# Simultaneous inversion of permittivity and conductivity employing a nonperturbative approach

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## ABSTRACT

We present a new inversion algorithm for the simultaneous reconstruction of permittivity and conductivity that recasts the nonlinear inversion as the solution of a coupled set of linear equations. The algorithm is iterative and proceeds through the minimization of two cost functions. At the initial step, the data is matched through the reconstruction of the radiating or minimum-norm scattering currents; subsequent steps refine the nonradiating scattering currents and the material properties inside the scatterer. We give recipes for constructing basis functions for the nonradiating components of the scattering currents. The method is illustrated through the reconstruction of a large constrast square cylinder from multiple-source measurements at a single frequency.

## 1. INTRODUCTION

Most practical algorithms for inverse scattering assume that the scatterer is a small perturbation of a known background medium. Included in this category are methods that use the ordinary or the distorted-wave Born approximation, either in a single, linear imaging step or in an iterative, non-linear search for the best model.[1] In general, all perturbative methods methods have difficulty when the perturbation is large; i.e., when the scatterer is large in size or when its material properties differ from those of the known medium, or starting model, by a large amount.[2]

This paper describes some improvements in an inverse scattering algorithm called the source-type integral equation (STIE) method, which is intended to deal with large perturbations. We will describe the method for electromagnetic scattering, but it applies to any scattering problem governed by the (scalar or vector) Helmholtz equation. Consider first a single scattering experiment, with one incident field at one frequency. Both the data and the field inside the scatterer can be *linearly* related to (ficticious) sources in the scatterer. In electromagnetics, these sources are scattering currents, which are the product of the field and the difference in material properties between the scatterer and an arbitrary background medium.[3] If the scattering currents could be computed directly from the data, then the field inside the scatterer could be computed from the scattering currents. Dividing the scattering currents by the field would give the material properties directly. Moreover, there is no apparent need with this method to assume that the contrast in material properties is small.

The direct use of the STIE method is frustrated by the non-uniqueness of the inverse source problem that must be solved to compute the scattering currents from the data. Inverse source problems are nonunique because of the existence of *nonradiating* sources.[4–5] In electromagnetics, nonradiating sources are current distributions that generate zero electric and magnetic field outside their domain of support. If the induced currents in a single scattering experiment have any nonradiating components, then data from the experiment contain no information about these components. It is therefore impossible, without other information, to reconstruct the full scattering currents and the field inside the scatterer. This problem can not be overcome by simply adding more experiments—different incident fields and frequencies—because the induced scattering currents change with the incident field and the frequency.

Nevertheless, it is known that inverse scattering problems with multiple experiments can have unique solutions.[6] The reason is that, although the induced currents change with each experiment, the unknown material properties do not change (The material properties certainly do not depend on the incident field, and often their frequency dependence can be given a simple explicit form.) The combination of different incident fields or different frequencies makes the inversion for the material properties a well-posed (but nonlinear) problem. The STIE method can be modified ("renormalized") to incorporate this feature.[2] The resulting algorithm is iterative, but appears to be robust and computationally efficient for large perturbations. An important part of the STIE method described here is the explicit construction and use of basis functions for the nonradiating part of the induced currents. Although these currents contribute nothing to the data, they are the key to establishing self-consistent total fields and material properties inside the scatterer. In fact, with the STIE method, the data is inverted only once for the radiating part of the induced currents; subsequent iterations refine just the nonradiating components.

## 2. BASIC EQUATIONS

We assume a two-dimensional (2-D) geometry which only sustains either a TE or a TM polarization. We will limit ourselves to the TE polarization with an electric line current source of unit strength, situated at $\bar{r}_s = x_s\hat{x} + z_s\hat{z}$ ($z_s < 0$). In this case, the nonvanishing field components are $H_x$, $H_z$ and $E_y$ which are functions of $x$ and $z$ only.

We assume that the unknown medium is embedded inside a known homogeneous background medium ($\epsilon_1$) and that the unknown region extends from $z = 0$ to $z = L$. Let

$$E(\bar{r}, \bar{r}_s) = E_o(\bar{r}, \bar{r}_s) + E_s(\bar{r}, \bar{r}_s), \tag{1}$$

where $\bar{r} = x\hat{x} + z\hat{z}$,

$$\nabla^2 E_o(\bar{r}, \bar{r}_s) + k_1^2 E_o(\bar{r}, \bar{r}_s) = i\omega\mu_o\, J_i(\bar{r}, \bar{r}_s), \tag{2}$$

$$\nabla^2 E_s(\bar{r}, \bar{r}_s) + k_1^2 E_s(\bar{r}, \bar{r}_s) = i\omega\mu_o\, J(\bar{r}, \bar{r}_s), \tag{3}$$

and $k_1^2 = \omega^2\mu_o\epsilon_1$.

Here $J_i(\bar{r}, \bar{r}_s)$ is the impressed current, $J_i(\bar{r}, \bar{r}_s) = \delta(\bar{r} - \bar{r}_s)$, and $J(\bar{r}, \bar{r}_s)$ is the induced scattering current inside the slab

$$J(\bar{r}, \bar{r}_s) = Q(\bar{r})E(\bar{r}, \bar{r}_s), \tag{4}$$

where

$$Q(\bar{r}) = i\omega\,[\epsilon(\bar{r}) - \epsilon_1]. \tag{5}$$

$Q$ is the unknown to be determined.

Equations (1)-(3) can be cast as an integral equation,

$$E(\bar{r},\bar{r}_s) = E_o(\bar{r},\bar{r}_s) + \int_0^L dz' \int_{-\infty}^{\infty} dx'\, g(\bar{r},\bar{r}')J(\bar{r}',\bar{r}_s). \tag{6}$$

Here $E_o(\bar{r},\bar{r}_s) = g(\bar{r},\bar{r}_s)$, where $g(\bar{r},\bar{r}')$ is the two-dimensional Green function

$$g(\bar{r},\bar{r}') = \frac{\omega\mu_o}{4} H_0^{(1)}\left(k_1|\bar{r}-\bar{r}'|\right). \tag{7}$$

Equation (6) can also be written in the spectral ($k_x$) domain by a Fourier transformation along the $x$-direction, which gives

$$\tilde{E}(k_x,z,\bar{r}_s) = \tilde{E}_o(k_x,z,\bar{r}_s) + \int_0^L dz'\, G(k_x,z,z')\tilde{J}(k_x,z',\bar{r}_s). \tag{8}$$

$G(k_x,z,z')$ is now the one-dimensional Green function,

$$G(k_x,z,z') = \frac{\omega\mu_o}{2k_{1z}}\, e^{ik_{1z}|z-z'|}, \tag{9}$$

where $k_{1z}^2 = k_1^2 - k_x^2$. In general, $k_{1z}$ is complex, $k_{1z} = k'_{1z} + ik''_{1z}$, with $k'_{1z} = \mathrm{Re}\{k_{1z}\}$ and $k''_{1z} = \mathrm{Im}\{k_{1z}\} \geq 0$. For any $k_x$, eq. (8) corresponds to an excitation by a plane wave whose wavenumber has the components $k_x$ along the $x$-axis and $k_{1z}$ along the $z$-axis.

## 3. INVERSE SOURCE AND SCATTERING PROBLEMS

Equation (6) (or equivalently 8) is the basic integral equation for the inverse scattering problem. It involves the unknown material property $Q$ through the scattering currents $J$ that are induced inside the scattering volume by the total electric field $E$ (eq. 4). The inverse scattering problem is to determine $Q$ from measurements of the field $E$ outside the scatterer, for different receiver positions $\bar{r}$, different source positions $\bar{r}_s$, and different frequencies $\omega$.

Viewed as equations for $Q$, eqs. (4) and (6) are nonlinear because the total field inside the scatterer, which multiplies $Q$ to give the scattering currents, also depends on $Q$. Inversion for $Q$ using the source-type integral equation treats the nonlinear inverse problem in two steps. The first step is the inversion of eq. (6) for the induced scattering currents $J$. Inversion for $J$ is a *linear* inverse source problem that only involves the fields recorded *outside* the scatterer. The second step involves calculation of a self-consistent total electric field $E$ and material property $Q$ *inside* the scatterer. In this section, we concentrate on the solution of the inverse source problem.

Inversion for $J$ is conveniently done in the $k_x$ domain with eq. (8),

$$\int_0^L dz'\, G(k_x,z=0,z')\tilde{J}(k_x,z',\bar{r}_s) = S(k_x,\bar{r}_s), \tag{10}$$

where the data $S$ is the scattered field recorded at the surface $z=0$,

$$S(k_x,\bar{r}_s) = \tilde{E}(k_x,z=0,\bar{r}_s) - \tilde{E}_o(k_x,z=0,\bar{r}_s). \tag{11}$$

This integral equation of the first kind for $J$ does not have a unique solution. A particular solution is easily found by substituting for $J$ in eq. (10) a function of the form

$$\tilde{J}_R(k_x, z, \bar{r}_s) = j(k_x, \bar{r}_s) f(k_x, z, \bar{r}_s), \tag{12}$$

and rearranging, which gives

$$j(k_x, \bar{r}_s) = \frac{S(k_x, \bar{r}_s)}{\int_0^L dz' \; G(k_x, z = 0, z') f(k_x, z', \bar{r}_s)} \,, \tag{13}$$

provided

$$\int_0^L dz' \; G(k_x, z = 0, z') f(k_x, z', \bar{r}_s) \neq 0 \,. \tag{14}$$

Thus, for any function $f(k_x, z, \bar{r}_s)$ satisfying eq. (14), eq. (10) has the solution

$$\tilde{J}_R(k_x, z, \bar{r}_s) = \frac{S(k_x, \bar{r}_s)}{\int_0^L dz' \; G(k_x, z = 0, z') f(k_x, z', \bar{r}_s)} f(k_x, z, \bar{r}_s) \,. \tag{15}$$

Since $f$ is arbitrary (aside from the restriction of eq. 14), there are infinitely many solutions to eq. (10); i.e., there are infinitely many source distributions $\tilde{J}_R$ that generate the data $S$. Also, since eq. (10) is linear, the difference of any two of its particular solutions is a source distribution that generates zero field at the measurement plane $z = 0$:

$$\int_0^L dz' \; G(k_x, z = 0, z') \tilde{J}_{NR}(k_x, z', \bar{r}_s) = 0 \,. \tag{16}$$

Such a source distribution $\tilde{J}_{NR}$ is called a *nonradiating source*. The set of all solutions to eq. (16) is called the annihilator of the kernel $G(k_x, z = 0, z')$. Clearly, an arbitrary linear combination of elements in the annihilator can be added to any particular solution of eq. (10), and the result will still satisfy eq. (10). Thus, the data $S(k_x, \bar{r}_s)$ contain no information about the part of the induced source distribution $\tilde{J}$ that belongs to the annihilator.

Although the solution to eq. (10) by itself is not unique, it is possible to pick out unique solutions by imposing additional conditions. An example is the solution that has a minimum $L_2$-norm, which is obtained by setting

$$f(k_x, z', \bar{r}_s) = G^*(k_x, z = 0, z'), \tag{17}$$

in eq. (15), where $*$ denotes complex conjugation. The minimum-norm solution is thus

$$\tilde{J}_{MN}(k_x, z', \bar{r}_s) = \frac{S(k_x, \bar{r}_s)}{\int_0^L dz'' \; |G(k_x, z = 0, z'')|^2} G^*(k_x, z = 0, z'). \tag{18}$$

$\tilde{J}_{MN}$ is orthogonal in the $L_2$-sense to all nonradiating sources; indeed,

$$\int_0^L dz\ \tilde{J}^*_{MN}(k_x, z, \bar{r}_s)\tilde{J}_{NR}(k_x, z, \bar{r}_s)$$

$$= \frac{S^*(k_x, \bar{r}_s)}{\int_0^L dz'\ |G(k_x, z=0, z')|^2} \int_0^L dz'\ G(k_x, z=0, z')\tilde{J}_{NR}(k_x, z', \bar{r}_s) \ \equiv 0$$

The minimum-norm solution can be viewed as an expansion of the source distribution $J_{MN}$ in the basis functions $G^*(k_x, z=0, z)$ (which are indexed by $k_x$). The scattering currents given by eq. (18) always lie in the space spanned by the $G^*(k_x, z=0, z)$. These basis functions can not, in general, fully represent the actual induced currents in the scatterer. The induced currents will usually have a nonradiating part $\tilde{J}_{NR}(k_x, z, \bar{r}_s)$, which must be included to solve the inverse scattering problem. Thus, the general solution to eq. (10) can be written

$$\tilde{J}(k_x, z, \bar{r}_s) = \tilde{J}_{MN}(k_x, z, \bar{r}_s) + \tilde{J}_{NR}(k_x, z, \bar{r}_s), \tag{19}$$

where $\tilde{J}_{NR}(k_x, z, \bar{r}_s)$ is a superposition of the solutions to eq. (16). The $L_2$-norm of this general solution is

$$\begin{aligned} ||\tilde{J}(k_x, z, \bar{r}_s)||^2 &= \langle \tilde{J}(k_x, z, \bar{r}_s), \tilde{J}(k_x, z, \bar{r}_s)\rangle = \int_0^L dz\ |\tilde{J}(k_x, z, \bar{r}_s)|^2 \\ &= ||\tilde{J}_{MN}(k_x, z, \bar{r}_s)||^2 + ||\tilde{J}_{NR}(k_x, z, \bar{r}_s)||^2, \end{aligned} \tag{20}$$

where the second equality comes from eq. (10) and shows explicitly that $\tilde{J}_{MN}(k_x, z, \bar{r}_s)$ is the minimum $L_2$-norm solution of the inverse source problem.

The minimum-norm solution has an interesting physical interpretation. In the spatial domain,

$$J_{MN}(\bar{r}, \bar{r}_s) = \int_{-\infty}^{\infty} dk_x\ e^{ik_x x} \frac{1}{N^2(k_x)} G^*(k_x, z=0, z) \left[\tilde{E}(k_x, z=0, \bar{r}_s) - \tilde{E}_o(k_x, z=0, \bar{r}_s)\right], \tag{21}$$

where

$$N^2(k_x) = \int_0^L dz'\ |G(k_x, z=0, z')|^2. \tag{22}$$

Equation (21) can be interpreted as the backpropagation of the scattered field from the measurement surface ($z = 0$) into the background medium.

## 4. THE ELECTRIC FIELD INSIDE THE SCATTERER

The electric field inside the slab can be decomposed into three terms—the incident field $E_o$, the field generated by the minimum-norm (or radiating) induced current distribution $E_{MN}$, and the field generated by the non-radiating induced current distribution $E_{NR}$:

$$\tilde{E}(k_x, z, \bar{r}_s) = \tilde{E}_o(k_x, z, \bar{r}_s) + \tilde{E}_{MN}(k_x, z, \bar{r}_s) + \tilde{E}_{NR}(k_x, z, \bar{r}_s).$$

Note that $\tilde{E}_{NR}$, the electric field of the nonradiating currents, will be non-zero *inside* the scatterer. Each of the above fields can be computed by operating on their current distributions with the Green function; specifically, for $E_{NR}$ and $E_{MN}$,

$$\tilde{E}_{NR}(k_x, z, \bar{r}_s) = \int_0^L dz'\, G(k_x, z, z')\tilde{J}_{NR}(k_x, z', \bar{r}_s), \tag{23}$$

$$\tilde{E}_{MN}(k_x, z, \bar{r}_s) = \frac{S(k_x, \bar{r}_s)}{N^2(k_x)} \int_0^L dz'\, G(k_x, z, z')G^*(k_x, z=0, z'). \tag{24}$$

## 5. CHOICE OF THE NONRADIATING CURRENTS

In the previous section, the current and electric fields inside the scatterer were related through the Green function $G$. These fields are also related by the constitutive equation involving the unknown $Q$:

$$Q(\bar{r};\omega)E(\bar{r},\bar{r}_s;\omega) = J(\bar{r},\bar{r}_s;\omega), \qquad \text{or} \qquad Q(\bar{r};\omega) = J(\bar{r},\bar{r}_s;\omega)/E(\bar{r},\bar{r}_s;\omega), \tag{25}$$

where the dependence on frequency $\omega$ is now explicit. In practice, of course, the fields will only be known at discrete frequencies, which later will be indicated by $\omega_j$. We now assume that $Q$ has the following specific frequency dependence,

$$Q(\bar{r},\omega) = i\omega\, \delta\epsilon(\bar{r}) - \delta\sigma(\bar{r}), \tag{26}$$

where $\epsilon(\bar{r})$ and $\sigma(\bar{r})$ do not depend on frequency. As discussed before, $E(\bar{r},\bar{r}_s;\omega)$ and $J(\bar{r},\bar{r}_s;\omega)$ can be split into different parts,

$$E(\bar{r},\bar{r}_s;\omega) = E_o(\bar{r},\bar{r}_s;\omega) + E_{MN}(\bar{r},\bar{r}_s;\omega) + E_{NR}(\bar{r},\bar{r}_s;\omega), \tag{27}$$

$$J(\bar{r},\bar{r}_s;\omega) = J_{MN}(\bar{r},\bar{r}_s;\omega) + J_{NR}(\bar{r},\bar{r}_s;\omega). \tag{28}$$

$J_{MN}$ and $E_{MN}$ are the minimum-norm current and its associated electric field, whereas $J_{NR}$ and $E_{NR}$ are the non-radiating current and its associated field. Finally, $E_o$ is the electric field in the background medium that is induced by the impressed source.

Equations (25)-(26) suggest a way of choosing the nonradiating sources in the inverse source problem (which has been embedded in the scattering problem by using the source-type integral equation). First, recall that $J_{MN}$, the minimun-norm current in eq. (28) is fixed by the data (see eq. 21); thus, its electric field inside the scatterer $E_{MN}$ is also fixed. Up till now, the non-radiating current $J_{NR}$ has only been constrained by the requirement that it give zero electric field at the measurement plane (16). There are of course infinitely many non-radiating current distributions. But eqs. (25) and (26) also imply that, inside the scatterer, the electric field $E_{NR}$ of the non-radiating currents, when added to $E_o$ and $E_{MN}$, should give a total electric field that divides the scattering current $J$ to produce a $Q$ that does not depend on the position of the source $\bar{r}_s$ and depends on frequency only through the factor $i\omega$ (eq. 26). Thus, eqs. (25)-(26) require that

$$\frac{\partial}{\partial x_s}\left[\frac{J(\bar{r},\bar{r}_s;\omega)}{E(\bar{r},\bar{r}_s;\omega)}\right] = 0, \tag{29}$$

$$\frac{\partial}{\partial \omega}\left[\mathrm{Re}\left\{\frac{J(\bar{r},\bar{r}_s;\omega)}{E(\bar{r},\bar{r}_s;\omega)}\right\}\right] = 0, \text{ and} \tag{30}$$

$$\frac{\partial}{\partial \omega}\left[\mathrm{Im}\left\{\frac{1}{\omega}\frac{J(\bar{r},\bar{r}_s;\omega)}{E(\bar{r},\bar{r}_s;\omega)}\right\}\right] = 0. \tag{31}$$

## 6. BASIS FUNCTIONS FOR NONRADIATING CURRENTS

To solve for the non-radiating part of the induced current using the criteria described above, we first expand the current in a suitable set of basis functions. We give two constructions for these basis functions. First, we construct a set of global basis functions out of elements that are individually radiating, but are combined in a way to produce zero field at the receiver locations. Second, we show how to construct a set of localized basis functions, each of which is individually non-radiating.

### 6.1. Invisible sources

In the first approach, we assume that there are $N$ receiver locations $\bar{r}_k$, $k = 1,\ldots,N$, and choose a set of $M$ radiating basis functions $\tilde{\psi}_i$, $i = 1,\ldots,M$, with $M > N$. Let $L = M - N$, and split these basis functions into two sets:

$$\tilde{\psi}_i^{(1)}(\bar{r}),\ i = 1,\ldots,N, \qquad \text{and} \qquad \tilde{\psi}_i^{(2)}(\bar{r}),\ i = 1,\ldots,L.$$

Let $\tilde{\phi}_i$ be the electric field generated by each basis function,

$$\tilde{\phi}_i^{(1,2)}(\bar{r};\omega) = \int d\bar{r}'\, g(\bar{r},\bar{r}';\omega)\,\tilde{\psi}_i^{(1,2)}(\bar{r}'),$$

where the dependence of the Green function (and thus $\tilde{\phi}$) on frequency $\omega$ is now explicit. Next, form the $N \times N$ and $N \times L$ matrices $\Phi^{(1)}$ and $\Phi^{(2)}$, whose elements are given by

$$\Phi_{ki}^{(1)} = \tilde{\phi}_i^{(1)}(\bar{r}_k;\omega) = \int d\bar{r}'\, g(\bar{r}_k,\bar{r}';\omega)\,\tilde{\psi}_i^{(1)}(\bar{r}'), \qquad k = 1,\ldots,N, \quad i = 1,\ldots,N\,, \tag{32}$$

$$\Phi_{ki}^{(2)} = \tilde{\phi}_i^{(2)}(\bar{r}_k;\omega) = \int d\bar{r}'\, g(\bar{r}_k,\bar{r}';\omega)\,\tilde{\psi}_i^{(2)}(\bar{r}'), \qquad k = 1,\ldots,N, \quad i = 1,\ldots,L\,. \tag{33}$$

Thus, the $ki$-th element of the matrix $\Phi^{(1)}$ ($\Phi^{(2)}$) is just the electric field at the $k-$th receiver location that is generated by the $i-$th basis function in the first (second) set of basis functions.

If the basis functions are chosen properly, the $N \times N$ matrix $\Phi^{(1)}$ will have an inverse, which we denote by $\Phi^{-(1)}$. Define new basis functions, $\psi_i$, $i = 1, \ldots, L$, as the following linear combination of the original basis functions:

$$\psi_i(\bar{r}, \omega) = \tilde{\psi}_i^{(2)}(\bar{r}) - \sum_n \sum_m \tilde{\psi}_n^{(1)}(\bar{r}) \, \Phi_{nm}^{-(1)} \, \Phi_{mi}^{(2)} . \tag{34}$$

Here the dependence on $\omega$ enters through the matrices $\Phi^{(1)}$ and $\Phi^{(2)}$. It is easy to show that these new basis functions give zero electric field at the receiver locations $\bar{r}_k$. We call a source constructed in this way an "invisible" source, because its electric field is not visible (i.e., is zero) only at the discrete set of receiver locations. Its field can and usually will be non-zero at other locations outside the scatterer. Nonradiating sources, which radiate zero field everywhere outside their domain of support, are a subset of the invisible sources.

With the above method, we can construct $L = M - N$ non-radiating basis functions and use them to expand the nonradiating part of the current distribution,

$$J_{NR}(\bar{r}, \bar{r}_s; \omega_j) = \sum_i a_i^{(sj)} \, \psi_i(\bar{r}; \omega_j), \tag{35}$$

with coefficients $a_i^{(sj)}$. The superscript indicates that in the inverse scattering problem these coefficients can depend both on a set of source positions, indexed by $s$, and a set of frequencies, indexed by $j$. The coefficients will be chosen to satisfy the criteria on $J_{NR}$ described in the previous section. Finally, note that each of the new basis functions generates a (non-zero) electric field $\phi_i$ inside the scatterer,

$$\phi_i(\bar{r}; \omega_j) = \int d\bar{r}' \, g(\bar{r}, \bar{r}'; \omega_j) \, \psi_i(\bar{r}'; \omega_j); \tag{36}$$

thus

$$E_{NR}(\bar{r}, \bar{r}_s; \omega_j) = \sum_i a_i^{(sj)} \, \phi_i(\bar{r}; \omega_j). \tag{37}$$

### 6.2. Nonradiating sources

We now describe a second way of constructing basis functions for $J_{NR}$ that is based on a general method of constructing nonradiating sources. These basis functions will be strictly nonradiating (i.e., their electric field will vanish identically outside their domain of support). They are thus less general than the invisible basis functions described above, but they will have some other convenient properties.

Let $\phi(\bar{r})$ be a continuous function of $\bar{r}$ with continuous first partial derivatives and let $\phi(\bar{r})$ vanish outside some domain $V$. Then the source distribution $\psi$, where

$$\psi(\bar{r}; \omega) = -\left[\nabla^2 \phi(\bar{r}) + k_1^2 \phi(\bar{r})\right], \tag{38}$$

is nonradiating,[4]

$$\int d\bar{r}' \, g(\bar{r}, \bar{r}'; \omega) \, \psi(\bar{r}'; \omega) = 0, \qquad \bar{r} \notin V. \tag{39}$$

This property is easily proved by substituting eq. (38) for $\psi$ into the integral, integrating by parts with Green's theorem, and using the fact that $\phi$ vanishes outside $V$. Equations (38)-(39) immediately suggest the following method of generating non-radiating basis functions for the source-type integral equation. Let $\phi_i(\bar{r})$, $i = 1, \ldots, L$, be a linearly independent set of functions that have continuous first partial derivatives and that vanish outside the scatterer. Generate the set of basis functions $\psi_i(\bar{r};\omega)$, $i = 1, \ldots, L$, by

$$\psi_i(\bar{r};\omega) = -\left[\nabla^2\phi_i(\bar{r}) + k_1^2\phi_i(\bar{r})\right]. \tag{40}$$

The $\psi_i$ are then suitable basis functions for the expansion

$$J_{NR}(\bar{r},\bar{r}_s;\omega_j) = \sum_i a_i^{(sj)}\, \psi_i(\bar{r};\omega_j).$$

Although these nonradiating basis functions are less general than the nonvisible basis functions, each function $\psi_i(\bar{r};\omega)$ has the property that its electric field is just $\phi_i(\bar{r})$; viz., the original function that was operated on in eq. (40) to generate $\psi_i$. Clearly, this property can save computations, compared to (36). Also, it is easy to construct functions $\phi_i$ that are non-zero only in a small sub-domain of the scatterer. Use of such local basis functions would then resemble a finite-element representation of $E_{NR}$ and $J_{NR}$.

There are several ways to construct local basis functions with the required properties; we give an example based on piecewise polynomial ("cubic Bessel") interpolation.[7] Let the domain of the scatterer be covered with a uniform rectangular grid,

$$\bar{r}_{mn} = (x_m, z_n), \qquad x_m = x_o + mh, \;\; z_n = z_o + nh,$$

where $m = 0, \ldots, M$, and $n = 0, \ldots, N$. A function $\phi_{mn}(\bar{r})$ will be associated with each node as the product,

$$\phi_{mn}(\bar{r}) = \phi(x - x_m)\phi(z - z_n), \tag{41}$$

where $\phi$ is the piecewise cubic polynomial,

$$\phi(y) = \varphi_0(y) + \frac{1}{2h}\left[\varphi_1(y - h) - \varphi_1(y + h)\right],$$

with

$$\varphi_0(y) = \begin{cases} (|y/h| - 1)^2\,(2|y/h| + 1)\,, & -h < y < h\,, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi_1(y) = \begin{cases} y\,(|y/h| - 1)^2\,, & -h < y < h\,, \\ 0 & \text{otherwise.} \end{cases}$$

The piecewise cubic polynomial $\phi$ is continuous with continuous first derivative. Its support is the interval $-2h < y < 2h$; $\phi(0) = 1$, $\phi(-h) = \phi(h) = 0$, whereas $\phi'(0) = 0$, $\phi'(-h) = -\phi'(h) = 1$.

Thus, an expansion of the form

$$f(y) = \sum_i f_i \, \phi(y - y_i),$$

where $y_i = ih$, is continuous with continuous first derivative; its values at the nodes $y_i$ are the sample values, $f(y_i) = f_i$; and its derivatives at the nodes are the central-difference approximations, $f'(y_i) = (f_{i+1} - f_{i-1})/(2h)$. The functions $\phi_{mn}$ have similar properties in two dimensions. These functions then generate nonradiating basis functions through

$$\psi_{mn}(\bar{r};\omega) = -\left[\nabla^2\phi_{mn}(\bar{r}) + k_1^2\phi_{mn}(\bar{r})\right], \tag{42}$$

and lead to the expansion

$$J_{NR}(\bar{r},\bar{r}_s;\omega_j) = \sum_{mn} a_{mn}^{(sj)} \, \psi_{mn}(\bar{r};\omega_j). \tag{43}$$

The coefficients $a_{mn}$ now are just the values of $E_{NR}$ at the nodes $\bar{r}_{mn}$. This construction for rectangular domains can be extended to general domains with finite-element methods. Note, finally, that the nodes can be numbered with a single index $i$ so that the expansion (43) resembles (35).

## 7. INVERSION ALGORITHM

To implement the inversion for the non-radiating part of the source distribtuion and, ultimately, the inversion for $Q$, we define two cost functions. The first cost function is designed to incorporate conditions (29)-(31). It accounts for the variation of the current $J$ and the electric field $E$ with both source position and frequency, while allowing the material property $Q$ to be independent of source position and to have a simple frequency dependence:

$$\begin{aligned} C_1(Q',Q'') &= \sum_{s,j} |J(\bar{r},\bar{r}_s;\omega_j) - Q(\bar{r};\omega_j)E(\bar{r},\bar{r}_s;\omega_j)|^2 \\ &= \sum_{s,j} \left|J(\bar{r},\bar{r}_s;\omega_j) - [Q'(\bar{r}) + i\omega_j Q''(\bar{r})]E(\bar{r},\bar{r}_s;\omega_j)\right|^2, \end{aligned} \tag{44}$$

where $Q'(\bar{r}) = -\delta\sigma(\bar{r})$ and $Q''(\bar{r}) = \delta\epsilon(\bar{r})$.

The second cost function accounts for the spatial variation of $J$, $E$, and $Q$ inside the scatterer:

$$\begin{aligned} C_2(a_i^{(sj)}) &= \int d\bar{r} \, |J(\bar{r},\bar{r}_s;\omega_j) - Q(\bar{r};\omega_j)E(\bar{r},\bar{r}_s;\omega_j)|^2 \\ &= \int d\bar{r} \, \Bigg| J_{MN}(\bar{r},\bar{r}_s;\omega_j) + \sum_i a_i^{(sj)}\psi_i(\bar{r};\omega_j) \\ &\qquad -Q(\bar{r};\omega_j)\left[E_o(\bar{r},\bar{r}_s;\omega_j) + E_{MN}(\bar{r},\bar{r}_s;\omega_j) + \sum_i a_i^{(sj)}\phi_i(\bar{r};\omega_j)\right]\Bigg|^2 . \end{aligned} \tag{45}$$

The integration above is over the domain of the scatterer. The second equality comes from decomposing $J$ and $E$ into their different parts and using the expansions (35) and (37) for $J_{NR}$ and $E_{NR}$..

Minimization of the first cost function by varying $Q$ gives

$$Q(\bar{r};\omega_j) = \frac{\sum_{s,i} \mathrm{Re}\{J(\bar{r},\bar{r}_s;\omega_i)E^*(\bar{r},\bar{r}_s;\omega_i)\}}{\sum_{s,i} |E(\bar{r},\bar{r}_s;\omega_i)|^2} + i\omega_j \frac{\sum_{s,i} \omega_i \,\mathrm{Im}\{J(\bar{r},\bar{r}_s;\omega_i)E^*(\bar{r},\bar{r}_s;\omega_i)\}}{\sum_{s,i} \omega_i^2 \,|E(\bar{r},\bar{r}_s;\omega_i)|^2} . \quad (46)$$

This formula for $Q(\bar{r};\omega_j)$ uses all the data—all source positions, receiver positions (in the computation of the backpropagated fields), and frequencies. The current $J$ and electric field $E$ on the right hand side include the non-radiating part of the current distribution and its associated electric field inside the scatterer, which depend on the expansion coefficients $a_i^{(sj)}$ and have not yet been fixed.

These coefficients are chosen to minimize the second cost function. Minimization of $C_2$ by varying $a_i^{(sj)}$ gives the following matrix equation for each source position $\bar{r}_s$ and frequency $\omega_j$,

$$\sum_i B_{li}(\omega_j)\, a_i^{(sj)} = -c_l(\bar{r}_s;\omega_j). \quad (47)$$

Here $B(\omega_j)$ is a Hermitian matrix with elements

$$B_{li}(\omega_j) = \int d\bar{r}\, [\phi_l^*(\bar{r};\omega_j) - Q^*(\bar{r};\omega_j)\phi_l^*(\bar{r};\omega_j)][\psi_i(\bar{r};\omega_j) - Q(\bar{r};\omega_j)\phi_i(\bar{r};\omega_j)], \quad (48)$$

$a$ is a vector of the expansion coefficients $a_i^{(sj)}$, and $c$ is the vector with elements

$$\begin{aligned} c_l(\bar{r}_s;\omega_j) \;=\; & \int d\bar{r}\; \{J_{MN}(\bar{r},\bar{r}_s;\omega_j) - Q(\bar{r},\bar{r}_s;\omega_j)[E_o(\bar{r},\bar{r}_s;\omega_j) + E_{MN}(\bar{r},\bar{r}_s;\omega_j)]\} \\ & \times [\psi_l^*(\bar{r};\omega_j) - Q^*(\bar{r};\omega_j)\phi_l^*(\bar{r};\omega_j)] . \end{aligned} \quad (49)$$

The matrix $B(\omega_j)$ depends only on the frequencies of operation $\omega_j$; thus, the use of different source positions at the same frequency does not involve much extra computation. Also if localized basis functions are used, then $B(\omega_j)$ is sparse.

Ideally, the two cost functions would be minimized simultaneously by varying both $Q$ and $a_i^{(sj)}$, but this simultaneous minimization is a nonlinear problem. The minimization by varying $Q$ and $a_i^{(sj)}$ sequentially suggests the following iterative scheme for the full nonlinear minimization:

Step (0): Approximate $J(\bar{r},\bar{r}_s;\omega_j)$ and $E(\bar{r},\bar{r}_s;\omega_j)$ by

$$J(\bar{r},\bar{r}_s;\omega_j) = J_{MN}(\bar{r},\bar{r}_s;\omega_j) ,$$

$$E(\bar{r},\bar{r}_s;\omega_j) = E_o(\bar{r},\bar{r}_s;\omega_j) + E_{MN}(\bar{r},\bar{r}_s;\omega_j) .$$

Step (1): Invert for $Q(\bar{r};\omega_j)$ from eq. (46) employing all data available.

Step (2): Invert for the coefficients $a_i^{(sj)}$ from eq. (47).

Step (3): Compute the total induced current $J(\bar{r}, \bar{r}_s; \omega_j)$ and the internal electric field $E(\bar{r}, \bar{r}_s; \omega_j)$ and return to Step (1).

The iteration is continued until changes in the profile are less than some specified tolerance.

## 8. RESULTS

Figure 1 shows a model problem that we have used to test the inversion algorithm described above. The scatterer (in black) is a 2-D square cylinder of dimensions $\lambda \times \lambda$, where $\lambda$ is the wavelength of the background medium (experiments were simulated at a single frequency). The contrast in material properties is 1:5. Synthetic data for an electric line source excitation of the medium were generated by the method of moments. The data are the $y$-component of the electric field at five positions equally spaced out to a distance $\lambda$ from the transmitter. The transmitter-receiver array is moved over a distance $4\lambda$, centered about the scatterer, in 40 steps.

The hatched region in Figure 1 shows the "probed medium," i.e., the region where the material properties were allowed to vary in the inversion. The probed medium is divided into a grid of $7\times7$ points. The unknowns are the permittivity and conductivity (expressed as the loss tangent) of the probed medium. The inversion uses non-visible basis functions constructed from sinc functions. Except for numerical error, the data were noise-free.

Figure 2 shows the reconstructions after the zeroth, first, and last (tenth) iteration of the algorithm. The smooth reconstruction at the zeroth step is obtained from just the minimum-norm inversion for the scattering currents. The reconstruction after the first step already reasonably defines the scatterer. Further iterations sharper the picture somewhat; the final result is a spatially filtered version of the actual model. This filtering is characteristic of any inverse scattering problem that operates at a single frequency.

## 9. CONCLUSIONS

The STIE method described here casts the nonlinear inverse scattering problem as the solution of a coupled set of linear equations. The method is iterative, but does not proceed by matching the data better at successive steps. Instead, the data is matched at the initial step through the solution of an inverse source problem for the minimum-norm or radiating part of the scattering currents. The iterative part of the algorithm then seeks total fields and material properties inside the scatterer that are consistent with the incident field and the minimum-norm scattering currents. The iterations proceed by adjusting the material properties and the nonradiating components of the scattering currents. A key part of the algorithm is the explicit construction of basis functions for the nonradiating (or invisible) currents. We have illustrated the method through the simultaneous inversion of permittivity and conductivity in a simple 2-D problem.

Several issues still must be addressed with STIE method. First is the sensitivity of the method to noise in the data and to the number of unknowns that are used in the solutions for both the minimum-norm and nonradiating currents. Next is the sufficiency of the constraints that are imposed on the nonradiating sources to provide a unique reconstruction. Finally, there is need for an estimate of the rate of convergence of the method for large contrasts.

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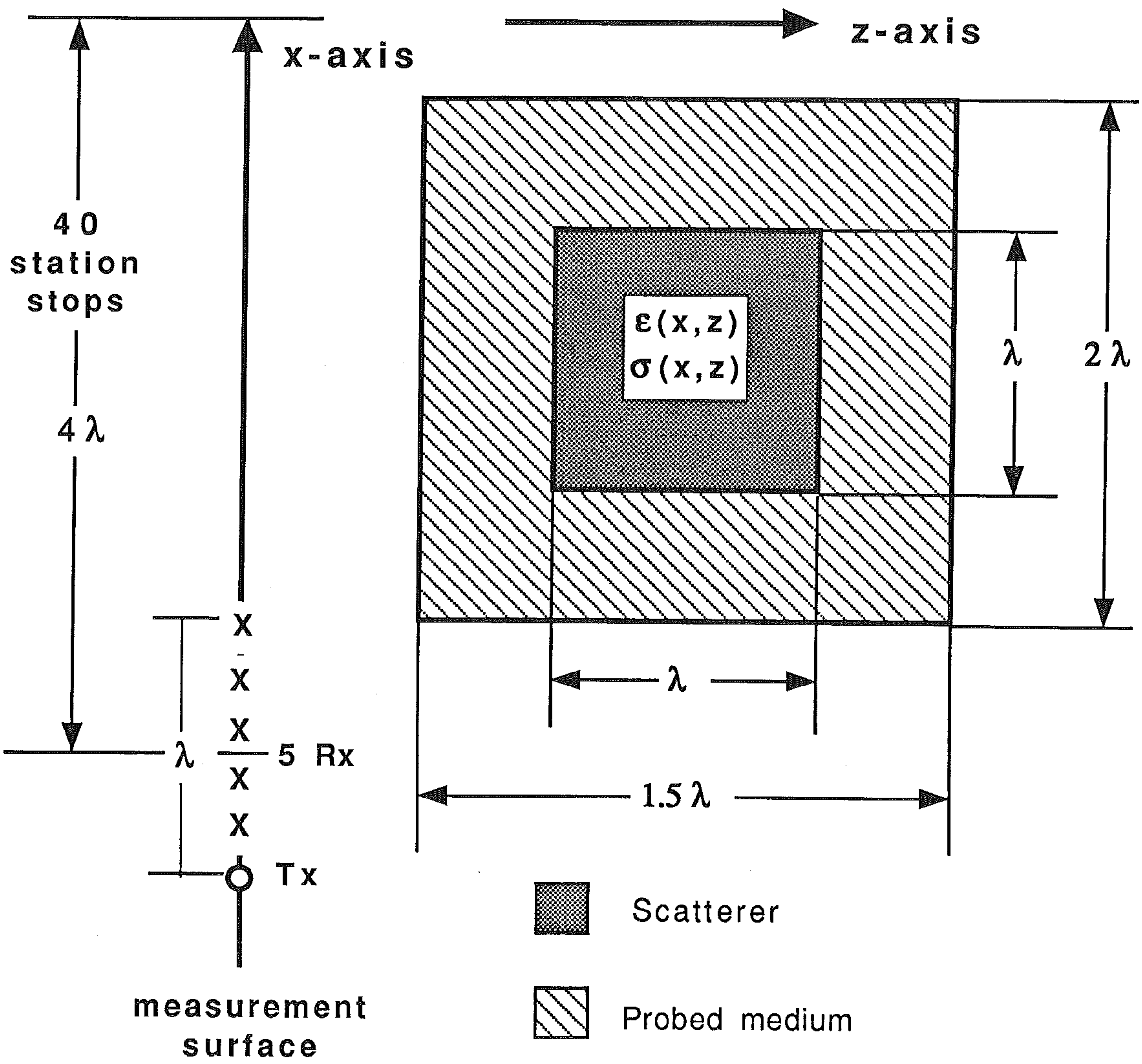


Figure 1. Schematic of the Inversion Problem

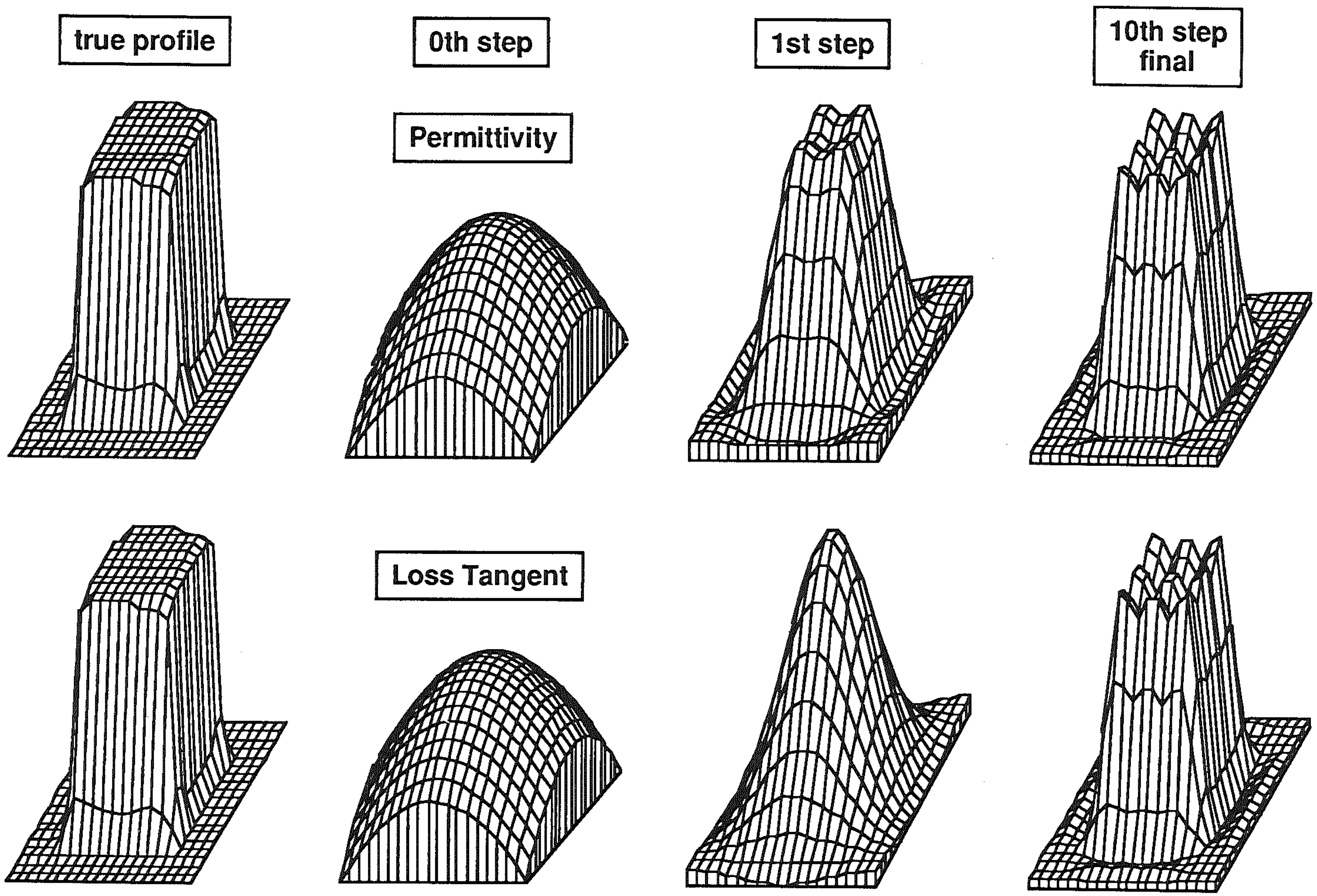


Figure 2. Reconstructed permittivity and conductivity maps.
Contrast = 1:5, number of grid points = 7 x7 .