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THEORETICAL DETERMINATION OF THE SURFACE MOTION OF A UNIFORM ELASTIC HALF-SPACE PRODUCED BY A DILATATIONAL, IMPULSIVE, POINT SOURCE

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RESUME

A une profondeur h au-dessous de la surface d'un demi-espace élastique uniforme, une source ponctuelle émet une onde de dilatation impulsive. Le mouvement de la surface libre est calculé au moyen d'une variante appropriée (et simplifiée) de la méthode de Cagniard. On exprime les composantes horizontale et verticale du vecteur déplacement, chacune au moyen d'une intégrale simple prise sur un intervalle fini. Ces intégrales sont écrites sous une forme qui permet aisément leur évaluation numérique.

SUMMARY

At a depth h below the surface of a uniform elastic half-space a point source emits an impulsive, dilatational, wave. The motion of the free surface is determined by using a modified (and simplified) version of Cagniard's method. The horizontal and vertical component of the displacement vector are each expressed in terms of a one-dimensional integral over a finite interval ; these integrals are written in a form that easily permits their numerical evaluation.

I - INTRODUCTION.

The propagation of impulsive wave motion in an elastic solid has been investigated by a large number of authors. Because of its importance in theoretical seismology, the solid under consideration is frequently assumed to consist of several layers with different elastic properties (model of the earth), while, in order to confine the attention to the basic phenomena, the impulsive source is taken to be a concentrated one. For a comprehensive bibliography on this subject the reader is referred to a recent book by Ewing, Jardetzky and Press [1].

A few papers that are of immediate interest to the present work will be mentioned. Lamb [2] calculated the surface motion due to a source (a point source as well as a two-dimensional line source) located at the free surface of an elastic half-space. Cagniard [3] considered the generalization of this problem to the case of a compressional point source located in an elastic half-space, rigidly coupled to another half-space with different elastic properties. In Cagniard's method one or more integral transforms with respect to the space and time variables play an important role. Independent of this latter author, Pekeris [4] developed a slightly different technique for solving problems in connection with the radiation from impulsive sources. This technique has been applied by Pekeris to the seismic surface pulse [5] and the seismic buried pulse [6] problem, where the sources are taken to be concentrated vertical forces. The final solution of the latter case has been given by Pekeris and Lifson [7]. The same technique has been applied by Pekeris and Alterman [8] to the electromagnetic radiation from an antenna with impulsive current distribution placed at the interface of two non-conducting media and by Pekeris and Longman [9] to the propagation of explosive sound in a layered liquid.

Apart from these three-dimensional problems several two-dimensional analogues have been investigated. We mention Garvin's paper [10] on the motion generated by a compressional buried line source and Sauter's study on the motion generated by a two-dimensional pressure distribution applied to the free surface of a liquid half-space [11] or an elastic half-space [12].

Another class of two-dimensional elastic wave propagation problems is furnished by the diffraction of a plane pulse by a semi-infinite crack or a semi-infinite baffle. Some of these problems have been solved by the present author [13] with the aid of a slight modification of Cagniard's technique (for elastodynamic "half-plane problems", see also a paper by Maue [14] and a report by Miles [15]).

In the present paper it is shown that, if the aforementioned modification adapted to two-dimensional problems is taken as a guidance, a technique can be developed which is suited to three-dimensional problems and which is considerably simpler than the ones used by Cagniard and by Pekeris. As an example we determine the displacement at the free surface of an elastic half-space generated by a buried compressional point source. The horizontal and the vertical component of the displacement vector under consideration are each expressed in terms of a one-dimensional integral over a finite interval, which integral is cast into a form suitable for numerical computation.

II - STATEMENT OF THE PROBLEM AND METHOD OF SOLUTION.

We consider wave motions of small amplitude in a homogeneous, isotropic, semi-infinite elastic solid. The properties of the elastic solid are characterized by its density ρ and its Lamé constants λ and μ. A Cartesian coordinate system x, y, z is chosen such that the elastic solid occupies the half-space $0 < z < \infty$. A point in space will be located by either its Cartesian coordinates or its cylindrical coordinates r, φ, z defined through

$$x = r \cos \varphi,$$
$$y = r \sin \varphi, \qquad \text{(II. 1)}$$
$$z = z,$$

with $0 \leqq r < \infty$, $0 \leqq \varphi < 2\pi$ and, for points inside the solid or at its surface, $0 \leqq z < \infty$.

Let $\tau_{xx}, \ldots, \tau_{zz}$ denote the components of the stress tensor. At the free surface $z = 0$ we have $\tau_{xz} = 0$, $\tau_{yz} = 0$, $\tau_{zz} = 0$ for all values of x and y. The motion in the solid is characterized by the displacement vector $u = (u_x, u_y, u_z)$. The components of the stress tensor are related to the components of the displacement vector through

$$\tau_{xx} = \lambda \operatorname{div} u + 2\mu \frac{\partial u_x}{\partial x},$$
$$\tau_{yy} = \lambda \operatorname{div} u + 2\mu \frac{\partial u_y}{\partial y},$$
$$\tau_{zz} = \lambda \operatorname{div} u + 2\mu \frac{\partial u_z}{\partial z},$$
$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \qquad \text{(II. 2)}$$
$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right),$$
$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right).$$

At $x = 0$, $y = 0$, $z = h$ a point source generates a spherically symmetric compressional wave. The source starts to act at the instant $t = 0$ and it is assumed that prior to this instant the medium is at rest. The total displacement is written as

$$u = u^i + u^r, \qquad \text{(II. 3)}$$

where u^i (the incident wave) is the displacement that would exist if the medium were unbounded, while u^r (the reflected wave) accounts for the reflection of the incident wave against the free surface and is defined as the difference between the actual displacement and u^i. At any interior point of the solid u^r is assumed to be continuous together with its first and second order partial derivatives. The vector u^r satisfies the homogeneous elastodynamic wave equation [16]

$$(\lambda + \mu)\ \mathrm{grad\ div}\ \mathbf{u}^r + \mu \Delta\ \mathbf{u}^r - \rho\ \frac{\partial^2\ \mathbf{u}^r}{\partial t^2} = 0, \qquad \text{(II. 4)}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the three-dimensional Laplacian. Equation (II. 4) can also be written in the form

$$v_P^2\ \mathrm{grad\ div}\ \mathbf{u}^r - v_S^2\ \mathrm{curl\ curl}\ \mathbf{u}^r - \frac{\partial^2\ \mathbf{u}^r}{\partial t^2} = 0, \qquad \text{(II. 5)}$$

in which $v_P = \{(\lambda + 2\mu)/\rho\}^{\frac{1}{2}}$ is the compressional or P-wave velocity and $v_S = (\mu/\rho)^{\frac{1}{2}}$ is the shear or S-wave velocity. The incident wave is given by [17]

$$\mathbf{u}^i = -\ \mathrm{grad}\ \frac{f(t - R/v_P)}{4\pi R}, \qquad \text{(II. 6)}$$

where

$$R = [x^2 + y^2 + (z-h)^2]^{\frac{1}{2}}. \qquad \text{(II. 7)}$$

The function f(t) determines the strength of the source as a function of time as can be seen from the equation

$$\Delta\ \mathbf{u}^i - \frac{1}{v_P^2}\ \frac{\partial^2\ \mathbf{u}^i}{\partial t^2} = \mathrm{grad}\ \ \delta(x,\ y,\ z-h)\ f(t)\,(\mathrm{curl}\ \mathbf{u}^i = 0), \qquad \text{(II. 8)}$$

where $\delta(x, y, z-h)$ denotes, in a usual notation, the three-dimensional delta function. According to our assumptions $f(t) = 0$, when $-\infty < t < 0$.

All field quantities occurring in the problem are now subjected to a one-sided Laplace transform with respect to time ; e.g.

$$F(s) = \int_0^\infty \exp(-st)\ f(t)\ dt. \qquad \text{(II. 9)}$$

Similarly, $\mathbf{U}_x, \ldots, \mathbf{U}_z$ and $T_{xx}, \ldots, T_{zz}$ denote the Laplace transforms of $u_x, \ldots, u_z$ and $\tau_{xx}, \ldots, \tau_{zz}$, respectively. Following Cagniard [3], s is restricted to real positive values large enough to ensure the convergence of integrals of the type (II. 9) (it is tacitly assumed that the behaviour of the relevant functions as $t \longrightarrow \infty$ is such that such a number can be found; then any larger value of s also serves the purpose). Since, in particular, $\mathbf{u}^r$ and $\partial \mathbf{u}^r/\partial t$ are continuous, $\mathbf{U}^r = \mathbf{U}^r(x, y, z; s)$ satisfies the differential equation

$$(\lambda + \mu)\ \mathrm{grad\ div}\ \mathbf{U}^r + \mu\Delta\ \mathbf{U}^r - \rho\, s^2\, \mathbf{U}^r = 0. \qquad \text{(II. 10)}$$

The next step is to introduce the two-dimensional Fourier transform of $\mathbf{U}^r(x, y, z; s)$ with respect to x and y. Let

$$\mathcal{U}^r(\alpha, \beta; z; s) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \exp[is(\alpha x + \beta y)]\, U^r(x, y, z; s)\, dx, \qquad \text{(II. 11)}$$

in which the (real) factor s in the argument of the exponential function has been included for convenience. If $\mathcal{U}^r$ were known, U^r could be determined from the inversion integral

$$U^r(x,y,z;s) = \frac{s^2}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \exp[-is(\alpha x + \beta y)]\, \mathcal{U}^r(\alpha, \beta; z; s)\, d\alpha. \qquad \text{(II. 12)}$$

The corresponding representation of $U^i(x, y, z; s)$ is known to be [18]

$$U^i(x, y, z; s) =$$

$$= \frac{s^2 F(s)}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \exp[-s(i\alpha x + i\beta y + \gamma_p |z-h|)].\, (i\alpha, i\beta, \pm\gamma_p)\, \frac{1}{2\gamma_p}\, d\alpha, \qquad \text{(II. 13)}$$

in which

$$\gamma_p = \gamma_p(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/v_p^2)^{\frac{1}{2}} \quad (\mathrm{Re}\ \gamma_p \geqq 0), \qquad \text{(II. 14)}$$

and where the upper and lower sign in (II. 13) apply to the regions $z > h$ and $z < h$, respectively.

If, similarly, $\mathcal{T}_{xx}, \ldots, \mathcal{T}_{zx}$ denote the two-dimensional Fourier transforms of $T_{xx}, \ldots, T_{zx}$ with respect to x and y the boundary conditions reduce to $\mathcal{T}_{xz}(\alpha, \beta; 0; s) = 0$, $\mathcal{T}_{yz}(\alpha, \beta; 0; s) = 0$ and $\mathcal{T}_{zz}(\alpha, \beta; 0; s) = 0$.

In order to determine $\mathcal{U}^r$ we substitute (II. 12) in the differential equation (II. 10). The result is a system of three simultaneous ordinary differential equations for the components of $\mathcal{U}^r$ with z as independent variable. The solution of these equations that remains bounded as $z \longrightarrow \infty$ can be written as

$$\mathcal{U}_x^r = F(s)\,[i\alpha\,\mathcal{A}\exp(-s\gamma_p z) + \gamma_s \mathcal{B}\exp(-s\gamma_s z)]\,\exp(-s\gamma_p h),$$

$$\mathcal{U}_y^r = F(s)\,[i\beta\,\mathcal{A}\exp(-s\gamma_p z) + \gamma_s \mathcal{C}\exp(-s\gamma_s z)]\,\exp(-s\gamma_p h), \qquad \text{(II. 15)}$$

$$\mathcal{U}_z^r = F(s)\,[\gamma_p \mathcal{A}\exp(-s\gamma_p z) - (i\alpha\mathcal{B} + i\beta\mathcal{C})\exp(-s\gamma_s z)]\,\exp(-s\gamma_p h),$$

where γ_p is given by (II. 14) and γ_s is given by

$$\gamma_s = \gamma_s(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/v_s^2)^{\frac{1}{2}} \quad (\mathrm{Re}\ \gamma_s \geqq 0). \qquad \text{(II. 16)}$$

The functions $\mathcal{A} = \mathcal{A}(\alpha, \beta)$, $\mathcal{B} = \mathcal{B}(\alpha, \beta)$ and $\mathcal{C} = \mathcal{C}(\alpha, \beta)$ follow from the boundary conditions at the free surface. It is found that

$$\mathcal{A} = -\frac{(\alpha^2 + \beta^2 + 1/2v_s^2)^2 + (\alpha^2 + \beta^2)\gamma_p\gamma_s}{2\gamma_p \mathcal{D}(\alpha, \beta)}, \qquad \text{(II. 17)}$$

$$\mathcal{B} = \frac{i\alpha\,(\alpha^2 + \beta^2 + 1/2v_s^2)}{\mathcal{D}(\alpha, \beta)}, \tag{II. 18}$$

$$\mathcal{C} = \frac{i\beta\,(\alpha^2 + \beta^2 + 1/2v_s^2)}{\mathcal{D}(\alpha, \beta)}, \tag{II. 19}$$

where

$$\mathcal{D}(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/2v_s^2)^2 - (\alpha^2 + \beta^2)\,\gamma_p\,\gamma_s. \tag{II. 20}$$

From these results the total displacement at the free surface will be determined. If $\mathrm{U}(x, y, 0; s)$ is written as

$$\mathrm{U}(x, y, 0; s) = \frac{s^2F(s)}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \exp\,[-s(i\alpha x + i\beta y + \gamma_p h)]\,\mathcal{W}(\alpha, \beta)\,d\alpha, \tag{II. 21}$$

the components of $\mathcal{W}$ are

$$\mathcal{W}_x = \frac{1}{2v_s^2}\,\frac{i\alpha\gamma_s}{\mathcal{D}(\alpha, \beta)}, \tag{II. 22}$$

$$\mathcal{W}_y = \frac{1}{2v_s^2}\,\frac{i\beta\gamma_s}{\mathcal{D}(\alpha, \beta)}, \tag{II. 23}$$

$$\mathcal{W}_z = -\frac{1}{2v_s^2}\,\frac{\alpha^2 + \beta^2 + 1/2v_s^2}{\mathcal{D}(\alpha, \beta)}. \tag{II. 24}$$

Equation (II. 21) shows that $\mathrm{U}(x, y, 0; s)$ is of the form

$$\mathrm{U}(x, y, 0; s) = s^2F(s)\ \mathrm{G}(x, y, 0; s), \tag{II. 25}$$

where

$$\mathrm{G}(x, y, 0; s) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \exp\,[-s\,(i\alpha x + i\beta y + \gamma_p h)\,\mathcal{W}(\alpha, \beta)\,d\alpha. \tag{II.26}$$

In the next section it will be shown that the integral on the right-hand side of (II. 26) can be transformed into

$$\mathrm{G}(x, y, 0; s) = \int_{R/v_p}^{\infty} \exp(-s\tau)\ g(x, y, 0, \tau)\,d\tau, \tag{II. 27}$$

where only *real* values of τ occur in the integration and where R is given by (II. 7) (R = distance from the source to the point of observation). Now we observe that $s^2F(s)\exp(-s\tau)$ is the Laplace transform of a function that vanishes when $t < \tau$ and equals $d^2f(t-\tau)/dt^2$ when $\tau < t$. Using the notation $d^2f/dt^2 = f''$ we finally obtain for the surface displacement as a function of position and time the expression

$$u(x, y, 0, t) = \begin{cases} 0 & (0 < t < R/v_P), \\ \int_{R/v_P}^{t} f''(t-\tau)\; g(x, y, 0, \tau)\, d\tau & (R/v_P < t < \infty). \end{cases} \tag{II. 28}$$

From the foregoing analysis it is clear that $g(x, y, 0, t)$ can be regarded as the surface displacement in case $f(t)$ is given by $f(t) = t$ $(t > 0)$.

III - DETERMINATION OF THE FUNCTION $g(x, y, 0, \tau)$.

In the present section it will be shown that the transformations outlined in [18, Section 3] lead to an expression for $g(x, y, 0, \tau)$ in the form of a one-dimensional integral over a finite interval. In the integral on the right-hand side of (II. 26) we introduce new variables of integration ω and q through

$$\alpha = \omega \cos \varphi - q \sin \varphi , \tag{III. 1}$$

$$\beta = \omega \sin \varphi + q \cos \varphi . \tag{III. 2}$$

Since $d\alpha\, d\beta = d\omega\, dq$, we obtain

$$G(x, y, 0; s) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} \exp\left[-s(i\omega r + \gamma_P h)\right] \mathcal{W}\, d\omega, \tag{III. 3}$$

in which, as $\alpha^2 + \beta^2 = \omega^2 + q^2$,

$$\gamma_{P,S} = (\omega^2 + q^2 + 1/v_{P,S}^2)^{\frac{1}{2}} \qquad (\mathrm{Re}\, \gamma_{P,S} \geqq 0). \tag{III. 4}$$

Next we introduce the variable $p = i\omega$ and regard p as a complex variable, while q is kept real. The result is

$$G(x, y, 0; s) = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} dq \int_{-i\infty}^{i\infty} \exp\left[-s(pr + \gamma_P h)\right] \mathcal{W}\, dp. \tag{III. 5}$$

Further, we introduce the polar components of $G(x, y, 0; s)$. Using the expressions (II. 22) - (II. 24) for $\mathcal{W}$ we obtain

$$G_r(x, y, 0; s) = \frac{1}{2\pi^2 i} \int_{0}^{\infty} dq \int_{-i\infty}^{i\infty} \exp\left[-s(pr + \gamma_P h)\right] \mathcal{W}_r\, dp, \tag{III. 6}$$

$$G_\phi(x, y, 0; s) = 0, \tag{III. 7}$$

$$G_z(x, y, 0; s) = \frac{1}{2\pi^2 i} \int_{0}^{\infty} dq \int_{-i\infty}^{i\infty} \exp\left[-s(pr + \gamma_P h)\right] \mathcal{W}_z\, dp, \tag{III. 8}$$

in which

$$\mathcal{W}_r = \frac{p\gamma_S}{2v_S^2\, \varpi}, \tag{III. 9}$$

$$\mathcal{W}_z = \frac{p^2 - (q^2 + 1/2v_S^2)}{2v_S^2\, \mathcal{D}}, \tag{III. 10}$$

with

$$\mathcal{D} = (q^2 + 1/2v_S^2 - p^2)^2 + (p^2 - q^2)\, \gamma_P\, \gamma_S\,, \tag{III. 11}$$

$$\gamma_{P,S} = (q^2 + 1/v_{P,S}{}^2 - p^2)^{\frac{1}{2}} \qquad (\mathrm{Re}\ \gamma_{P,S} \geqq 0). \tag{III. 12}$$

In the complex p-plane the integrands in (III. 6) and (III. 8) have branch points at $p = \pm\, \Omega_P(q)$ and at $p = \pm\, \Omega_S(q)$ and, furthermore, simple poles at $p = \pm\, \Omega_R(q)$, where

$$\Omega_{P,S,R}(q) = (q^2 + 1/v_{P,S,R}^2)^{\frac{1}{2}} \tag{III. 13}$$

and where $v = v_R$ (v_R = Rayleigh wave velocity) is the real, positive, root of the equation

$$(\tfrac{1}{2}v^2/v_S^2 - 1)^2 - (1 - v^2/v_P^2)^{\frac{1}{2}}\, (1 - v^2/v_S^2)^{\frac{1}{2}} = 0. \tag{III. 14}$$

In view of subsequent deformations of the path of integration we take $\mathrm{Re}\ \gamma_P \geqq 0$ and $\mathrm{Re}\ \gamma_S \geqq 0$ not only on the imaginary p-axis but everywhere in the p-plane. This implies that branch cuts are introduced along $\mathrm{Im}\ p = 0$, $\Omega_P(q) < |\mathrm{Re}\ p| < \infty$ and along $\mathrm{Im}\ p = 0$, $\Omega_S(q) < |\mathrm{Re}\ p| < \infty$. It can easily be verified that, by virtue of Cauchy's theorem and Jordan's lemma [19], the integral along the imaginary p-axis in (III. 6) and (III. 8) can be replaced by an integral along the hyperbola Γ_P defined by

$$p = (r/R^2)\, \tau \pm i(h/R^2)\, [\tau^2 - R^2\, \Omega_P^2\, (q)]^{\frac{1}{2}}\ (R\, \Omega_P(q) < \tau < \infty), \tag{III. 15}$$

in which the square root is taken positive. The upper and lower sign in (III. 15) refer to the part of Γ_P located in the upper and lower half of the p-plane, respectively. Along Γ_P we have

$$\gamma_P = (h/R^2)\tau \mp i(r/R^2)\ [\tau^2 - R^2 \Omega_P^2\, (q)]^{\frac{1}{2}} \tag{III. 16}$$

and

$$\frac{\partial p}{\partial \tau} = \pm \frac{i\, \gamma_P}{[\tau^2 - R^2\, \Omega_P^2\, (q)]^{\frac{1}{2}}}. \tag{III. 17}$$

In (III. 15), (III. 16) and (III. 17) the upper and lower signs belong together. Taking into account the symmetry of the path of integration with respect to the real axis and introducing τ as variable of integration we obtain, since q, s and τ are real,

$$G_{r,z}(x, y, 0\,; s) =$$

$$= \frac{1}{\pi^2} \int_0^\infty dq \int_{R\Omega_P(q)}^\infty \exp(-s\tau)\, \frac{1}{[\tau^2 - R^2\, \Omega_P^2\, (q)]^{\frac{1}{2}}}\ \mathrm{Re}\,\{\mathcal{W}_{r,z}\gamma_P\}\ d\tau. \tag{III.18}$$

Interchanging the order of integration we have

$$G_{r,z}(x, y, 0 ; s) = \int_{R/v_p}^{\infty} \exp(-s\tau) \, d\tau \, .$$

$$\int_0^{(\tau^2/R^2 - 1/v_p^2)^{\frac{1}{2}}} \frac{1}{\pi^2} \mathrm{Re} \, \{\mathscr{W}_{r,z} \, \gamma_p\} \frac{1}{[\tau^2 - R^2 \Omega_p^2 (q)]^{\frac{1}{2}}} dq. \qquad \text{(III. 19)}$$

The integral on the right-hand side of (III. 19) has the form announced in Section 2, eq. (II. 27). Consequently, the polar components of the vector $g(x, y, 0, \tau)$ are given by

$$g_r(x, y, 0, \tau) = \frac{1}{\pi^2 R} \int_0^{\frac{1}{2}\pi} \mathrm{Re} \, \{ \mathscr{W}_r \, \gamma_p \} \, d\Psi \, , \qquad \text{(III. 20)}$$

$$g_\phi(x, y, 0, \tau) = 0, \qquad \text{(III. 21)}$$

$$g_z(x, y, 0, \tau) = \frac{1}{\pi^2 R} \int_0^{\frac{1}{2}\pi} \mathrm{Re} \, \{ \mathscr{W}_z \gamma_p \} \, d\Psi \, , \qquad \text{(III. 22)}$$

where a new variable of integration Ψ has been introduced through

$$q = (\tau^2/R^2 - 1/v_p^2)^{\frac{1}{2}} \sin \Psi \qquad (0 \leqq \Psi \leqq \tfrac{1}{2}\pi). \qquad \text{(III. 23)}$$

In the right-hand sides of (III. 20) and (III. 22) we have to substitute for p and γ_p the values (compare (III. 15) and (III. 16))

$$p = (r/R^2)\tau + i(h/R^2) (\tau^2 - R^2/v_p^2)^{\frac{1}{2}} \cos \Psi \, , \qquad \text{(III. 24)}$$

$$\gamma_p = (h/R^2)\tau - i(r/R^2) (\tau^2 - R^2/v_p^2)^{\frac{1}{2}} \cos \Psi \, , \qquad \text{(III. 25)}$$

while q is given by (III. 23). In all these expressions $R/v_p < \tau < \infty$ and $R = (r^2 + h^2)^{\frac{1}{2}}$.

IV - CONCLUDING REMARKS.

The problem of determining the surface displacement due to a compressional point source located in a homogeneous, isotropic, elastic half-space has been reduced to the evaluation of the integrals in (II. 28), (III. 20) and (III. 22). In (II. 28) the function $f''(t-\tau)$ takes into account the time dependence of the strength of the source, while the vector $g(x, y, 0, \tau)$, given by (III. 20) - (III. 22), depends on the geometry of the boundary value problem and the physical properties of the elastic solid.

The modification of Cagniard's method as outlined in the present paper can be applied to determine the motion generated by any type of point source (compressional, shear, or mixed) located either inside or at the surface of a homogeneous, isotropic, half-spece.

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DISCUSSION

M. DAVIDS : M. de Hoop a mis en évidence quelques nouveaux aspects, utiles et importants, de la méthode de Cagniard. En général, on développe la fonction de Green en une série infinie dont chaque terme représente une onde réfléchie par quelque point frontière et contient également les facteurs exponentiels exigés par la transformation de Laplace. Ce développement présente une autre particularité : c'est qu'à une époque donnée T seul un nombre fini n d'ondes réfléchies sont parvenues à un point déterminé, de sorte que tous les termes d'ordre supérieur à n sont nuls, et la question de la convergence de la série ne se pose pas. Nous avons utilisé ce développement pour certaines recherches sur des plaques métalliques. J'aimerais savoir si la méthode de l'auteur se prête à un développement de ce genre, et si elle permet de simplifier les calculs pratiques.

M. de HOOP : La méthode modifiée de Cagniard que nous avons exposée peut s'appliquer à un milieu non homogène composé d'un nombre fini de couches, à condition que chacune de ces couches soit homogène et isotrope. Le cas d'ondes SH produites par une source linéaire enterrée dans une couche unique surmontant un demi-espace (ondes de Love) a été résolu par L. Knopoff (J. Geophysical Res. *63* (1958), 619-630). On peut considérer le résultat comme la somme d'un nombre infini de contributions, dont un nombre fini seulement est non nul : les termes de la série correspondent à des ondes réfléchies sur les interfaces. Cependant, la solution exacte montre qu'en outre on doit y ajouter un autre mouvement ondulatoire, qui représente "l'avant-coureur de Love" ("Love precursor") et qui est analogue à l'onde conique de Cagniard, excitée par le plan de séparation des deux milieux.

M. COULOMB : Le problème à 3 dimensions de la naissance des ondes de Love a été traité par Jeffreys dans les Geilands Beiträge zur Geophysik par la méthode opérationnelle, bien avant les travaux de Knopoff sur le cas à deux dimensions cité par M. de Hoop.

M. BIOT : Une autre technique qui évite l'emploi de variables complexes est celle des modes normaux. La méthode n'a fait l'objet jusqu'ici que d'une publication dans le domaine de l'acoustique du dièdre (Biot et Tolstoy, J. A. S. A.) L'exemple traité illustre la simplicité de la méthode. D'autres travaux non publiés indiquent que la méthode s'applique aux solides avec les mêmes avantages.

M. de HOOP : Je regrette de ne pas connaître ces travaux, mais n'en suis pas étonné puisqu'ils n'ont pas été publiés. On peut gagner en élégance en utilisant les distributions et les solutions fondamentales de l'équation d'ondes, sans sortir du domaine réel, mais je ne pense pas que la solution finale serait plus simple que celle que j'ai présentée.

M. LEVINE : K. B. Broberg (Royal Institute of Technology, Stockholm) a étudié les ondes de contrainte excitées dans une plaque par un impact sur l'une de ses faces, et a mesuré le déplacement superficiel au-dessous du point d'impact ; dans un tel problème, la technique de Cagniard est particulièrement simple, et donne des résultats qui sont en bon accord avec l'expérience. Je voudrais demander, d'autre part, si la méthode de Cagniard a été appliquée à quelques problèmes avec des paramètres continument variables ?

M. de HOOP : A ma connaissance, la méthode de Cagniard n'a pas été appliquée au cas d'un certain nombre de couches hétérogènes au-dessus d'un demi-espace ; ce problème est étudié en ce moment à Delft, mais on n'a pas encore obtenu de résultats satisfaisants. Car si l'on prend la transformation de Laplace par rapport à la variable t et la transformation de Fourier bidimensionnelle par rapport aux variables x et y, on doit encore résoudre une équation différentielle ordinaire par rapport à la variable z. Dans le cas où la solution de cette équation différentielle peut être donnée sous forme d'une série finie ou d'une intégrale contenant des fonctions du type exponentiel, on peut espérer que la méthode de Cagniard soit applicable.