ELASTIC WAVE PROPAGATION
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ELASTODYNAMIC TIME-DOMAIN RECIPROCITY THEOREMS FOR SOLIDS WITH RELAXATION

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Time-domain reciprocity theorems of the time-convolution and the time-correlation type for elastodynamic wave fields in linear, time-invariant, and locally reacting solids are discussed. Inhomogeneity, anisotropy, and arbitrary relaxation effects, both of the active (anti-causal) and passive (causal) kind, are included. The analysis is entirely carried out in space-time, without intermediate recourse to the frequency or the wavevector domains. The application to inverse source problems is briefly indicated.

1. INTRODUCTION

A wave field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two admissible physical states that could occur in one and the same domain in space-time. The present investigation deals with time-convolution and time-correlation reciprocity theorems for elastodynamic wave fields in time-invariant configurations that are linear and locally reacting in their elastodynamic behavior. The space-time geometry in which the two admissible states occur, is the Cartesian product $D \times R$ of a time-invariant spatial domain $D \subset R^3$ and the real time axis R. Further, the constitutive parameters of the media present in the two states are time invariant and independent of the wave field values. No further restrictions are imposed. Inhomogeneity and arbitrary anisotropy are included, as well as arbitrary relaxation effects. Both the time-convolution and the time-correlation type of reciprocity theorem have an important field of application in inverse source and related problems. These applications will, from a general point of view, briefly be indicated.

The position of observation in R^3 is specified by the coordinates $\{x_1,x_2,x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin $\underline{O}$ and the three mutually perpendicular base vectors $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ of unit length each. In the indicated order the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors in R^3 is employed and the summation convention applies. The corresponding lower-case Latin subcripts are to be assigned the values $\{1,2,3\}$. Whenever appropriate, the position vector will be denoted by $\mathbf{x} = x_p\mathbf{i}_p$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_p denotes the differentiation with respect to x_p, ∂_t denotes the differentiation with respect to t.

The reciprocity theorems will be derived for bounded domains D. In the analysis also the boundary ∂D of D occurs. The unit vector along the normal to ∂D is denoted by ν_m; it points away from D.

2. SOME PROPERTIES OF THE TIME CONVOLUTION AND THE TIME CORRELATION OF SPACE-TIME FUNCTIONS

Let $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ be two transient space-time functions. By this we mean that the functions are absolutely integrable on the entire $t \in R$. Then, the time convolution of f_1 and f_2 is defined as

$$C(f_1,f_2;x,\tau) = \int_{t\in R} f_1(x,t)f_2(x,\tau - t)dt \tag{2.1}$$

and the time correlation of f_1 and f_2 as

$$R(f_1,f_2;x,\tau) = \int_{t\in R} f_1(x,t)f_2(x,t - \tau)dt. \tag{2.2}$$

Let $\bar{f}$ denote the time-reversed of f, i.e.,

$$\bar{f}(x,t) = f(x,-t), \tag{2.3}$$

then, it follows from (2.1) - (2.3) that

$$R(f_1,f_2;x,\tau) = C(f_1,\bar{f}_2;x,\tau). \tag{2.4}$$

Using (2.1), we obtain the property

$$C(\bar{f}_1,f_2;x,\tau) = \bar{C}(f_1,\bar{f}_2;x,\tau). \tag{2.5}$$

For the time derivative of the time convolution the rules

$$\partial_\tau C(f_1,f_2;x,\tau) = C(f_1,\partial_t f_2;x,\tau) = C(\partial_t f_1,f_2;x,\tau) \tag{2.6}$$

apply. In view of the property

$$\overline{\partial_t \bar{f}} = - \partial_t \bar{f}, \tag{2.7}$$

the time derivatives of the time correlation are taken care of by using

$$\partial_\tau C(f_1,\bar{f}_2;x,\tau) = C(\partial_t f_1,\bar{f}_2;x,\tau) = - C(f_1,\overline{\partial_t \bar{f}_2};x,\tau). \tag{2.8}$$

For the incorporation of relaxation effects in the reciprocity theorems we also need the time convolution of three space-time functions. For this, either of the definitions

$$C(f_1,f_2,f_3;x,\tau) = C(f_1,C(f_2,f_3);x,\tau) = C(C(f_1,f_2),f_3;x,\tau) \tag{2.9}$$

holds. In view of its simpler properties, the time convolution concept is used throughout the entire subsequent derivations, i.e., both for the time convolution and for the time correlation reciprocity theorems.

3. PROPERTIES OF THE ELASTODYNAMIC WAVE FIELD IN THE CONFIGURATION

In each subdomain of the configuration where the elastodynamic properties vary continuously with position, the elastodynamic wave field quantities are continuously differentiable and satisfy the equations

$$- \Delta_{k,m,p,q}\partial_m\tau_{p,q} + \dot{\Phi}_k = f_k, \tag{3.1}$$

$$\Delta_{i,j,m,r}\partial_m v_r - \dot{e}_{i,j} = h_{i,j}, \tag{3.2}$$

where $\tau_{p,q}$ = stress (Pa), v_r = particle velocity ($m\cdot s^{-1}$), $\dot{\Phi}_k$ = mass flow density rate ($kg\cdot m^{-2}\cdot s^{-2}$), $\dot{e}_{i,j}$ = deformation rate (s^{-1}), f_k = volume source density of force ($N\cdot m^{-3}$), $h_{i,j}$ = volume source density of strain rate (s^{-1}), $\Delta_{i,j,m,r} = (\delta_{i,m}\delta_{j,r} + \delta_{i,r}\delta_{j,m})/2$, and $\delta_{i,m}$ is the symmetrical unit tensor of rank two (Kronecker tensor). Equations (3.1) and (3.2) are supplemented by the constitutive relations. For a linear, time-invariant, locally reacting solid these are, using the notation of (2.1),

$$\dot{e}_{i,j}(\mathbf{x},t) = \partial_t C(\kappa_{i,j,p,q},\tau_{p,q};\mathbf{x},t), \tag{3.3}$$

$$\dot{\Phi}_k(\mathbf{x},t) = \partial_t C(\gamma_{k,r},v_r;\mathbf{x},t), \tag{3.4}$$

where $\kappa_{i,j,p,q}$ = compliance relaxation function ($Pa^{-1}\cdot s^{-1}$), and $\gamma_{k,r}$ = inertia relaxation function ($kg\cdot m^{-3}\cdot s^{-1}$). In (3.3) and (3.4), inhomogeneity, anisotropy and relaxation of the solid are included. If $\{\kappa_{i,j,p,q},\gamma_{k,r}\}(\mathbf{x},\tau) = 0$ when $\tau<0$, the solid at $\mathbf{x}$ is causal. If $\kappa_{i,j,p,q}(\mathbf{x},\tau) = s_{i,j,p,q}(\mathbf{x})\delta(\tau)$, $\gamma_{k,r}(\mathbf{x},\tau) = \rho_{k,r}(\mathbf{x})\delta(\tau)$, where $\delta(\tau)$ is the unit impulse (Dirac distribution), the solid is instantaneously reacting, and $s_{i,j,p,q}$ and $\rho_{k,r}$ are its compliance and its (tensorial) volume density of mass, respectively. If $\{\kappa_{i,j,p,q},\gamma_{k,r}\}(\mathbf{x},\tau) = 0$ when $\tau>0$, the solid is anticausal or effectual. For our reciprocity theorems no specific type of relaxation function is presupposed. It is assumed that $\kappa_{i,j,p,q}$ and $\gamma_{k,r}$ are piecewise continuous functions of position. At an interface between two different solids, at which we assume the solids to be in rigid contact, the constitutive parameters jump by finite amounts, but the traction (i.e., the normal component of the stress) and the particle velocity are continuous. If an elastically impenetrable object is present, either the traction (at a void) or the particle velocity (at an immovable rigid object) has zero value at its boundary.

The two states that occur in the reciprocity theorems will be denoted by the superscripts 'a' and 'b', respectively.

4. THE RECIPROCITY THEOREM OF THE TIME-CONVOLUTION TYPE

The reciprocity theorem of the time-convolution type follows upon considering the interaction quantity $\Delta_{m,r,p,q}[C(-\tau^a_{p,q},v^b_r;\mathbf{x},\tau) - C(-\tau^b_{p,q},v^a_r;\mathbf{x},\tau)]$. Using (3.1) - (3.4) for each of the two states we arrive at

$$\Delta_{m,r,p,q}\partial_m[C(-\tau^a_{p,q},v^b_r;\mathbf{x},\tau) - C(-\tau^b_{p,q},v^a_r;\mathbf{x},\tau)]$$
$$= \partial_\tau C(\gamma^b_{r,k} - \gamma^a_{k,r},v^a_r,v^b_k;\mathbf{x},\tau) - \partial_\tau C(\kappa^b_{p,q,i,j} - \kappa^a_{i,j,p,q},\tau^a_{p,q},\tau^b_{i,j};\mathbf{x},\tau)$$
$$+ C(f^a_r,v^b_r;\mathbf{x},\tau) + C(-\tau^a_{p,q},h^b_{p,q};\mathbf{x},\tau) - C(f^b_r,v^a_r;\mathbf{x},\tau) - C(-\tau^b_{p,q},h^a_{p,q};\mathbf{x},\tau). \tag{4.1}$$

Equation (4.1) is the local form of the time-convolution reciprocity theorem. The first two terms at the right-hand side are representative for the differences in the properties of the solids present in the two states. If $\gamma^b_{r,k}(\mathbf{x},\tau) = \gamma^a_{k,r}(\mathbf{x},\tau)$ and $\kappa^b_{p,q,i,j}(\mathbf{x},\tau) = \kappa^a_{i,j,p,q}(\mathbf{x},\tau)$ for all $\tau \in R$, these terms vanish and the two media are denoted as each other's adjoints. Note that the adjoint of a causal (effectual) medium is a causal (effectual) one. Integrating (4.1) over the subdomains of D where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\int_{\mathbf{x}\in\partial D} \Delta_{m,r,p,q}\nu_m[C(-\tau^a_{p,q},v^b_r;\mathbf{x},\tau) - C(-\tau^b_{p,q},v^a_r;\mathbf{x},\tau)]dA$$
$$= \int_{\mathbf{x}\in D} [\partial_\tau C(\gamma^b_{r,k} - \gamma^a_{k,r},v^a_r,v^b_k;\mathbf{x},\tau)$$
$$- \partial_\tau C(\kappa^b_{p,q,i,j} - \kappa^a_{i,j,p,q},\tau^a_{p,q},\tau^b_{i,j};\mathbf{x},\tau)]dV$$
$$+ \int_{\mathbf{x}\in D} [C(f^a_r,v^b_r;\mathbf{x},\tau) + C(-\tau^a_{p,q},h^b_{p,q};\mathbf{x},\tau)$$
$$- C(f^b_r,v^a_r;\mathbf{x},\tau) - C(-\tau^b_{p,q},h^a_{p,q};\mathbf{x},\tau)]dV. \tag{4.2}$$

Equation (4.2) is the global form, for the domain D, of the time-convolution reciprocity theorem. Note that in the left-hand side the contributions from interfaces between different solids present in D have cancelled and that the contributions from the boundaries of elastically impenetrable objects present in D have vanished in view of the boundary conditions stated in Section 3.

5. THE RECIPROCITY THEOREM OF THE TIME-CORRELATION TYPE

The reciprocity theorem of the time-correlation type follows upon considering the interaction quantity $\Delta_{m,r,p,q}[R(-\tau^a_{p,q},v^b_r;\mathbf{x},\tau) + R(-\tau^b_{p,q},v^a_r;\mathbf{x},-\tau)]$ $= \Delta_{m,r,p,q}[C(-\tau^a_{p,q},\bar{v}^b_r;\mathbf{x},\tau) + C(-\bar{\tau}^b_{p,q},v^a_r;\mathbf{x},\tau)]$. Using (3.1) - (3.4) for each of the two states, we arrive at

$$\Delta_{m,r,p,q}\partial_m[C(-\tau^a_{p,q},\bar{v}^b_r;\mathbf{x},\tau) + C(-\bar{\tau}^b_{p,q},v^a_r;\mathbf{x},\tau)]$$
$$= \partial_\tau C(\bar{\gamma}^b_{r,k} - \gamma^a_{k,r},v^a_r,\bar{v}^b_k;\mathbf{x},\tau) + \partial_\tau C(\bar{\kappa}^b_{p,q,i,j} - \kappa^a_{i,j,p,q},\tau^a_{p,q},\bar{\tau}^b_{i,j};\mathbf{x},\tau)$$
$$+ C(f^a_r,\bar{v}^b_r;\mathbf{x},\tau) + C(-\tau^a_{p,q},\bar{h}^b_{p,q};\mathbf{x},\tau) + C(\bar{f}^b_r,v^a_r;\mathbf{x},\tau) + C(-\bar{\tau}^b_{p,q},h^a_{p,q};\mathbf{x},\tau). \tag{5.1}$$

Equation (5.1) is the local form of the time-correlation reciprocity theorem. The first two terms at the right-hand side are representative for the differences in the properties of the solids present in the two states. If $\bar{\gamma}^{b}_{r,k}(\mathbf{x},\tau) = \gamma^{a}_{k,r}(\mathbf{x},\tau)$ and $\bar{\kappa}^{b}_{p,q,i,j}(\mathbf{x},\tau) = \kappa^{a}_{i,j,p,q}(\mathbf{x},\tau)$ for all $\tau \in \mathbb{R}$, these terms vanish and the two media are denoted as each other's time-reverse adjoints. Note that the time-reverse adjoint of a causal (effectual) medium is an effectual (causal) one. Integrating (5.1) over the subdomains of D where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\begin{aligned}
&\int_{\mathbf{x}\in\partial D} \Delta_{m,r,p,q}\nu_m[C(-\tau^{a}_{p,q},\bar{v}^{b}_{r};\mathbf{x},\tau) + C(-\bar{\tau}^{b}_{p,q},v^{a}_{r};\mathbf{x},\tau)]dA \\
&= \int_{\mathbf{x}\in D} [\partial_\tau C(\bar{\gamma}^{b}_{r,k} - \gamma^{a}_{k,r},v^{a}_{r},\bar{v}^{b}_{k};\mathbf{x},\tau) \\
&\quad + \partial_\tau C(\bar{\kappa}^{b}_{p,q,i,j} - \kappa^{a}_{i,j,p,q},\tau^{a}_{p,q},\bar{\tau}^{b}_{i,j};\mathbf{x},\tau)]dV \\
&\quad + \int_{\mathbf{x}\in D} [C(f^{a}_{r},\bar{v}^{b}_{r};\mathbf{x},\tau) + C(-\tau^{a}_{p,q},\bar{h}^{b}_{p,q};\mathbf{x},\tau) \\
&\quad + C(\bar{f}^{b}_{r},v^{a}_{r};\mathbf{x},\tau) + C(-\bar{\tau}^{b}_{p,q},h^{a}_{p,q};\mathbf{x},\tau)]dV. \qquad (5.2)
\end{aligned}$$

Equation (5.2) is the global form, for the domain D, of the time-correlation reciprocity theorem. Note that in the left-hand side the contributions from interfaces between different media present in D have cancelled and that the contributions from the boundaries of elastically impenetrable objects present in D have vanished in view of the boundary conditions stated in Section 3.

6. APPLICATION TO INVERSE PROBLEMS

In this section we briefly indicate the application of (4.2) and (5.2) to the elastodynamic inverse source problem. In the inverse source problem the elastodynamic wave field in State 'a' is taken to be one that is radiated by the unknown source distributions $\{f^{T}_{r},h^{T}_{p,q}\}$. Let $D^{T} \subset \mathbb{R}^3$ be their spatial support. The radiated wave field $\{-\tau^{T}_{p,q},v^{T}_{r}\}$ is measured in some, accessible, observational domain $D^{\Omega} \subset \mathbb{R}^3$. The intersection of D^{T} and D^{Ω} is empty (Figure 1). State 'b' is taken to be a computational state, denoted as the 'observational' one. The corresponding wave field $\{-\tau^{\Omega}_{p,q},v^{\Omega}_{r}\}$ that would be radiated by known sources with distributions $\{f^{\Omega}_{r},h^{\Omega}_{p,q}\}$ is computed and its interaction with the measured elastodynamic wave field in D^{Ω} is evaluated. In general, one could say that the introduction of the observational state is representative for the processing of the measured data. Since only the interaction in D^{Ω} is considered, it makes no sense to take the support of $\{f^{\Omega}_{r},h^{\Omega}_{p,q}\}$ larger than D^{Ω}. Finally, the solid in the observational state is taken to be either the adjoint (for the application of (4.2)) or the time-reverse adjoint (for the application of (5.2)) of the one in which the unknown sources radiate. The reciprocity relations are now applied to the domain interior to the closed surface S^{Ω} that is taken such that D^{T} and D^{Ω} are located in its interior. Then, through the reciprocity relations, the known interactions in D^{Ω} are related to the source distributions to be reconstructed. Exploiting these relationships, several reconstruction algorithms can be developed. As to the role of S^{Ω}, we observe that in practice one is as a rule interested only in causal media. Then, it is advantageous to choose, in the application of (4.2), the wave fields causal as well. Given the fact that S^{Ω} surrounds all sources, the integral over S^{Ω} can be shown to be zero. In the application of (5.2), however, effectual (or anticausal) wave fields are involved in all cases and the integral over S^{Ω} differs from zero. This difference in the roles of the surface integrals in the two cases has been pointed out by Bojarski [1].

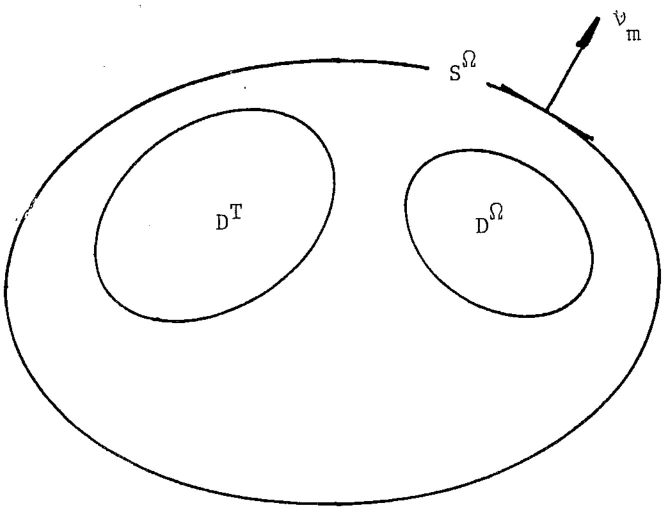

FIGURE 1

Configuration ilustrative for the inverse source problem: unknown acoustic sources radiate in D^{T}; the elastodynamic wave field is measured in D^{Ω} and on S^{Ω}.

REFERENCES

[1] Bojarski, N.N., "Generalized reaction principles and reciprocity theorems for the wave equations, and the relationship between the time-advanced and time-retarded fields", Journal of the Acoustical Society of America 74 (1), 281-285, 1983.