shear wave both originating at the edge of the screen. Moreover, in the regions $0 \leqslant \theta < \arccos(v_S/v_P)$ and $2\pi - \arccos(v_S/v_P) < \theta \leqslant 2\pi$ there is a diffracted wave whose wave front is a plane travelling with the velocity v_S. The wave front of the latter wave can be considered as the envelope of the cylindrical shear waves emitted by secondary sources at the screen which have been excited by the diffracted compressional wave. It is the two-dimensional analogue of Cagniard's "onde conique"; in the German literature it is known as the "Kopfwelle". The wave fronts are shown in Fig. 7.

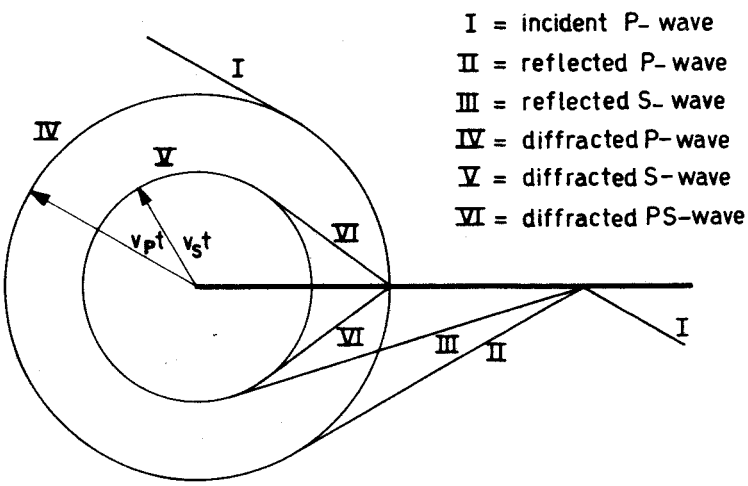

Fig. 7. Wave fronts for diffraction of a plane P-wave by a perfectly rigid half-plane.

13. FACTORIZATION OF THE KERNEL FUNCTION K(p)

The function K(p), introduced in Section 12, eq. (12.26), and given by

$$K(p) = \frac{2}{v_S^{-2}+v_P^{-2}} \left[p^2 + \gamma_P(p)\gamma_S(p) \right], \tag{13.1}$$

is nowhere zero or real and negative. Furthermore,

$$K(p) = 1 + O(p^{-2}) \quad \text{as} \quad |p| \to \infty. \tag{13.2}$$

Application of Cauchy's theorem yields

$$\log K(p) = \frac{1}{2\pi i} \oint_C \log K(w) \frac{dw}{w-p}, \tag{13.3}$$

where C is a closed contour in the w-plane, surrounding the pole w=p. In accordance with the choice of sign of the square roots in Section 12, the integrand is made single-valued by introducing branch cuts at Im w=0, $1/v_P < |\text{Re } w| < 1/v_S$ and taking the principal value of the logarithm. For the moment, it is assumed that p is not a real number such that $1/v_P < |p| < 1/v_S$. By virtue of the asymptotic behaviour as $|w| \to \infty$, the contour C may be deformed into the loops C^+ and C^- around the branch cuts (Fig. 8). The factorization is then carried out by writing

$$\log K(p) = \log K^+(p) + \log K^-(p), \tag{13.4}$$

where

$$\log K^+(p) = \frac{1}{2\pi i} \int_{C^+} \log K(w) \frac{dw}{w-p} \tag{13.5}$$

and

$$\log K^-(p) = \frac{1}{2\pi i} \int_{C^-} \log K(w) \frac{dw}{w-p}. \tag{13.6}$$

The right-hand side of (13.5) can be transformed into the real integral

$$\log K^+(p) = -\frac{1}{\pi} \int_{1/v_P}^{1/v_S} \arctan\left[\frac{(w^2 - 1/v_P^2)^{\frac{1}{2}} (1/v_S^2 - w^2)^{\frac{1}{2}}}{w^2}\right] \frac{dw}{w+p}. \tag{13.7}$$

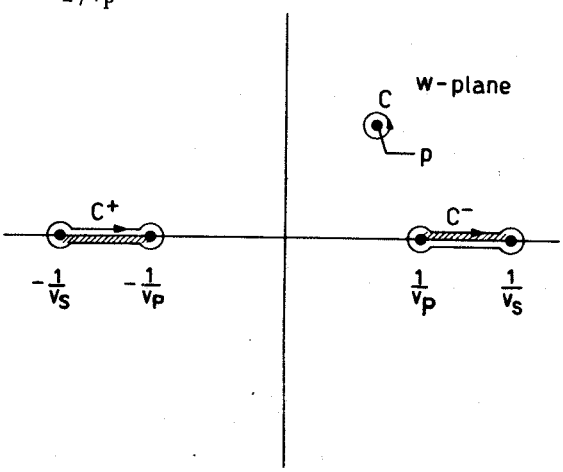

Fig. 8. Contours of integration for factorization of the kernel functions K(p) and L(p).

Further, $K^-(p)$ follows from the relation $K^-(p) = K^+(-p)$.

When p is a number just above or just below the real axis such that $-1/v_S < \mathrm{Re}\, p < -1/v_P$, we have

$$\log K^+(p) = \pm i \arctan\left[\frac{(p^2-1/v_P^2)^{\frac{1}{2}}(1/v_S^2-p^2)^{\frac{1}{2}}}{p^2}\right] -$$

$$-\frac{1}{\pi} P \int_{1/v_P}^{1/v_S} \arctan\left[\frac{(w^2-1/v_P^2)^{\frac{1}{2}}(1/v_S^2-w^2)^{\frac{1}{2}}}{w^2}\right] \frac{dw}{w+p}, \qquad (13.8)$$

where the upper sign applies when p is just above the real axis and the lower sign when p is just below the real axis. The integral on the right-hand side of (13.8) has to be taken in the sense of a Cauchy's principal value, which is indicated by the "P" in front of the integral sign. In the same way as before, $K^-(p)$ follows from the relation $K^-(p)=K^+(-p)$.

Finally, we remark that for numerical computation it may be useful to introduce the new variable of integration α through

$$w^2 = \frac{1}{2}\left(\frac{1}{v_S^2}+\frac{1}{v_P^2}\right) - \frac{1}{2}\left(\frac{1}{v_S^2}-\frac{1}{v_P^2}\right)\cos\alpha, \qquad (13.9)$$

where $0 \leqslant \alpha \leqslant \pi$. The result of this substitution is easily obtained and will not be given here.

14. DIFFRACTION OF A PLANE P-PULSE BY A PERFECTLY WEAK HALF-PLANE

Consider the two-dimensional problem of the diffraction of a plane compressional pulse by a perfectly weak half-plane coinciding with $z=0$, $0<x<\infty$. The incident wave is represented by

$$u_x^i(x,z,t) = \cos\theta_P\, f[t-(x/v_P)\cos\theta_P - (z/v_P)\sin\theta_P], \qquad (14.1)$$

$$u_z^i(x,z,t) = \sin\theta_P\, f[t-(x/v_P)\cos\theta_P - (z/v_P)\sin\theta_P], \qquad (14.2)$$

where θ_P is the angle of incidence $(0 \leqslant \theta_P \leqslant \pi/2)$ and $f(t)=0$ when $t<0$. In terms of the Laplace transforms with respect to time we have

$$U_x^i(x,z;s) = F(s)\cos\theta_P \exp[-(s/v_P)(x\cos\theta_P + z\sin\theta_P)], \qquad (14.3)$$

$$U_z^i(x,z;s) = F(s)\sin\theta_P \exp[-(s/v_P)(x\cos\theta_P + z\sin\theta_P)], \qquad (14.4)$$

where

$$F(s) = \int_0^\infty \exp(-st) f(t)\,dt. \qquad (14.5)$$

From (7.23) we obtain an expression for the Laplace transforms of the components of the scattered wave; the different terms in this expression can be arranged in the following form

$$U_x^s(x,z;s) = -\frac{v_S^2}{\pi s^2}\frac{\partial^3}{\partial x^2 \partial z}\int_0^\infty K_0(sR/v_P)[U_x]_-^+ d\xi -$$
$$-\frac{v_S^2}{\pi s^2}\left(\frac{s^2}{2v_S^2}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}\int_0^\infty K_0(sR/v_P)[U_z]_-^+ d\xi -$$
$$-\frac{v_S^2}{\pi s^2}\left(\frac{s^2}{2v_S^2}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial z}\int_0^\infty K_0(sR/v_S)[U_x]_-^+ d\xi +$$
$$+\frac{v_S^2}{\pi s^2}\frac{\partial^3}{\partial x \partial z^2}\int_0^\infty K_0(sR/v_S)[U_z]_-^+ d\xi, \tag{14.6}$$

$$U_z^s(x,z;s) = -\frac{v_S^2}{\pi s^2}\frac{\partial^3}{\partial x \partial z^2}\int_0^\infty K_0(sR/v_P)[U_x]_-^+ d\xi -$$
$$-\frac{v_S^2}{\pi s^2}\left(\frac{s^2}{2v_S^2}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial z}\int_0^\infty K_0(sR/v_P)[U_z]_-^+ d\xi +$$
$$+\frac{v_S^2}{\pi s^2}\left(\frac{s^2}{2v_S^2}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}\int_0^\infty K_0(sR/v_S)[U_x]_-^+ d\xi -$$
$$-\frac{v_S^2}{\pi s^2}\frac{\partial^3}{\partial x^2 \partial z}\int_0^\infty K_0(sR/v_S)[U_z]_-^+ d\xi, \tag{14.7}$$

where $[U_x]_-^+ = U_x(\xi,+0;s)-U_x(\xi,-0;s)$, $[U_z]_-^+ = U_z(\xi,+0;s)-U_z(\xi,-0;s)$ and

$$R = \{(x-\xi)^2+z^2\}^{\frac{1}{2}} \geqslant 0. \tag{14.8}$$

Again, it is anticipated that the diffraction problem under consideration will be solved with the aid of two-sided Laplace transforms with respect to x. To this aim we introduce the functions

$$\int_0^\infty \exp(-sp\xi)[U_x]_-^+ d\xi = -\frac{F(s)}{s}C(p), \quad (-(1/v_P)\cos\theta_P < \mathrm{Re}\, p), \tag{14.9}$$

$$\int_0^\infty \exp(-sp\xi)[U_z]_-^+ d\xi = -\frac{F(s)}{s}D(p), \quad (-(1/v_P)\cos\theta_P < \mathrm{Re}\, p). \tag{14.10}$$

The indicated domain of regularity of $C(p)$ and $D(p)$ follows from the asymptotic behaviour of $[U_x]_-^+$ and $[U_z]_-^+$ as $\xi\to\infty$. The physical assumption that the scattered wave predicted from the geometrical solution of the diffraction problem is predominant leads to $[U_x]_-^+ \sim O[\exp\{-(s\xi/v_P)\cos\theta_P\}]$ and $[U_z]_-^+ \sim O[\exp\{-(s\xi/v_P)\cos\theta_P\}]$ as $\xi\to\infty$. Multiplication of (14.6) and (14.7) by $\exp(-spx)$ and integration over all x leads, with the aid of (9.8) and the convolution theorem, to

$$\int_{-\infty}^{\infty} \exp(-spx) U_x^s(x,z;s)dx =$$

$$= \frac{F(s)}{s} v_S^2 \left[\left\{ \mp p^2 \gamma_P C(p) + (1/2v_S^2 - p^2) p\, D(p) \right\} \frac{\exp(-s\gamma_P |z|)}{\gamma_P} + \right.$$

$$\left. + \left\{ \mp (1/2v_S^2 - p^2)\, \gamma_S\, C(p) - p\, \gamma_S^2 D(p) \right\} \frac{\exp(-s\gamma_S |z|)}{\gamma_S} \right], \quad (14.11)$$

$$\int_{-\infty}^{\infty} \exp(-spx) U_z^s(x,z;s)dx =$$

$$= \frac{F(s)}{s} v_S^2 \left[\left\{ p\gamma_P^2 C(p) \mp (1/2v_S^2 - p^2) \quad \gamma_P\, D(p) \right\} \frac{\exp(-s\gamma_P |z|)}{\gamma_P} + \right.$$

$$\left. + \left\{ -(1/2v_S^2 - p^2) p\, C(p) \mp p^2 \gamma_S\, D(p) \right\} \frac{\exp(-s\gamma_S |z|)}{\gamma_S} \right], \quad (14.12)$$

where $\gamma_P(p)$ and $\gamma_S(p)$ are given by (12.16) and (12.17). The upper sign in (14.11) and (14.12) applies when $z>0$ and the lower sign when $z<0$.

From the stress-strain relation we determine $T_{xz}^s(x,z;s)$ and $T_{zz}^s(x,z;s)$. For their two-sided Laplace transforms with respect to x we obtain

$$\int_{-\infty}^{\infty} \exp(-spx) T_{xz}^s(x,z;s)dx =$$

$$= 2\mu v_S^2 F(s) \left[\left\{ p^2 \gamma_P^2\, C(p) \mp (1/2v_S^2 - p^2) p\, \gamma_P\, D(p) \right\} \frac{\exp(-s\gamma_P |z|)}{\gamma_P} + \right.$$

$$\left. + \left\{ (1/2v_S^2 - p^2)^2\, C(p) \pm (1/2v_S^2 - p^2) p\gamma_S\, D(p) \right\} \frac{\exp(-s\gamma_S |z|)}{\gamma_S} \right],$$

$$(14.13)$$

$$\int_{-\infty}^{\infty} \exp(-spx) T_{zz}^s(x,z;s)dx = 2\mu v_S^2 F(s) \cdot$$

$$\cdot \left[\left\{ \mp (1/2v_S^2 - p^2) p\gamma_P\, C(p) + (1/2v_S^2 - p^2)^2\, D(p) \right\} \frac{\exp(-s\gamma_P |z|)}{\gamma_P} + \right.$$

$$\left. + \left\{ \pm (1/2v_S^2 - p^2) p\gamma_S\, C(p) + p^2 \gamma_S^2 D(p) \right\} \frac{\exp(-s\gamma_S |z|)}{\gamma_S} \right], \quad (14.14)$$

where the upper sign applies when $z>0$ and the lower sign when $z<0$. In the limit $z=0$ these equations reduce to

$$\int_{-\infty}^{\infty} \exp(-spx) T_{xz}^s(x,0;s)dx =$$

$$= 2\mu v_S^2 F(s) \quad \left[(1/2v_S^2 - p^2)^2 + p^2 \gamma_P \gamma_S \right] \frac{C(p)}{\gamma_S}, \quad (14.15)$$

$$\int_{-\infty}^{\infty} \exp(-spx) T_{zz}^{s}(x, 0; s) dx =$$

$$= 2\mu v_S^2 F(s) \quad [(1/2v_S^2 - p^2)^2 + p^2 \gamma_P \gamma_S] \frac{D(p)}{\gamma_P}. \tag{14.16}$$

By virtue of the boundary conditions we have $T_{xz}^{s}(x, 0; s) = -T_{xz}^{i}(x, 0; s)$ and $T_{zz}^{s}(x, 0; s) = -T_{zz}^{i}(x, 0; s)$ when $0<x<\infty$; hence,

$$\int_{0}^{\infty} \exp(-spx) T_{xz}^{s}(x, 0; s) dx = \frac{\mu \sin 2\theta_P F(s)}{v_P(p-p_o)},$$

$$(-(1/v_P)\cos\theta_P < \operatorname{Re} p), \tag{14.17}$$

$$\int_{0}^{\infty} \exp(-spx) T_{zz}^{s}(x, 0; s) dx = -\frac{(\lambda+2\mu)\cos 2\theta_S F(s)}{v_P(p-p_o)},$$

$$(-(1/v_P)\cos\theta_P < \operatorname{Re} p), \tag{14.18}$$

where $p_o = -(1/v_P)\cos\theta_P$ and θ_S follows from Snell's law

$$(1/v_P)\cos\theta_P = (1/v_S)\cos\theta_S, \quad (0<\arccos(v_S/v_P)\leqslant\theta_S\leqslant\pi/2). \tag{14.19}$$

Further, we introduce the functions

$$\int_{-\infty}^{0} \exp(-spx) T_{xz}^{s}(x, 0; s) dx = F(s)\, A(p), \quad (\operatorname{Re} p<1/v_P), \tag{14.20}$$

$$\int_{-\infty}^{0} \exp(-spx) T_{zz}^{s}(x, 0; s) dx = F(s)\, B(p), \quad (\operatorname{Re} p<1/v_P). \tag{14.21}$$

The indicated domain of regularity follows from the asymptotic behaviour of $T_{xz}^{s}(x, 0; s)$ and $T_{zz}^{s}(x, 0; s)$ as $x\to-\infty$; this behaviour can be determined from (14.6) and (14.7) by substituting the asymptotic expansion of K_o and using the stress-strain relation. Substitution of (14.17), (14.18), (14.20) and (14.21) in (14.15) and (14.16) leads to

$$A(p) + \frac{\mu \sin 2\theta_P}{v_P(p-p_o)} = \mu v_S^2 \left(\frac{1}{v_S^2} - \frac{1}{v_P^2}\right)\left(\frac{1}{v_R^2} - p^2\right) L(p) \frac{C(p)}{\gamma_S(p)},$$

$$(-(1/v_P)\cos\theta_P<\operatorname{Re} p<1/v_P), \tag{14.22}$$

$$B(p) - \frac{(\lambda+2\mu)\cos 2\theta_S}{v_P(p-p_o)} = \mu v_S^2 \left(\frac{1}{v_S^2} - \frac{1}{v_P^2}\right)\left(\frac{1}{v_R^2} - p^2\right) L(p) \frac{D(p)}{\gamma_P(p)},$$

$$(-(1/v_P)\cos\theta_P<\operatorname{Re} p<1/v_P), \tag{14.23}$$

where $p=1/v_R$ denotes the real and positive root of

$$(1/2v_S^2 - p^2)^2 + p^2 \gamma_P(p)\, \gamma_S(p) = 0; \tag{14.24}$$

v_R is called the Rayleigh wave velocity and is associated with surface waves along the free boundary of an elastic half-space. The kernel function L(p) is given by

$$L(p) = \frac{2}{v_S^{-2}-v_P^{-2}} \frac{(1/2v_S^2-p^2)^2+p^2\gamma_P(p)\gamma_S(p)}{(1/v_R^2-p^2)} . \tag{14.25}$$

The only singularities of L(p) are branch points at $p = \pm 1/v_P$ and $p = \pm 1/v_S$. Its behaviour at infinity is found to be

$$L(p) = 1 + O(p^{-2}) \text{ as } |p| \to \infty. \tag{14.26}$$

Eqs. (14.22) and (14.23) hold in the indicated strip of regularity common to all transforms involved.

In order to apply the Wiener-Hopf technique, L(p) is written in the form

$$L(p) = L^+(p)L^-(p), \tag{14.27}$$

where $L^+(p)$ and its reciprocal are regular in the right half-plane $-1/v_P < \text{Re } p$ and $L^-(p)$ and its reciprocal are regular in the left half-plane $\text{Re } p < 1/v_P$. Furthermore, we make this factorization unique by requiring

$$L^+(p) = 1 + O(p^{-1}) \text{ as } |p| \to \infty \tag{14.28}$$

and

$$L^-(p) = 1 + O(p^{-1}) \text{ as } |p| \to \infty. \tag{14.29}$$

Explicit expressions for $L^+(p)$ and $L^-(p)$ are derived in Section 15. Similarly, we write

$$\gamma_P(p) = \gamma_P^+(p)\, \gamma_P^-(p), \tag{14.30}$$

where $\gamma_P^+(p)$ and $\gamma_P^-(p)$ are given by (12.32) and (12.33) respectively; a similar factorization holds for $\gamma_S(p)$.

Eqs. (14.22) and (14.23) are now rewritten as

$$\frac{A(p)\,\gamma_S^-(p)}{(1/v_R-p)L^-(p)} + \frac{\mu \sin 2\theta_P}{v_P(p-p_o)} \left(\frac{\gamma_S^-(p)}{(1/v_R-p)L^-(p)} - \frac{\gamma_S^-(p_o)}{(1/v_R-p_o)L^-(p_o)}\right) =$$
$$= \mu v_S^2 \left(\frac{1}{v_S^2} - \frac{1}{v_P^2}\right)(1/v_R+p)L^+(p) \frac{C(p)}{\gamma_S^+(p)} - \frac{\mu \sin 2\theta_P\, \gamma_S^-(p_o)}{v_P(p-p_o)(1/v_R-p_o)L^-(p_o)} \tag{14.31}$$

$$\frac{B(p)\,\gamma_P^-(p)}{(1/v_R-p)L^-(p)} - \frac{(\lambda+2\mu)\cos 2\theta_S}{v_P(p-p_o)} \left(\frac{\gamma_P^-(p)}{(1/v_R-p)L^-(p)} - \frac{\gamma_P^-(p_o)}{(1/v_R-p_o)L^-(p_o)}\right) =$$
$$= \mu v_S^2 \left(\frac{1}{v_S^2} - \frac{1}{v_P^2}\right)(1/v_R+p)L^+(p) \frac{D(p)}{\gamma_P^+(p)} + \frac{(\lambda+2\mu)\cos 2\theta_S\, \gamma_P^-(p_o)}{v_P(p-p_o)(1/v_R-p_o)L^-(p_o)} \tag{14.32}$$

The usual reasoning leads to the solution

$$C(p) = \frac{v_P}{v_P^2 - v_S^2} \frac{\gamma_S^+(p)\,\gamma_S^-(p_o)\sin 2\theta_P}{(p-p_o)(1/v_R+p)(1/v_R-p_o)L^+(p)L^-(p_o)}, \tag{14.33}$$

$$D(p) = -\frac{v_P^2}{v_S^2}\frac{v_P}{v_P^2-v_S^2} \frac{\gamma_P^+(p)\,\gamma_P^-(p_o)\cos 2\theta_S}{(p-p_o)(1/v_R+p)(1/v_R-p_o)L^+(p)L^-(p_o)}. \tag{14.34}$$

Through these expressions the functions C(p) and D(p) are, in principle, determined. Next we turn our attention to the transient solution of the problem. In accordance with (14.11) and (14.12), we write the Laplace transforms of the cartesian components of the scattered wave in the form of the following Mellin inversion integrals

$$\begin{aligned} U_x^s(x,z;s) = {} & \frac{v_S^2F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp[spx - s\gamma_P(p)\,|z|]}{\gamma_P(p)} \cdot \\ & \cdot \{\mp p^2\gamma_P(p)C(p) + (1/2v_S^2 - p^2)pD(p)\}\,dp + \\ & + \frac{v_S^2F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp[spx - s\gamma_S(p)\,|z|]}{\gamma_S(p)} \cdot \\ & \cdot \{\mp (1/2v_S^2 - p^2)\gamma_S(p)C(p) - p\,\gamma_S^2(p)D(p)\}\,dp, \end{aligned} \tag{14.35}$$

$$\begin{aligned} U_z^s(x,z;s) = {} & \frac{v_S^2F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp[spx - s\gamma_P(p)\,|z|]}{\gamma_P(p)} \cdot \\ & \cdot \{p\gamma_P^2(p)C(p) \mp (1/2v_S^2 - p^2)\gamma_P(p)D(p)\}\,dp + \\ & + \frac{v_S^2F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp[spx - s\gamma_S(p)\,|z|]}{\gamma_S(p)} \cdot \\ & \cdot \{-(1/2v_S^2 - p^2)pC(p) \mp p^2\gamma_S(p)D(p)\}\,dp, \end{aligned} \tag{14.36}$$

where $-(1/v_P)\cos\theta_P < c < 1/v_P$. The integrands are made single-valued by introducing branch cuts at Im $p=0$, $1/v_P < |\mathrm{Re}\,p| < \infty$ and at Im $p=0$, $1/v_S < |\mathrm{Re}\,p| < \infty$ and choosing $\mathrm{Re}\,\gamma_P(p) \geqslant 0$ and $\mathrm{Re}\,\gamma_S(p) \geqslant 0$ everywhere in the cut p-plane (Fig. 9).

The *diffracted* wave is introduced in exactly the same way as outlined in Section 12. The final representation of its polar components in the form of a composition product is obtained by transformations identical to the ones given in Section 12. Therefore, we only give the results and leave the details of the calculations to the reader. In terms of the polar coordinates r and θ ($0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$) we have

$$\begin{aligned} & u_r^d(r,\theta,t) = \\ & = \left\{\int_{r/v_P}^{t} f(t-\tau)[\varphi_r^{(P)}(r,\theta,\tau) + \psi_r^{(P)}(r,\theta,\tau)]\,d\tau\right\} H(t - r/v_P) + \end{aligned}$$

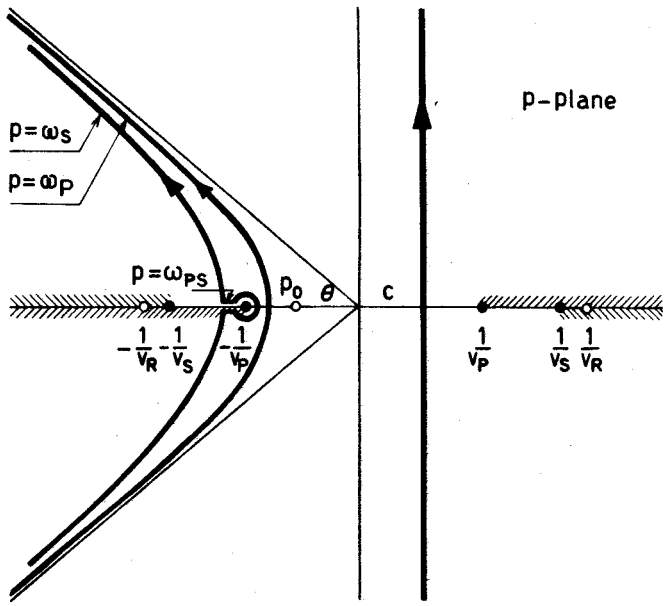

Fig. 9. Paths of integration for diffraction of a plane P-wave by a perfectly weak half-plane.

$$+\left\{\int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)[\varphi_r^{(PS)}(r,\theta,\tau)+\psi_r^{(PS)}(r,\theta,\tau)]d\tau\right\}H(t-t_{PS})+$$
$$+\left\{\int_{r/v_S}^{t} f(t-\tau)[\varphi_r^{(S)}(r,\theta,\tau)+\psi_r^{(S)}(r,\theta,\tau)]d\tau\right\}H(t-r/v_S), \tag{14.37}$$

$$u_\theta^d(r,\theta,t) =$$
$$=\left\{\int_{r/v_P}^{t} f(t-\tau)[\varphi_\theta^{(P)}(r,\theta,\tau)+\psi_\theta^{(P)}(r,\theta,\tau)]d\tau\right\}H(t-r/v_P)+$$
$$+\left\{\int_{t_{PS}}^{\min(t,\,r/v_S)} f(t-\tau)[\varphi_\theta^{(PS)}(r,\theta,\tau)+\psi_\theta^{(PS)}(r,\theta,\tau)]d\tau\right\}H(t-t_{PS}$$
$$+\left\{\int_{r/v_S}^{t} f(t-\tau)[\varphi_\theta^{(S)}(r,\theta,\tau)+\psi_\theta^{(S)}(r,\theta,\tau)]d\tau\right\}H(t-r/v_S), \tag{14.38}$$

where

$$\begin{cases}\varphi_r^{(P)}(r,\theta,t) = -\dfrac{v_S^2}{\pi r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\mathrm{Re}\{(1/2v_S^2-\omega_P^2)D(\omega_P)\}H(t-r/v_P),\\ \varphi_\theta^{(P)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}\mathrm{Im}\{(1/2v_S^2-\omega_P^2)D(\omega_P)\}\ H(t-r/v_P),\end{cases} \tag{14.39}$$

$$\begin{cases} \psi_r^{(P)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\mathrm{Re}\{\omega_P\gamma_P(\omega_P)C(\omega_P)\}H(t-r/v_P), \\ \psi_\theta^{(P)}(r,\theta,t) = -\dfrac{v_S^2}{\pi r}\mathrm{Im}\{\omega_P\gamma_P(\omega_P)C(\omega_P)\}H(t-r/v_P), \end{cases} \tag{14.40}$$

$$\begin{cases} \varphi_r^{(PS)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}\mathrm{Im}\{\omega_{PS}\gamma_S(\omega_{PS})D(\omega_{PS})\}[H(t-t_{PS}) - H(t-r/v_S)], \\ \varphi_\theta^{(PS)}(r,\theta,t) = -\dfrac{v_S^2}{\pi r}(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\mathrm{Im}\{\omega_{PS}\gamma_S(\omega_{PS})D(\omega_{PS})\}[H(t-t_{PS})-H(t-r/v_S)], \end{cases} \tag{14.41}$$

$$\begin{cases} \psi_r^{(PS)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}\mathrm{Im}\{(1/2v_S^2-\omega_{PS}^2)C(\omega_{PS})\}[H(t-t_{PS}) - H(t-r/v_S)], \\ \psi_\theta^{(PS)}(r,\theta,t) = -\dfrac{v_S^2}{\pi r}(r^2/v_S^2t^2-1)^{-\frac{1}{2}}\mathrm{Im}\{(1/2v_S^2-\omega_{PS}^2)C(\omega_{PS})\}[H(t-t_{PS})-H(t-r/v_S)], \end{cases} \tag{14.42}$$

$$\begin{cases} \varphi_r^{(S)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}\mathrm{Im}\{\omega_S\gamma_S(\omega_S)D(\omega_S)\}H(t-r/v_S), \\ \varphi_\theta^{(S)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\mathrm{Re}\{\omega_S\gamma_S(\omega_S)D(\omega_S)\}H(t-r/v_S), \end{cases} \tag{14.43}$$

$$\begin{cases} \psi_r^{(S)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}\mathrm{Im}\{(1/2v_S^2-\omega_S^2)C(\omega_S)\}H(t-r/v_S), \\ \psi_\theta^{(S)}(r,\theta,t) = \dfrac{v_S^2}{\pi r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}}\mathrm{Re}\{(1/2v_S^2-\omega_S^2)C(\omega_S)\}H(t-r/v_S) \end{cases} \tag{14.44}$$

and

$$\omega_P(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2-1/v_P^2)^{\frac{1}{2}}\sin\theta, \tag{14.45}$$

$$\omega_{PS}(r,\theta,t) = -(t/r)\cos\theta + (1/v_S^2-t^2/r^2)^{\frac{1}{2}}\sin\theta + i\delta, \quad (\delta\to 0), \tag{14.46}$$

$$\omega_S(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2-1/v_S^2)^{\frac{1}{2}}\sin\theta, \tag{14.47}$$

$$t_{PS} = (r/v_P)\cos\theta + (r/v_S)(1-v_S^2/v_P^2)^{\frac{1}{2}}\sin\theta, \tag{14.48}$$

with $0\leqslant\theta\leqslant\pi$. The second term on the right-hand sides of (14.37) and (14.38) is only present in the region $0\leqslant\theta<\arccos(v_S/v_P)$. In the region $\pi\leqslant\theta\leqslant 2\pi$ the results follow from the appropriate symmetry relations. The functions $\varphi(r,\theta,t)$ have been chosen such that they represent waves that are symmetrical with respect to $z=0$; consequently, $\varphi_r(r,\theta,t) = \varphi_r(r,2\pi-\theta,t)$, $\varphi_\theta(r,\theta,t) = -\varphi_\theta(r,2\pi-\theta,t)$. The functions $\psi(r,\theta,t)$ have been chosen such that they represent waves that are antisymmetrical with respect to $z=0$; consequently, $\psi_r(r,\theta,t) = -\psi_r(r,2\pi-\theta,t)$, $\psi_\theta(r,\theta,t) = \psi_\theta(r,2\pi-\theta,t)$.

The *geometrical* solution of the diffraction problem is obtained by superimposing the incident wave upon the contribution from the pole $p=p_0$. The cartesian components of the relevant displacement are given by

$$u_x^{geom}(r,\theta,t) =$$

$$= \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta < 2\pi-\theta_S), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PS}\sin\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & \\ & (2\pi-\theta_S < \theta < 2\pi-\theta_P), \\ \cos\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PP}\cos\theta_P\, f[t-(r/v_P)\cos(\theta+\theta_P)] + & \\ \quad + R_{PS}\sin\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_P < \theta \leqslant 2\pi), \end{cases} \tag{14.49}$$

$$u_z^{geom}(r,\theta,t) =$$

$$= \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta < 2\pi-\theta_S), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] + R_{PS}\cos\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & \\ & (2\pi-\theta_S < \theta < 2\pi-\theta_P), \\ \sin\theta_P\, f[t-(r/v_P)\cos(\theta-\theta_P)] - R_{PP}\sin\theta_P\, f[t-(r/v_P)\cos(\theta+\theta_P)] + & \\ \quad + R_{PS}\cos\theta_S\, f[t-(r/v_S)\cos(\theta+\theta_S)], & (2\pi-\theta_P < \theta \leqslant 2\pi), \end{cases} \tag{14.50}$$

where

$$R_{PP} = -\frac{\cos^2 2\theta_S - (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S}{\cos^2 2\theta_S + (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S} \tag{14.51}$$

and

$$R_{PS} = -\frac{2(v_S/v_P)\sin 2\theta_P \cos 2\theta_S}{\cos^2 2\theta_S + (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S} \tag{14.52}$$

are the amplitudes of the reflected P- and S-wave respectively, when a plane P-wave is incident upon a perfectly weak plane boundary.

At the values $\theta=\theta_P$, $\theta=2\pi-\theta_S$ and $\theta=2\pi-\theta_P$ a special investigation is required. At these values the geometrical solution is taken as the arithmetical mean of the limiting values at either side of the ray under consideration. Further, the diffracted waves are taken as the limiting values which are obtained by first substituting the

relevant value of θ and afterwards approaching the lower limits of integration from above. With the expressions thus generalized, the total wave motion is everywhere the superposition of the geometrical solution and the diffracted waves.

From the results it is clear that in the first place the diffracted waves consist of a cylindrical compressional and a cylindrical shear wave, both originating at the edge of the screen. Moreover, in the regions $0 \leqslant \theta < \text{arc}\cos(v_S/v_P)$ and $2\pi - \text{arc}\cos(v_S/v_P) < \theta \leqslant 2\pi$, there appears a PS-conversion wave whose wave front is a plane, travelling with the velocity v_S. Further, due to the presence of the pole $p = -1/v_R$, there is a singularity in the displacement on both sides of the screen; this singularity travels with the Rayleigh wave velocity v_R. The wave fronts are shown in Fig. 10.

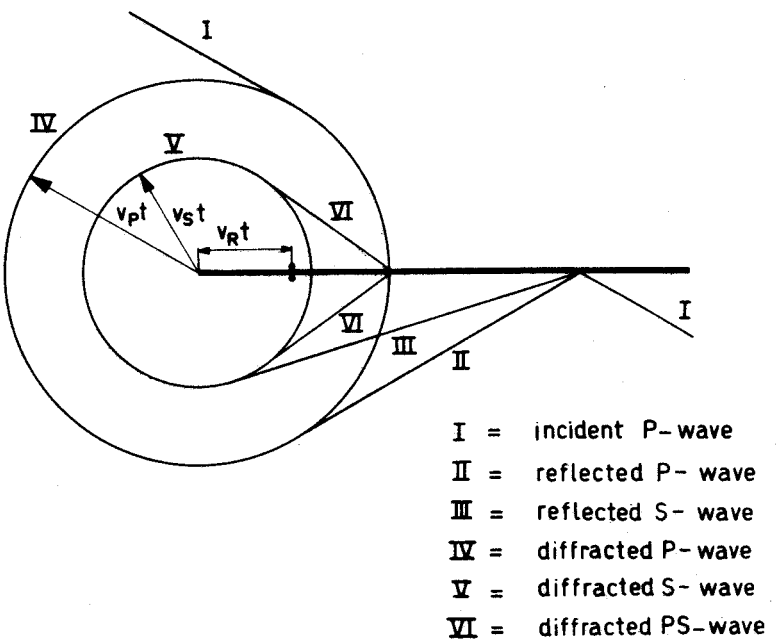

Fig. 10. Wave fronts for diffraction of a plane P-wave by a perfectly weak half-plane.

15. FACTORIZATION OF THE KERNEL FUNCTION L(p)

The function L(p) introduced in Section 14, eq. (14.25), and given by

$$L(p) = \frac{2}{1/v_S^2 - 1/v_P^2} \frac{(1/2v_S^2 - p^2)^2 + p^2 \gamma_P(p)\gamma_S(p)}{(1/v_R^2 - p^2)} \tag{15.1}$$

is nowhere zero or real and negative. Furthermore,

$$L(p) = 1 + O(p^{-2}) \text{ as } |p| \to \infty. \tag{15.2}$$

Application of Cauchy's theorem yields

$$\log L(p) = \frac{1}{2\pi i} \oint_C \log L(w) \frac{dw}{w-p}, \tag{15.3}$$

where C is a closed contour in the w-plane, surrounding the pole w=p. In accordance with the choice of sign of the square roots, the integrand is made single-valued by introducing branch cuts at Im w=0, $1/v_P < |\text{Re } w| < 1/v_S$ and taking the principal value of the logarithm. For the moment it is assumed that p is not a real number such that $1/v_P < |p| < 1/v_S$. By virtue of the asymptotic behaviour as $|w| \to \infty$, the contour C may be deformed into the loops C^+ and C^- around the branch cuts (Fig. 8, Section 13). The factorization is then carried out by writing

$$\log L(p) = \log L^+(p) + \log L^-(p), \tag{15.4}$$

where

$$\log L^+(p) = \frac{1}{2\pi i} \int_{C^+} \log L(w) \frac{dw}{w-p} \tag{15.5}$$

and

$$\log L^-(p) = \frac{1}{2\pi i} \int_{C^-} \log L(w) \frac{dw}{w-p}. \tag{15.6}$$

The right-hand side of (15.5) can be transformed into the real integral

$$\log L^+(p) = -\frac{1}{\pi} \int_{1/v_P}^{1/v_S} \arctan \left[\frac{w^2(w^2-1/v_P^2)^{\frac{1}{2}}(1/v_S^2-w^2)^{\frac{1}{2}}}{(1/2v_S^2-w^2)^2} \right] \frac{dw}{w+p}. \tag{15.7}$$

Further, $L^-(p)$ follows from the relation $L^-(p)=L^+(-p)$.

When p is a number just above or just below the real axis such that $-1/v_S < \text{Re } p < -1/v_P$, we have

$$\log L^+(p) = \pm i \arctan \left[\frac{p^2(p^2-1/v_P^2)^{\frac{1}{2}}(1/v_S^2-p^2)^{\frac{1}{2}}}{(1/2v_S^2-p^2)^2} \right] -$$

$$-\frac{1}{\pi} P \int_{1/v_P}^{1/v_S} \arctan \left[\frac{w^2(w^2-1/v_P^2)^{\frac{1}{2}}(1/v_S^2-w^2)^{\frac{1}{2}}}{(1/2v_S^2-w^2)^2} \right] \frac{dw}{w+p}, \tag{15.8}$$

where the upper sign applies when p is just above the real axis and the lower sign when p is just below the real axis. The integral on the right-hand side of (15.8) has to be taken in the sense of a Cauchy's principal value, which is indicated by a "P" in front of

the integral sign. In the same way as before, $L^-(p)$ follows from the relation $L^-(p)=L^+(-p)$.

For numerical computation it is useful to introduce in this case, too, the variable of integration α through

$$w^2 = \frac{1}{2}\left(\frac{1}{v_S^2}+\frac{1}{v_P^2}\right)-\frac{1}{2}\left(\frac{1}{v_S^2}-\frac{1}{v_P^2}\right)\cos\alpha, \tag{15.9}$$

where $0\leqslant\alpha\leqslant\pi$. The result of this substitution is easily obtained and will not be given here.

16. DIFFRACTION OF A PLANE P-PULSE BY A HALF-PLANE AS A SALTUS PROBLEM

The present section deals with some examples of the saltus problem formulation of the diffraction of a plane compressional wave by a half-plane coinciding with $z=0$, $0<x<\infty$. The incident wave is given by

$$u_x^i(x,z,t) = \cos\theta_P\, f[t-(x/v_P)\cos\theta_P-(z/v_P)\sin\theta_P], \tag{16.1}$$

$$u_z^i(x,z,t) = \sin\theta_P\, f[t-(x/v_P)\cos\theta_P-(z/v_P)\sin\theta_P], \tag{16.2}$$

where θ_P is the angle of incidence $(0\leqslant\theta_P\leqslant\pi/2)$ and $f(t)=0$ when $t<0$. The amounts by which the traction and the displacement jump across the screen are expressed in terms of their Laplace transforms with respect to time. The procedure by means of which the geometrical solution and the diffracted waves are obtained is the same as the one outlined in Section 12 and Section 14. Accordingly, we introduce the following functions

$$\int_0^\infty \exp(-sp\xi)\,[T_{xz}]_-^+\,d\xi = F(s)A(p), \tag{16.3}$$

$$\int_0^\infty \exp(-sp\xi)\,[T_{zz}]_-^+\,d\xi = F(s)B(p), \tag{16.4}$$

$$\int_0^\infty \exp(-sp\xi)\,[U_x]_-^+\,d\xi = -\frac{F(s)}{s}C(p), \tag{16.5}$$

$$\int_0^\infty \exp(-sp\xi)\,[U_z]_-^+\,d\xi = -\frac{F(s)}{s}D(p), \tag{16.6}$$

where

$$F(s) = \int_0^\infty \exp(-st)f(t)dt. \tag{16.7}$$

In the first place the amounts by which the traction and the displacement jump across the screen are taken as if the geometrical solution of the diffraction by a perfectly rigid half-plane were the exact solution. This leads to

$$[T_{xz}]_-^+ = -\rho v_S R_{PS}^{(I)} sF(s)\exp[-(s/v_P)\xi \cos\theta_P], \tag{16.8}$$

$$[T_{zz}]_-^+ = \rho v_P (1+R_{PP}^{(I)}) sF(s)\exp[-(s/v_P)\xi \cos\theta_P], \tag{16.9}$$

$$[U_x]_-^+ = 0, \tag{16.10}$$

$$[U_z]_-^+ = 0, \tag{16.11}$$

where

$$R_{PP}^{(I)} = -\frac{\cos(\theta_S+\theta_P)}{\cos(\theta_S-\theta_P)}, \tag{16.12}$$

$$R_{PS}^{(I)} = -\frac{\sin 2\theta_P}{\cos(\theta_S-\theta_P)}. \tag{16.13}$$

Consequently,

$$A^{(I)}(p) = -\rho v_S R_{PS}^{(I)}/(p-p_o), \tag{16.14}$$

$$B^{(I)}(p) = \rho v_P (1+R_{PP}^{(I)})/(p-p_o), \tag{16.15}$$

$$C^{(I)}(p) = 0, \tag{16.16}$$

$$D^{(I)}(p) = 0, \tag{16.17}$$

where $p_o=-(1/v_P)\cos\theta_P$. It can be verified that in this case the geometrical solution is identical with the geometrical solution of the diffraction of a plane P-pulse by a perfectly rigid half-plane, which is given by (12.105) and (12.106).

In the second place the amounts by which the traction and the displacement jump across the screen are taken as if the geometrical solution of the diffraction by a perfectly weak screen were the exact solution. This leads to

$$[T_{xz}]_-^+ = 0, \tag{16.18}$$

$$[T_{zz}]_-^+ = 0, \tag{16.19}$$

$$[U_x]_-^+ = -[(1+R_{PP}^{(II)})\cos\theta_P - R_{PS}^{(II)}\sin\theta_S]F(s)\exp[-(s/v_P)\xi\cos\theta_P], \tag{16.20}$$

$$[U_z]_-^+ = -[(1-R_{PP}^{(II)})\sin\theta_P + R_{PS}^{(II)}\cos\theta_S]F(s)\exp[-(s/v_P)\xi\cos\theta_P], \tag{16.21}$$

where

$$R_{PP}^{(II)} = -\frac{\cos^2 2\theta_S - (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S}{\cos^2 2\theta_S + (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S}, \tag{16.22}$$

$$R_{PS}^{(II)} = -\frac{2(v_S/v_P)\sin 2\theta_P \cos 2\theta_S}{\cos^2 2\theta_S + (v_S/v_P)^2 \sin 2\theta_P \sin 2\theta_S}. \tag{16.23}$$

Consequently,

$$A^{(II)}(p) = 0, \tag{16.24}$$

$$B^{(II)}(p) = 0, \tag{16.25}$$

$$C^{(II)}(p) = [(1+R_{PP}^{(II)})\cos\theta_P + R_{PS}^{(II)}\sin\theta_S]/(p-p_o), \tag{16.26}$$

$$D^{(II)}(p) = [(1-R_{PP}^{(II)})\sin\theta_P + R_{PS}^{(II)}\cos\theta_S]/(p-p_o). \tag{16.27}$$

In this case the geometrical solution is identical with the geometrical solution of the diffraction of a plane P-pulse by a perfectly weak half-plane, which is given by (14.49) and (14.50). The wave fronts for these two examples are shown in Fig. 11.

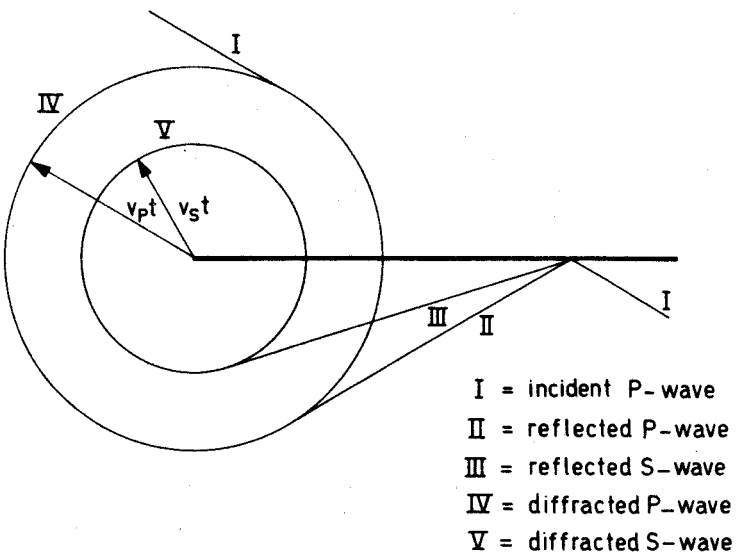

Fig. 11. Wave fronts for saltus problem diffraction of a plane P-wave by a half-plane.

As a final example the amounts by which the traction and the displacement jump across the screen are taken numerically equal to the corresponding values of the incident wave at the screen (Kirchhoff's assumptions). This leads to

$$[T_{xz}]_-^+ = \rho(v_S^2/v_P)\sin 2\theta_P \, sF(s)\exp[-(s/v_P)\xi\cos\theta_P], \tag{16.28}$$

$$[T_{zz}]_-^+ = -\rho v_P \cos 2\theta_S \, sF(s)\exp[-(s/v_P)\xi \cos \theta_P], \tag{16.29}$$

$$[U_x]_-^+ = -\cos \theta_P \, F(s)\exp[-(s/v_P)\xi \cos \theta_P], \tag{16.30}$$

$$[U_z]_-^+ = -\sin \theta_P \, F(s)\exp[-(s/v_P)\xi \cos \theta_P]. \tag{16.31}$$

Consequently,

$$A^{(K)}(p) = \rho (v_S^2/v_P)\sin 2\theta_P/(p-p_o), \tag{16.32}$$

$$B^{(K)}(p) = -\rho v_P \cos 2\theta_S/(p-p_o), \tag{16.33}$$

$$C^{(K)}(p) = \cos \theta_P/(p-p_o), \tag{16.34}$$

$$D^{(K)}(p) = \sin \theta_P/(p-p_o). \tag{16.35}$$

In this case the geometrical solution is given by

$$^{(K)}u_x^{geom}(r, \theta, t) = \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \cos \theta_P f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta \leqslant 2\pi), \end{cases} \tag{16.36}$$

$$^{(K)}u_z^{geom}(r, \theta, t) = \begin{cases} 0, & (0 \leqslant \theta < \theta_P), \\ \sin \theta_P f[t-(r/v_P)\cos(\theta-\theta_P)], & (\theta_P < \theta \leqslant 2\pi). \end{cases} \tag{16.37}$$

The structure of this solution explains why the Kirchhoff assumptions can be assumed to solve the diffraction by a perfectly absorbing screen (in optical terms a black screen). The wave fronts for this example are shown in Fig. 12.

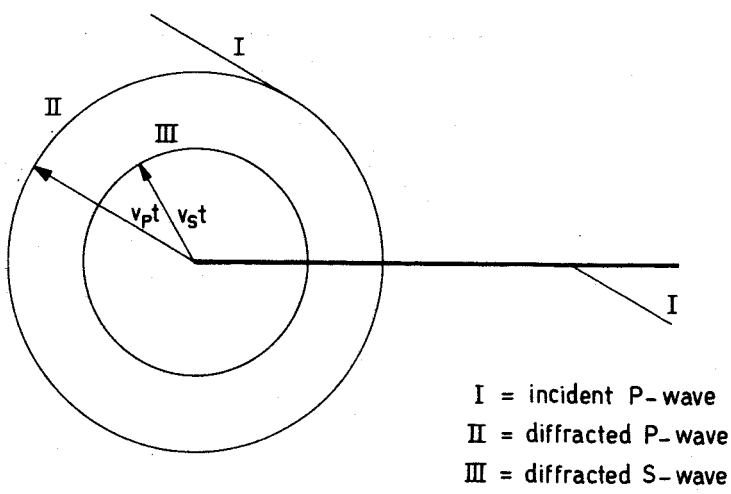

Fig. 12. Wave fronts for Kirchhoff diffraction of a plane P-wave by a half-plane.

In each of the aforementioned cases the diffracted waves are obtained from (12.38), (12.39), (14.35) and (14.36) in the way outlined in Section 12 and Section 14. The result is

$$u_r^d(r,\theta,t) =$$
$$= \left\{\int_{r/v_P}^{t} f(t-\tau)[\varphi_r^{(P)}(r,\theta,\tau)+\psi_r^{(P)}(r,\theta,\tau)]\,d\tau\right\}H(t-r/v_P) +$$
$$+ \left\{\int_{r/v_S}^{t} f(t-\tau)[\varphi_r^{(S)}(r,\theta,\tau)+\psi_r^{(S)}(r,\theta,\tau)]\,d\tau\right\}H(t-r/v_S), \tag{16.38}$$

$$u_\theta^d(r,\theta,t) =$$
$$= \left\{\int_{r/v_P}^{t} f(t-\tau)[\varphi_\theta^{(P)}(r,\theta,\tau)+\psi_\theta^{(P)}(r,\theta,\tau)]\,d\tau\right\}H(t-r/v_P) +$$
$$+ \left\{\int_{r/v_S}^{t} f(t-\tau)[\varphi_\theta^{(S)}(r,\theta,\tau)+\psi_\theta^{(S)}(r,\theta,\tau)]\,d\tau\right\}H(t-r/v_S); \tag{16.39}$$

where

$$\begin{cases} \varphi_r^{(P)}(r,\theta,t) = \dfrac{1}{\pi r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\,. \\ \cdot\ \mathrm{Re}\left\{\dfrac{1}{2\rho}\omega_P A(\omega_P) - v_S^2(1/2v_S^2-\omega_P^2)D(\omega_P)\right\}H(t-r/v_P), \\ \varphi_\theta^{(P)}(r,\theta,t) = \\ \quad = \dfrac{1}{\pi r}\,\mathrm{Im}\left\{-\dfrac{1}{2\rho}\omega_P A(\omega_P) + v_S^2(1/2v_S^2-\omega_P^2)D(\omega_P)\right\}H(t-r/v_P), \end{cases} \tag{16.40}$$

$$\begin{cases} \psi_r^{(P)}(r,\theta,t) = \dfrac{1}{\pi r}(1-r^2/v_P^2t^2)^{-\frac{1}{2}}\,. \\ \cdot\ \mathrm{Re}\left\{-\dfrac{1}{2\rho}\gamma_P(\omega_P)B(\omega_P) + v_S^2\,\omega_P\gamma_P(\omega_P)C(\omega_P)\right\}H(t-r/v_P), \\ \psi_\theta^{(P)}(r,\theta,t) = \\ \quad = \dfrac{1}{\pi r}\,\mathrm{Im}\left\{\dfrac{1}{2\rho}\gamma_P(\omega_P)B(\omega_P) - v_S^2\omega_P\gamma_P(\omega_P)C(\omega_P)\right\}H(t-r/v_P), \end{cases} \tag{16.41}$$

$$\varphi_r^{(S)}(r,\theta,t) =$$
$$= \frac{1}{\pi r}\,\mathrm{Im}\left\{\frac{1}{2\rho}\gamma_S(\omega_S)A(\omega_S) + v_S^2\omega_S\gamma_S(\omega_S)D(\omega_S)\right\}H(t-r/v_S),$$

$$\varphi_\theta^{(S)}(r,\theta,t) = \frac{1}{\pi r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}} \cdot$$

$$\cdot \operatorname{Re}\left\{\frac{1}{2\rho}\gamma_S(\omega_S)A(\omega_S) + v_S^2\omega_S\gamma_S(\omega_S)D(\omega_S)\right\} H(t-r/v_S), \tag{16.42}$$

$$\begin{cases} \psi_r^{(S)}(r,\theta,t) = \\ \quad = \frac{1}{\pi r}\operatorname{Im}\left\{\frac{1}{2\rho}\omega_S B(\omega_S) + v_S^2(1/2v_S^2-\omega_S^2)C(\omega_S)\right\} H(t-r/v_S), \\ \psi_\theta^{(S)}(r,\theta,t) = \frac{1}{\pi r}(1-r^2/v_S^2t^2)^{-\frac{1}{2}} \cdot \end{cases}$$

$$\cdot \operatorname{Re}\left\{\frac{1}{2\rho}\omega_S B(\omega_S) + v_S^2(1/2v_S^2-\omega_S^2)C(\omega_S)\right\} H(t-r/v_S), \tag{16.43}$$

in which

$$\omega_P(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2-1/v_P^2)^{\frac{1}{2}}\sin\theta, \tag{16.44}$$

$$\omega_S(r,\theta,t) = -(t/r)\cos\theta + i(t^2/r^2-1/v_S^2)^{\frac{1}{2}}\sin\theta, \tag{16.45}$$

with $0 \leqslant \theta \leqslant \pi$. In the region $\pi \leqslant \theta \leqslant 2\pi$ the results follow from the symmetry relations $\varphi_r(r,\theta,t)=\varphi_r(r,2\pi-\theta,t)$, $\varphi_\theta(r,\theta,t)=-\varphi_\theta(r,2\pi-\theta,t)$; $\psi_r(r,\theta,t)=-\psi_r(r,2\pi-\theta,t)$, $\psi_\theta(r,\theta,t)=\psi_\theta(r,2\pi-\theta,t)$. At the values $\theta=\theta_P$, $\theta=2\pi-\theta_S$ and $\theta=2\pi-\theta_P$ the expressions are taken in the usual sense.

From the expressions given above it is clear that the diffracted wave consists of a cylindrical P-wave and a cylindrical S-wave both originating at the edge of the screen. A striking feature is that there is no PS-conversion wave.

SUMMARY

The purpose of the present thesis is to give a mathematical treatment of elastodynamic diffraction problems. The method we give has been inspired by recent developments in acoustic and electromagnetic diffraction theory. Further, several problems concerning the diffraction of a plane pulse by a half-plane are solved and worked out in detail.

Chapter I gives a general introduction and a review of the literature.

In Chapter II we derive a representation theorem for the displacement vector in a homogeneous, isotropic, elastic solid. This representation theorem expresses the displacement at a point inside a closed surface in terms of the traction and the displacement at the surface. We derive both the two- and the three-dimensional form of the representation theorem; the result is valid for arbitrary time dependence.

With the aid of the representation theorem we formulate in Chapter III the problem of the diffraction by an obstacle of vanishing thickness, called a "screen" (a discussion on the corresponding electromagnetic problem is to be found in Section 6). Two classes of diffraction problems are considered, viz. boundary value problems and saltus problems. In the class of boundary value problems the screens are assumed to be either perfectly rigid (the displacement vanishes at the screen) or perfectly weak (the traction vanishes at the screen). These problems are reduced to solving certain (differential-)integral equations. In the class of saltus problems the amounts by which the traction and the displacement jump across the screen are prescribed. In this case the representation theorem directly leads to the solution.

Chapter IV deals with the two-dimensional diffraction of a plane SH-pulse by a half-plane. Solutions are given for the diffraction by a perfectly rigid and a perfectly weak half-plane. Moreover, several examples of the saltus problem formulation (including the Kirchhoff diffraction) are given.

Chapter V deals with the two-dimensional diffraction of a plane P-pulse by a half-plane. Solutions are given for the diffraction by a perfectly rigid and a perfectly weak half-plane. Moreover, several examples of the saltus problem formulation (including the Kirchhoff diffraction) are given.

The (differential-)integral equations occurring in Chapter IV and Chapter V are of the Wiener-Hopf type and are solved with the aid of the Wiener-Hopf technique. The transient solution to the problems discussed in Chapter IV and Chapter V is obtained by a modification of a technique originally developed by Cagniard.

SAMENVATTING

Dit proefschrift heeft tot onderwerp de wiskundige behandeling van elastodynamische diffraktieproblemen. De opzet is ontleend aan recente ontwikkelingen op het gebied van de akoestische en de elektromagnetische diffraktietheorie. Tevens worden enkele gevallen van de diffraktie van een vlakke golf aan een obstakel in de vorm van een halfvlak uitvoerig behandeld.

Hoofdstuk I geeft een algemene inleiding en een literatuuroverzicht.

In Hoofdstuk II wordt een representatietheorema voor de verplaatsingsvector in een homogeen, isotroop, elastisch medium afgeleid. Dit representatietheorema drukt de verplaatsingsvector in een punt binnen een gesloten oppervlak uit in de spanningsvector en de verplaatsingsvector op genoemd oppervlak. Zowel de tweedimensionale als de driedimensionale vorm van het representatietheorema worden afgeleid; het resultaat is geldig voor willekeurig met de tijd veranderende grootheden.

Met behulp van het representatietheorema wordt in Hoofdstuk III het vraagstuk van de diffraktie aan een oneindig dun obstakel, een „scherm", wiskundig geformuleerd (een bespreking van het overeenkomstige elektromagnetische vraagstuk vindt men in Paragraaf 6). Er worden twee klassen van diffraktieproblemen onderscheiden, nl. randwaardeproblemen en sprongwaardeproblemen. In de klasse van de randwaardeproblemen wordt ondersteld dat de schermen of volkomen star zijn (de verplaatsingsvector is nul op het scherm) of een scheur voorstellen (de spanningsvector is nul op het scherm). Deze vraagstukken worden gereduceerd tot het oplossen van (differentiaal-)integraalvergelijkingen. In de klasse van de sprongwaardeproblemen worden de sprongen in de spanningsvector en de verplaatsingsvector bij doorgang door het scherm bekend ondersteld. In dit geval leidt het representatietheorema direkt tot de oplossing.

In Hoofdstuk IV wordt de tweedimensionale diffraktie van een vlakke SH-golf aan een halfvlak beschouwd. Behalve de beide randwaardeproblemen worden ook enkele voorbeelden van de formulering als sprongwaardeprobleem (o.a. de diffraktie volgens Kirchhoff) behandeld.

In Hoofdstuk V wordt de tweedimensionale diffraktie van een vlakke P-golf aan een halfvlak beschouwd. Behalve de beide randwaardeproblemen worden ook enkele voorbeelden van de formulering als sprongwaardeprobleem (o.a. de diffraktie volgens Kirchhoff) behandeld.

De (differentiaal-)integraalvergelijkingen die in Hoofdstuk IV en Hoofdstuk V aan de orde komen zijn van het type van Wiener en Hopf en worden opgelost met behulp van de Wiener-Hopf tech-

niek. De overgangsverschijnselen worden berekend door gebruik te maken van een methode die afkomstig is van Cagniard en die in dit proefschrift verder is ontwikkeld.

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LEVENSBERICHT

De samensteller van dit proefschrift werd op 24 december 192' te Rotterdam geboren. Van 1940 tot 1945 doorliep hij aldaar de H.B.S. en in 1945 werd hem het einddiploma van de afdeling I uitgereikt. Van 1945 tot 1950 studeerde hij aan de Technische Hogeschool te Delft, waar hij op 4 juli 1950 het diploma voor elektrotechnisch ingenieur behaalde. Gedurende de cursus 1949-'50 was hij student-assistent bij prof.dr.ir. W.Th.Bähler.

Na het afstuderen bleef hij aan de Technische Hogeschool verbonden en verrichtte hij speurwerk onder leiding van prof.dr.ir J.P.Schouten op het gebied van de elektromagnetische diffraktietheorie. Als bewijs hiervan gaf de Afdeling der Elektrotechniek hem op 25 januari 1952 een verklaring.

Van 1952-1953 vervulde hij zijn militaire dienstplicht.

Na het beëindigen hiervan zette hij zijn onderzoekingen op het gebied van de diffraktietheorie voort op het Laboratorium voor theoretische elektrotechniek en elektromagnetische straling. Resultaten van deze onderzoekingen werden gepubliceerd in "Applied Scientific Research" en de "Proceedings van de Koninklijke Nederlandse Akademie van Wetenschappen". Gedurende deze periode doorliep hij verschillende rangen in de wetenschappelijke staf van de Technische Hogeschool. Met ingang van 1 januari 195' werd hij benoemd tot lector in de Afdeling der Elektrotechniek om onderwijs te geven in de theorie der elektriciteit.

Op uitnodiging van Professor Louis B.Slichter, directeur van het Institute of Geophysics, University of California, Los Angeles California, U.S.A., werkte hij van mei 1956 tot april 1957 bij genoemd instituut aan de diffraktietheorie voor golven in een elastisch medium. Deze onderzoekingen, die gefinancierd werden door het "Seismic Scattering Project", hebben geleid tot het samenstellen van dit proefschrift.