ELASTIC WAVE PROPAGATION
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A SPACE-TIME FINITE-ELEMENT METHOD FOR THE COMPUTATION OF THREE-DIMENSIONAL ELASTODYNAMIC WAVE FIELDS (THEORY)

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The theory of a space-time finite-element method for the numerical solution of elastodynamic wave problems in bounded time-invariant subdomains of three-dimensional space is developed. It is shown how the finite-element method can be regarded as to be based on a space-time elastodynamic reciprocity theorem of the time-correlation type. Particular local representations for the elastodynamic wave field are developed that can handle strongly inhomogeneous structures in which solid/solid interfaces are present.

1. INTRODUCTION

The theoretical essentials of a full space-time finite-element method, based on reciprocity, for the numerical solution of elastodynamic wave problems in bounded, time-invariant subdomains of three-dimensional space are developed. The elastodynamic wave fields are characterized by their particle velocity and stress, which are considered as their state variables. It is shown how the finite-element method can be regarded as to be based on a space-time elastodynamic reciprocity theorem of the time-correlation type [1]. In its turn this theorem is shown to be equivalent to a certain weighting procedure applied to the equation of motion and deformation rate equation. The medium in the configuration is taken to be linear, locally and instantaneously reacting, and time-invariant in its elastic behaviour. Arbitrary inhomogeneity and anisotropy are taken into account. Particular local representations for the elastodynamic wave field are developed that can handle strongly inhomogeneous structures in which solid/solid interfaces are present.

2. BASIC EQUATIONS OF ELASTODYNAMICS

The elastodynamic waves under consideration are small-amplitude disturbances. Position of observation in R^3 is specified by the Cartesian coordinates $\{x_1,x_2,x_3\}$. The subscript notation for Cartesian vectors and tensors in R^3 is employed and the summation convention applies. The corresponding lower-case Latin subscripts are to be assigned the values $\{1,2,3\}$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_p denotes differentiation with respect to x_p, ∂_t denotes the differentiation with respect to t. The geometrical configuration that we study is taken to be time-invariant. The medium in it is assumed to be linear, locally and instantaneously reacting, and time-invariant in its elastic behaviour. It may be arbitrarily inhomogeneous and anisotropic. The elastodynamic wave fields are characterized by their particle velocity v_r and stress τ_{pq}. The physical properties of the solid are characterized by its anisotropic volume density of mass ρ_{kr} and its compliance s_{ijpq}. In each subdomain of the configuration where the elastodynamic properties vary continuously with position, the elastodynamic field quantities are continuously differentiable and satisfy the (linearized) equation of motion

(2.1) $$-\Delta_{kmpq}\partial_m\tau_{pq} + \rho_{kr}\partial_t v_r = f_k,$$

and the (linearized) deformation rate equation

(2.2) $$\Delta_{ijmr}\partial_m v_r - s_{ijpq}\partial_t\tau_{pq} = h_{ij},$$

where f_k = volume source density of force, h_{ij} = volume density of strain rate and $\Delta_{kmpq} = (\delta_{kp}\delta_{mq} + \delta_{kq}\delta_{mp})/2$ is a unit tensor of rank four that specifically occurs in elastodynamics. It has the symmetry properties $\Delta_{kmpq} = \Delta_{mkpq} = \Delta_{mkqp} = \Delta_{kmqp}$. At interfaces between two different media the constitutive coefficients ρ_{kr} and s_{ijpq} in general jump by finite amounts. In all applications we shall assume that at a solid/solid interface the media are in rigid contact; then, the particle velocity v_r and the traction $t_k = \Delta_{kmpq}\nu_m\tau_{pq}$, where ν_m is the unit normal to the interface, are continuous across the interface. Finally, to have a unique solution of the wave problem in the bounded domain D (which later is identified with the domain of finite-element computation), we need initial conditions at some time $t = t_0$:

(2.3) $$v_r(\mathbf{x},t_0) = v_r^I(\mathbf{x}), \quad \text{when} \quad \mathbf{x} \in D, \quad \tau_{pq}(\mathbf{x},t_0) = \tau_{pq}^I(\mathbf{x}), \quad \text{when} \quad \mathbf{x} \in D,$$

and boundary conditions on the boundary surface ∂D of D. The simplest of these are explicit ones that apply to either a prescribed particle velocity or a prescribed traction, i.e.,

(2.4) $$v_r = v_r^B, \quad \text{when} \quad \mathbf{x} \in \partial D_1 \text{ and } t_k = t_k^B, \quad \text{when} \quad \mathbf{x} \in \partial D_2$$

where v_r^B and τ_{pq}^B are the prescribed values of the particle velocity and the stress at the parts ∂D_1 and ∂D_2 of ∂D respectively. The intersection of ∂D_1 and ∂D_2 is empty and the union of ∂D_1 and ∂D_2 is ∂D.

3. THE SPACE-TIME FINITE-ELEMENT METHOD

As the starting point for the construction of a finite-element formulation of the elastodynamic wave problem we introduce the space-time elastodynamic reciprocity theorem of the time-correlation type [1]. Of the two states occurring in this theorem, State A is identified with the actual wave field that is to be approximated, while State B is considered as a computational one that remains to be chosen appropriately. As regards the space-time geometry in which the two states occur, the time-invariance implies that this geometry is the Cartesian product $D \times R$ of a time-invariant spatial domain $D \subset R^3$ and the real time axis R. The theorem is applied to the bounded domain D that consists of subdomains $\{D^{(n)};\ n = 1,..,N\}$ where the medium properties vary continuously with position. The boundary surface of $D^{(n)}$ is denoted by $\partial D^{(n)}$. The required reciprocity relation is then

$$(3.1)\quad \sum_{n=1}^{N} \int_{t\in R} dt \int_{x\in\partial D^{(n)}} \Delta_{mrpq}\nu_m[v_r^A(x,t)\tau_{pq}^B(x,t-\tau) + v_r^B(x,t-\tau)\tau_{pq}^A(x,t)]dA$$
$$= \partial_\tau \int_{t\in R} dt \int_{x\in D} \tau_{ij}^B(x,t-\tau)[s_{ijpq}^A(x)-s_{pqij}^B(x)]\tau_{pq}^A(x,t)dV$$
$$+ \partial_\tau \int_{t\in R} dt \int_{x\in D} v_k^B(x,t-\tau)[\rho_{kr}^A(x)-\rho_{rk}^B(x)]v_r^A(x,t)dV$$
$$+ \int_{t\in R} dt \int_{x\in D} \tau_{ij}^B(x,t-\tau)h_{ij}^A(x,t) - v_k^B(x,t-\tau)f_k^A(x,t)dV$$
$$+ \int_{t\in R} dt \int_{x\in D} \tau_{pq}^A(x,t)h_{pq}^B(x,t-\tau) - v_r^A(x,t)f_r^B(x,t-\tau)dV.$$

In this theorem, the state quantities have been assumed to be piecewise continuously differentiable in $\{D^{(n)},\ n=1,..,N\}$. First of all we shall show that Equation (3.1) can, from a particular point of view, also be regarded as a "weighted" form of the equation of motion (2.1) and the deformation rate equation (2.2) pertaining to the State A. To this end, we take the quantities $\{v_k^B, \tau_{ij}^B\}$ of State B to be continuously differentiable functions in the subdomains $\{D^{(n)},\ n=1,..,N\}$, subject to the choice $\rho_{rk}^B = 0$, $s_{ijpq}^B = 0$, $f_r^B = -\Delta_{rmij}\partial_m\tau_{ij}^B$ and $h_{pq}^B = \Delta_{pqmk}\partial_m v_k^B$. Substituting these quantities into (3.1) and using Gauss' divergence theorem in the subdomains $\{D^{(n)};n=1,..,N\}$, of D where both sides are continuously differentiable, we end up with

$$(3.2)\quad \int_{t\in R} dt \int_{x\in D} v_k^B(x,t-\tau)[-\Delta_{kmpq}\partial_m\tau_{pq}^A(x,t) + \rho_{kr}^A(x)\partial_t v_r^A(x,t) - f_k^A(x,t)]dV$$
$$+ \int_{t\in R} dt \int_{x\in D} \tau_{ij}^B(x,t-\tau)[\Delta_{ijmr}\partial_m v_r^A(x,t) - s_{ijpq}^A(x)\partial_t\tau_{pq}^A(x,t)$$
$$- h_{ij}^A(x,t)]dV = 0.$$

Upon taking the functions $\tau_{ij}^B = 0$ (and hence $f_r^B = 0$) throughout $D \times R$ and $v_k^B \neq 0$, Equation (3.2) represents the weighted form of the equation of motion (2.1) over the space-time domain $D \times R$ with the arbitrary weighting function v_k^B, while if we choose the functions $v_k^B = 0$ (and hence $h_{pq}^B = 0$) throughout $D \times R$ and $\tau_{ij}^B \neq 0$, Equation (3.2) is nothing but the weighted form of the equation of deformation rate (2.2) over the space-time domain $D \times R$ with the arbitrary weighting function τ^B_{ij}.

Formulation (3.2) is used to set up a space-time finite-element method. In this formulation the A-field is identified with the actual wave field and the B-field with the weighting field. In view of the time-invariance of D, the space-time domain over which the finite-element method is applied, is discretized into a union of elementary subdomains that are cylindrical in the time direction. In these subdomains local functions are defined that are the product of a function of the spatial variables and a function of time. The local functions are subsequently combined to global expansion functions that are used in the approximation of the wave field.

When taking a reciprocity relation as the point of departure of setting up a numerical scheme, it seems more or less natural to treat the States A and B in an equivalent manner, which implies that each specimen of the sequences of functions into which State A is expanded are also taken as a specimen of the State B. As far as (3.2) is concerned, this implies that the sequence of weighting functions is taken to be the same as the sequence of expansion functions.

In the application of the finite-element method we further take the elastodynamic properties of the medium and the known volume sources to be constant in each elementary subdomain. The surface sources are taken to be piecewise constant in the elementary subdomains of the discretized outer boundary of computation.

4. THE EXPANSION FUNCTIONS

Our expansion functions that are used to locally represent the elastodynamic wave field quantities in an elementary subdomain of the configuration are the product of a function of the spatial coordinates and a function of time. In our discussion we shall concentrate on the expansion functions of spatial coordinates because in the spatial direction strong inhomogeneities may occur. We take polynomial expansion functions of degree one which are the lowest-degree polynomials by which physically non-existing surface source distributions on interfaces of discontinuity can numerically be avoided in the representations.

To obtain these linear interpolation functions, the spatial domain is discretized into a union of tetrahedra, the vertices of which coincide with the nodes of the discretization. The position vectors of the four vertices $\{P(0), P(1), P(2), P(3)\}$ of T are denoted by $\{x_i(0), x_i(1), x_i(2), x_i(3)\}$. The vectorial areas $\{A_i(0), A_i(1), A_i(2), A_i(3)\}$ of the faces of T that are directed along the outward normals to the faces of T are given by

(4.1) $A_i(0) = \varepsilon_{ijk}[x_j(1)x_k(2) + x_j(2)x_k(3) + x_j(3)x_k(1)]/2$, etc.

where ε_{ijk} is the completely antisymmetric unit tensor of rank three (Levi-Civita tensor). The volume of T is given by

(4.2) $V = \varepsilon_{ijk}[- x_i(0)x_j(1)x_k(2) + x_i(1)x_j(2)x_k(3) - x_i(2)x_j(3)x_k(0)$
$+ x_i(3)x_j(0)x_k(1)]/6.$

Let the position vector b_i of the barycenter of T be introduced through

(4.3) $b_i = (1/4)\sum_{I=0}^{3} x_i(I)$,

then the linear scalar interpolation function that equals unity when $x_i = x_i(I)$ for I=0,1,2,3, and equals zero in the remaining three vertices of T can be written as

(4.4) $\phi_i(I;x_i) = 1/4 - (3V)^{-1}(x_j-b_j)A_j(I)$.

Since $x_i = \sum_{I=0}^{3}x_i(I)\phi_i(I;x_i)$, with $\sum_{I=0}^{3}\phi(I;x_i) = 1$, the functions $\{\phi(I;x_i);$ $I=0,1,2,3\}$ are nothing but the barycentric coordinates in T.

For the local spatial interpolation of the elastodynamic wave field, we let ourselves guide by the same arguments as that have been used in [2] for the representation of three-dimensional electromagnetic fields, i.e., we want functions that guarantee the continuity of all field-components that are continuous across an interface, while leaving the non-continuous components free to jump by finite amounts. In the realm of elastodynamic wave fields this implies that we construct functions that automatically guarantee the continuity of all components of the particle velocity and the continuity of the normal components of the stress (i.e., the traction) across solid/solid interfaces of discontinuity, while it leaves the tangential components of the stress free to jump by finite amounts.

The local functions $\phi(I;x_i)$ are employed to interpolate the particle velocity v_r and the stress τ_{pq} in a tetrahedron. Let $\{v_r(I;t),\tau_{pq}(I;t);$ $I = 0,1,2,3\}$ denote the values of $\{v_r,\tau_{pq}\}$ when $x_i = x_i(I)$ is approached via the interior of T, then the local representations of $\{v_r,\tau_{pq}\}$ are

(4.5) $\{v_r(x_i;t),\tau_{pq}(x_i;t)\} = \sum_{I=0}^{3} \{v_r(I;t),\tau_r(I;t)\}\phi(I;x_i)$ for $x_i \in T$.

In case the face $A_i(J)$ of T coincides with a solid/solid interface, face interpolation is used for $\tau_{pq}(I;t)$, with $I \neq J$. We represent $\tau_{pq}(I;t)$ with respect to the local basis $\{-(3V)^{-1}[x_i(J)-x_i(I)]\}$, with $J \neq I$ as follows:

(4.6) $\tau_{pq}(I;t) = -(3V)^{-1}\sum_{J=0}^{3} T_p(I,J;t)[x_q(J)-x_q(I)]$,

in which $T_p(I,J;t) = \tau_{pq}(I;t)A_q(J)$. Apart from a factor, $T_p(I,J;t)$ is the traction on $A_q(J)$, with $J \neq I$, in $x_i(I)$.

For the global representation of the particle velocity we use nodal interpolation throughout D, with vector components along the axes of the background reference frame. With this the continuity of all components of the particle velocity across solid/solid interfaces is automatically guaranteed. For the global representation of the stress nodal interpolation is used in the subdomains of D where the medium properties vary continuously with position; near solid/solid interfaces, face interpolation is employed. In this way the continuity of the traction across solid/solid interfaces is automatically guaranteed, while the tangential components of the stress are free to jump by finite amounts.

5. CONCLUSIONS

The theory of a space-time finite-element method for the numerical computation of three-dimensional elastodynamic wave motions in bounded, time-invariant configurations that may be anisotropic and strongly inhomogeneous, is presented. It is shown that the finite-element method can be considered to be based on a space-time reciprocity theorem of the time-correlation type. Linear local spatial expansion functions for the representation of the particle velocity and the stress in an elementary subdomain of a discretized geometry are constructed that automatically guarantee the continuity requirements at interfaces of discontinuity in material properties in an elastic configuration while leaving non-continuous elastodynamic field components free to jump by finite amounts.

REFERENCES

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