Similarity analysis of the elastic wave motion in a dissipative solid under a global relaxation law

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A similarity analysis is carried out on the elastic wave motion in a dissipative solid under a global relaxation law. The solid is taken to be arbitrarily inhomogeneous and anisotropic. Global relaxation laws consisting of the combination of a frictional-force and a viscosity type are assumed to apply to the inertia and the compliance properties of the solid, i.e., the time derivatives in the first-order coupled wave equations are replaced by a relaxation operator with a constant coefficient. It is shown how in such a dissipative solid the elastic wave motion generated by arbitrary force type or deformation-rate type source distributions with bounded support is related to the wave motion in the lossless counterpart of the configuration. The similarity relation is based upon the Schouten-Van der Pol theorem in the theory of the one-sided Laplace transformation. As such, the similarity analysis generalizes earlier results obtained by Chao and Achenbach.

INTRODUCTION

The mathematical modeling of elastic wave phenomena is usually, in the first instance, carried out under the assumption of negligibly small losses in the solid in which the waves propagate. In practice, however, often attenuation and dispersion in the wave phenomena are observed. These must be attributed to some, in many cases as yet unknown, dissipation mechanism. Now, for each particular dissipation mechanism to be considered in the mathematical description of the wave propagation, the calculations or computations have to be carried out anew. Especially if one is still at the initial stage of investigating whether or not a particular dissipation mechanism can explain the observed attenuation and dispersion phenomena, this is an awkward situation. Out of this situation the question arose whether there exists some dissipation mechanism, with possibly one or two adjustable parameters, in the presence of which the propagated waves could be evaluated by some simple additional operation to be carried out on the already obtained answer for the lossless case. In the present contribution this is shown to be the case for a particular global relaxation law.

The solid under consideration is taken to be arbitrarily inhomogeneous and anisotropic. Global relaxation laws consisting of the combination of a frictional-force and a viscosity type are assumed to apply to the inertia and the compliance properties of the solid, respectively, i.e., the time derivatives in the first-order coupled elastic wave equations are replaced by relaxation operators with constant coefficients. It is shown how in such a dissipative solid the elastic wave motion generated by arbitrary force type or deformation-rate type source distributions with bounded support is related to the wave motion in the lossless counterpart of the configuration. The similarity relation is based upon the Schouten-Van der Pol theorem (Schouten 1934, Van der Pol 1934, Van der Pol and Bremmer 1950, Van der Pol 1960, Schouten 1961) in the theory of the one-sided Laplace transformation. As such, the similarity analysis generalizes results obtained by Chao and Achenbach (1964).

BASIC EQUATIONS FOR ELASTIC WAVE MOTION

The basic equations governing the linearized elastic wave motion generated by arbitrary sources in an inhomogeneous, anisotropic, dissipative solid are

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau_{p,q} + \partial_t C_t(\mu_{k,r}, v_r; \mathbf{x}, t) = f_k, \quad (1)$$

$$\Delta^{+}_{i,j,n,r}\partial_n v_r - \partial_t C_t(\chi_{i,j,p,q}, \tau_{p,q}; \mathbf{x}, t) = h_{i,j}. \quad (2)$$

Here, $\mathbf{x} = \{x_1, x_2, x_3\}$ are the Cartesian coordinates of a point of observation in three-dimensional Euclidean space $\mathcal{R}^3$, t is the time coordinate ($t \in \mathcal{R}$), ∂_m denotes differentiation with respect to x_m, ∂_t is a reserved symbol denoting differentiation with respect to t, and C_t denotes time convolution:

$$C_t(f, g; \mathbf{x}, t) = \int_{t'\in\mathcal{R}} f(\mathbf{x}, t')g(\mathbf{x}, t-t')\mathrm{d}t'. \quad (3)$$

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The quantities in Eqs (1) - (2) have the following meaning:

- $\tau_{p,q}$ = stress,
- v_r = particle velocity,
- f_k = volume source density of force,
- $h_{i,j}$ = volume source density of deformation rate,
- $\mu_{k,r}$ = inertia relaxation function,
- $\chi_{i,j,p,q}$ = compliance relaxation function.

Further,

$$\Delta^+_{i,j,p,q} = (1/2)(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}) \tag{4}$$

is the symmetric unit tensor of rank four and $\delta_{i,j}$ is the symmetric unit tensor of rank two (Kronecker tensor: $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \neq j$). The subscript notation of Cartesian vectors and tensors is used and the summation convention (with subscript range 1:3) applies. The causality of the reaction of the medium entails the causality property

$$\mu_{k,r}(\mathbf{x},t) = 0 \quad \text{and} \quad \chi_{i,j,p,q}(\mathbf{x},t) = 0 \quad \text{for } t < 0 \text{ and all } \mathbf{x}. \tag{5}$$

For a *lossless* solid, we have

$$\mu_{k,r}(\mathbf{x},t) = \rho_{k,r}(\mathbf{x})\delta(t), \tag{6}$$

$$\chi_{i,j,p,q}(\mathbf{x},t) = S_{i,j,p,q}(\mathbf{x})\delta(t), \tag{7}$$

where $\delta(t)$ is the one-dimensional Dirac distribution operative at $t = 0$, and

- $\rho_{k,r}$ = volume density of mass,
- $S_{i,j,p,q}$ = compliance.

For a solid with *frictional-force/viscosity* losses, we have

$$\mu_{k,r}(\mathbf{x},t) = \rho_{k,r}(\mathbf{x})\delta(t) + K_{k,r}(\mathbf{x})H(t), \tag{8}$$

$$\chi_{i,j,p,q}(\mathbf{x},t) = S_{i,j,p,q}(\mathbf{x})\delta(t) + \Gamma_{i,j,p,q}(\mathbf{x})H(t), \tag{9}$$

where $H(t)$ is the Heaviside unit step function ($H(t) = \{0, 1/2, 1\}$ for $\{t < 0, t = 0, t > 0\}$), and

- $K_{k,r}$ = coefficient of frictional force,
- $\Gamma_{i,j,p,q}$ = coefficient of viscosity.

In any subdomain of the configuration where the constitutive relaxation functions $\{\mu_{k,r}, \chi_{i,j,p,q}\}$ and the volume source densities $\{f_k, h_{i,j}\}$ are continuous, the acoustic wave quantities $\{\tau_{p,q}, v_r\}$ are continuously differentiable. Across an interface of discontinuity in mechanical properties where the solids on the two sides are in rigid contact, the elastic wave quantities satisfy the boundary conditions of the continuity type

$$\Delta^+_{k,m,p,q}\nu_m\tau_{p,q} \quad \text{and} \quad v_r \quad \text{continuous across interface,} \tag{10}$$

where ν_r is the unit vector along the normal to the interface. On the boundary surface of an elastodynamically impenetrable object, either of the following boundary conditions holds:

$$\lim_{h\downarrow 0} \Delta^+_{k,m,p,q}\nu_m\tau_{p,q}(\mathbf{x} + h\boldsymbol{\nu}, t) = 0 \tag{11}$$

on the boundary of a *void*, or

$$\lim_{h\downarrow 0} v_r(\mathbf{x} + h\boldsymbol{\nu}, t) = 0 \tag{12}$$

on the boundary of an *immovable perfectly rigid object*, where $\boldsymbol{\nu}$ is the unit vector along the outward normal to the boundary of the impenetrable object.

We assume that the sources start to act at the instant $t = 0$. The elastic wavefield that is causally related to the action of the sources then vanishes throughout the configuration for $t < 0$.

Through the point-source solutions (Green's functions) of the elastic wave problem, the wavefield quantities can be expressed in terms of the source densities. Let the latter have the bounded support $\mathcal{D}$, then

$$\begin{aligned} -\tau_{p,q}(\mathbf{x}',t) &= \int_{\mathbf{x}\in\mathcal{D}} [\mathcal{C}_t(G^{\tau f}_{p,q,k}, f_k; \mathbf{x}', \mathbf{x}, t) \\ &+ \mathcal{C}_t(G^{\tau h}_{p,q,i,j,}, h_{i,j}; \mathbf{x}', \mathbf{x}, t)]\mathrm{d}V, \end{aligned} \tag{13}$$

$$\begin{aligned} v_r(\mathbf{x}',t) &= \int_{\mathbf{x}\in\mathcal{D}} [\mathcal{C}_t(G^{vf}_{r,k}, f_k; \mathbf{x}', \mathbf{x}, t) \\ &+ \mathcal{C}_t(G^{vh}_{r,i,j}, h_{i,j}; \mathbf{x}', \mathbf{x}, t)]\mathrm{d}V, \end{aligned} \tag{14}$$

in which $\{G^{\tau f}_{p,q,k'}, G^{vf}_{r,k'}\} = \{G^{\tau f}_{p,q,k'}, G^{vf}_{r,k'}\}(\mathbf{x}', \mathbf{x}, t)$ satisfy the system of equations

$$-\Delta^+_{k,m,p,q}\partial'_m G^{\tau f}_{p,q,k'} + \partial_t \mathcal{C}_t(\mu_{k,r}, G^{vf}_{r,k'}; \mathbf{x}', \mathbf{x}, t)$$

$$= \delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}, t), \tag{15}$$

$$\Delta^+_{i,j,n,r}\partial'_n G^{vf}_{r,k'} - \partial_t \mathcal{C}_t(\chi_{i,j,p,q}, G^{\tau f}_{p,q,k'}; \mathbf{x}', \mathbf{x}, t)$$

$$= 0, \tag{16}$$

with a point source of force, and $\{G^{\tau h}_{p,q,i',j'}, G^{vh}_{r,i',j'}\} = \{G^{\tau h}_{p,q,i',j'}, G^{vh}_{r,i',j'}\}(\mathbf{x}', \mathbf{x}, t)$ satisfy the system of equations

$$-\Delta^+_{k,m,p,q}\partial'_m G^{\tau h}_{p,q,i',j'} + \partial_t \mathcal{C}_t(\mu_{k,r}, G^{vh}_{r,i',j'}; \mathbf{x}', \mathbf{x}, t)$$

$$= 0, \tag{17}$$

$$\Delta^{+}_{i,j,n,r}\partial'_n G^{vh}_{r,i',j'} - \partial_t \mathcal{C}_t(\chi_{i,j,p,q}, G^{\tau h}_{p,q,i',j'}; \mathbf{x}', \mathbf{x}, t)$$

$$= \Delta^{+}_{i,j,i',j'}\delta(\mathbf{x}' - \mathbf{x}, t), \tag{18}$$

with a point source of deformation rate. In these equations ∂'_m means differentiation with respect to x'_m and $\delta(\mathbf{x}' - \mathbf{x}, t)$ is the four-dimensional Dirac distribution operative at $\mathbf{x}' = \mathbf{x}$ and $t = 0$.

THE COMPLEX FREQUENCY DOMAIN ELASTIC WAVE EQUATIONS

The key issue of the similarity analysis to be carried out is found in the time Laplace transform domain or complex frequency domain. Therefore, we need the complex frequency domain counterparts of the equations of the previous section. With

$$\hat{v}_r(\mathbf{x}, s) = \int_{t=0}^{\infty} \exp(-st) v_r(\mathbf{x}, t) \mathrm{d}t \tag{19}$$

and similar relations for the other quantities, together with the properties $\hat{\partial}_t = s$ for zero-value initial conditions and

$$\hat{\mathcal{C}}_t(f, g; \mathbf{x}, t) = \hat{f}(\mathbf{x}, s)\hat{g}(\mathbf{x}, s), \tag{20}$$

Eqs (1) and (2) change into

$$-\Delta^{+}_{k,m,p,q}\partial_m \hat{\tau}_{p,q} + \hat{\zeta}_{k,r}\hat{v}_r = \hat{f}_k, \tag{21}$$

$$\Delta^{+}_{i,j,n,r}\partial_n \hat{v}_r - \hat{\eta}_{i,j,p,q}\hat{\tau}_{p,q} = \hat{h}_{i,j}, \tag{22}$$

where

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\hat{\mu}_{k,r}(\mathbf{x}, s), \tag{23}$$

$$\hat{\eta}_{i,j,p,q}(\mathbf{x}, s) = s\hat{\chi}_{i,j,p,q}(\mathbf{x}, s), \tag{24}$$

while Eq (10) changes into

$$\Delta^{+}_{k,m,p,q}\nu_m \hat{\tau}_{p,q} \quad \text{and} \quad \hat{v}_r \quad \text{continuous across interface,} \tag{25}$$

and Eqs (11) and (12) into

$$\lim_{h \downarrow 0} \nu_m \hat{\tau}_{p,q}(\mathbf{x} + h\boldsymbol{\nu}, s) = 0 \tag{26}$$

on the boundary of a *void* and

$$\lim_{h \downarrow 0} \hat{v}_r(\mathbf{x} + h\boldsymbol{\nu}, s) = 0, \tag{27}$$

on the boundary of an *immovable perfectly rigid object*, respectively.

Further, Eqs (13) and (14) change into

$$-\hat{\tau}_{p,q}(\mathbf{x}', s) = \int_{\mathbf{x}\in\mathcal{D}} [\hat{G}^{\tau f}_{p,q,k}(\mathbf{x}', \mathbf{x}, s)\hat{f}_k(\mathbf{x}, s)$$

$$+ \hat{G}^{\tau h}_{p,q,i,j}(\mathbf{x}', \mathbf{x}, s)\hat{h}_{i,j}(\mathbf{x}, s)]\mathrm{d}V, \tag{28}$$

$$\hat{v}_r(\mathbf{x}', s) = \int_{\mathbf{x}\in\mathcal{D}} [\hat{G}^{vf}_{r,k}(\mathbf{x}', \mathbf{x}, s)\hat{f}_k(\mathbf{x}, s)$$

$$+ \hat{G}^{vh}_{r,i,j}(\mathbf{x}', \mathbf{x}, s)\hat{h}_{i,j}(\mathbf{x}, s)]\mathrm{d}V, \tag{29}$$

in which $\{\hat{G}^{\tau f}_{p,q,k'}, \hat{G}^{vf}_{r,k'}\} = \{\hat{G}^{\tau f}_{p,q,k'}, \hat{G}^{vf}_{r,k'}\}(\mathbf{x}', \mathbf{x}, s)$ satisfy the system of equations

$$-\Delta^{+}_{k,m,p,q}\partial'_m \hat{G}^{\tau f}_{p,q,k'} + \hat{\zeta}_{k,r}\hat{G}^{vf}_{r,k'}$$

$$= \delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}), \tag{30}$$

$$\Delta^{+}_{i,j,n,r}\partial'_n \hat{G}^{vf}_{r,k'} - \hat{\eta}_{i,j,p,q}\hat{G}^{\tau f}_{p,q,k'} = 0, \tag{31}$$

with a point source of force, and $\{\hat{G}^{\tau h}_{p,q,i',j'}, \hat{G}^{vh}_{r,i',j'}\} = \{\hat{G}^{\tau h}_{p,q,i',j'}, \hat{G}^{vh}_{r,i',j'}\}(\mathbf{x}', \mathbf{x}, s)$ satisfy the system of equations

$$-\Delta^{+}_{k,m,p,q}\partial'_m \hat{G}^{\tau h}_{p,q,i',j'} + \hat{\zeta}_{k,r}\hat{G}^{vh}_{r,i',j'} = 0, \tag{32}$$

$$\Delta^{+}_{i,j,n,r}\partial'_n \hat{G}^{vh}_{r,i',j'} - \hat{\eta}_{i,j,p,q}\hat{G}^{\tau h}_{p,q,i',j'}$$

$$= \Delta^{+}_{i,j,i',j'}\delta(\mathbf{x}' - \mathbf{x}), \tag{33}$$

with a point source of deformation rate. In these equations ∂'_m means differentiation with respect to x'_m, $\delta(\mathbf{x} - \mathbf{x}')$ is the three-dimensional Dirac distribution operative at $\mathbf{x}' = \mathbf{x}$, and the property $\hat{\delta}(\mathbf{x} - \mathbf{x}', t) = \delta(\mathbf{x}' - \mathbf{x})$ has been used.

For a *lossless* solid, we have, on account of Eqs (6) and (7)

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\rho_{k,r}(\mathbf{x}), \tag{34}$$

$$\hat{\eta}_{i,j,p,q}(\mathbf{x}, s) = sS_{i,j,p,q}(\mathbf{x}), \tag{35}$$

while for a solid with *frictional-force/viscosity losses* we have, on account of Eqs (8) and (9),

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\rho_{k,r}(\mathbf{x}) + K_{k,r}(\mathbf{x}), \tag{36}$$

$$\hat{\eta}_{i,j,p,q}(\mathbf{x}, s) = sS_{i,j,p,q}(\mathbf{x}) + \Gamma_{i,j,p,q}(\mathbf{x}). \tag{37}$$

For a global relaxation law, the coefficients in Eqs (36) and (37) are interrelated through

$$K_{k,r}(\mathbf{x}) = \alpha\rho_{k,r}(\mathbf{x}) \quad \text{for all } \mathbf{x}, \tag{38}$$

$$\Gamma_{i,j,p,q}(\mathbf{x}) = \beta S_{i,j,p,q}(\mathbf{x}) \quad \text{for all } \mathbf{x}, \tag{39}$$

where α and β are arbitrary, non-negative constants. Under these conditions, Eqs (36) and (37) change into

$$\hat{\zeta}_{k,r} = (s + \alpha)\rho_{k,r}(\mathbf{x}), \tag{40}$$

$$\hat{\eta}_{i,j,p,q} = (s + \beta)S_{i,j,p,q}(\mathbf{x}). \tag{41}$$

These expressions will be used in our further similarity analysis. Obviously, $\alpha = \beta = 0$ corresponds to the lossless case.

SIMILARITY ANALYSIS IN A SOLID WITH GLOBAL FRICTIONAL-FORCE/VISCOSITY RELAXATION PARAMETERS

The similarity analysis will be carried out for the wave motion governed by the two-parameter system of elastic wave equations

$$-\Delta^+_{k,m,p,q}\partial_m \tau_{p,q} + \rho_{k,r}(\alpha + \partial_t)v_r = f_k, \quad (42)$$

$$\Delta^+_{i,j,n,r}\partial_n v_r - S_{i,j,p,q}(\beta + \partial_t)\tau_{p,q} = h_{i,j}. \quad (43)$$

The pertaining source-type integral representations for the wavefield quantities are written as (cf. Eqs (13) and (14))

$$-\tau_{p,q}(\mathbf{x}', t; \alpha, \beta) = \int_{\mathbf{x}\in\mathcal{D}} [\mathcal{C}_t(G^{\tau f}_{p,q,k}, f_k; \mathbf{x}', \mathbf{x}, t; \alpha, \beta) + \mathcal{C}_t(G^{\tau h}_{p,q,i,j}, h_{i,j}; \mathbf{x}', \mathbf{x}, t; \alpha, \beta)]\mathrm{d}V, \quad (44)$$

$$v_r(\mathbf{x}', t; \alpha, \beta) = \int_{\mathbf{x}\in\mathcal{D}} [\mathcal{C}_t(G^{vf}_{r,k}, f_k; \mathbf{x}', \mathbf{x}, t; \alpha, \beta) + \mathcal{C}_t(G^{vh}_{r,i,j}, h_{i,j}; \mathbf{x}', \mathbf{x}, t; \alpha, \beta)]\mathrm{d}V, \quad (45)$$

in which the Green's function satisfy the equations (cf. Eqs (15) - (18))

$$-\Delta^+_{k,m,p,q}\partial'_m G^{\tau f}_{p,q,k'} + \rho_{k,r}(\alpha + \partial_t)G^{vf}_{r,k'} = \delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}, t), \quad (46)$$

$$\Delta^+_{i,j,n,r}\partial'_n G^{vf}_{r,k'} - S_{i,j,p,q}(\beta + \partial_t)G^{\tau f}_{p,q,k'} = 0, \quad (47)$$

and

$$-\Delta^+_{k,m,p,q}\partial'_m G^{\tau h}_{p,q,i',j'} + \rho_{k,r}(\alpha + \partial_t)G^{vh}_{r,i',j'} = 0, \quad (48)$$

$$\Delta^+_{i,j,n,r}\partial'_n G^{vh}_{r,i',j'} - S_{i,j,p,q}(\beta + \partial_t)G^{\tau h}_{p,q,i',j'} = \Delta^+_{i,j,i',j'}\delta(\mathbf{x}' - \mathbf{x}, t). \quad (49)$$

The complex frequency domain counterparts of Eqs (46) - (49) are obtained upon taking the time Laplace transform of these equations; under this operation ∂_t is replaced by the factor s. The relevant result is rewritten as

$$-\Delta^+_{k,m,p,q}\partial'_m[(s+\beta)^{1/2}\hat{G}^{\tau f}_{p,q,k'}] + (s+\beta)^{1/2}(s+\alpha)^{1/2}\rho_{k,r}[(s+\alpha)^{1/2}\hat{G}^{vf}_{r,k'}] = (s+\beta)^{1/2}\delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}), \quad (50)$$

$$\Delta^+_{i,j,n,r}\partial'_n[(s+\alpha)^{1/2}\hat{G}^{vf}_{r,k'}] - (s+\alpha)^{1/2}(s+\beta)^{1/2}S_{i,j,p,q}[(s+\beta)^{1/2}\hat{G}^{\tau f}_{p,q,k'}] = 0, \quad (51)$$

and

$$-\Delta^+_{k,m,p,q}\partial'_m[(s+\beta)^{1/2}\hat{G}^{\tau h}_{p,q,i',j'}] + (s+\beta)^{1/2}(s+\alpha)^{1/2}\rho_{k,r}[(s+\alpha)^{1/2}\hat{G}^{vh}_{r,i',j'}] = 0, \quad (52)$$

$$\Delta^+_{i,j,n,r}\partial'_n[(s+\alpha)^{1/2}\hat{G}^{vh}_{r,i',j'}] - (s+\alpha)^{1/2}(s+\beta)^{1/2}S_{i,j,p,q}[(s+\beta)^{1/2}\hat{G}^{\tau h}_{p,q,i'j'}] = (s+\beta)^{1/2}\Delta^+_{i,j,i',j'}\delta(\mathbf{x}' - \mathbf{x}). \quad (53)$$

Upon comparing Eqs (50) - (53) with the ones for $\alpha = \beta = 0$ (i.e., the case of wave propagation in a lossless solid), it follows that

$$\hat{G}^{\tau f}_{p,q,k}(\mathbf{x}', \mathbf{x}, s; \alpha, \beta) = \hat{G}^{\tau f}_{p,q,k}[\mathbf{x}', \mathbf{x}, (s+\alpha)^{1/2}(s+\beta)^{1/2}; 0, 0], \quad (54)$$

$$(s+\beta)^{1/2}\hat{G}^{\tau h}_{p,q,i,j}(\mathbf{x}', \mathbf{x}, s; \alpha, \beta) = (s+\alpha)^{1/2} \times \hat{G}^{\tau h}_{p,q,i,j}[\mathbf{x}', \mathbf{x}, (s+\alpha)^{1/2}(s+\beta)^{1/2}; 0, 0], \quad (55)$$

$$(s+\alpha)^{1/2})\hat{G}^{vf}_{r,k}(\mathbf{x}', \mathbf{x}, s; \alpha, \beta) = (s+\beta)^{1/2} \times \hat{G}^{vf}_{r,k}[\mathbf{x}', \mathbf{x}, (s+\beta)^{1/2}(s+\alpha)^{1/2}; 0, 0], \quad (56)$$

$$\hat{G}^{vh}_{r,i,j}(\mathbf{x}', \mathbf{x}, s; \alpha, \beta)$$

$$= \hat{G}^{vh}_{r,i,j}[\mathbf{x}', \mathbf{x}, (s+\beta)^{1/2}(s+\alpha)^{1/2}; 0, 0]. \quad (57)$$

In the right-hand sides, the complex frequency domain Green's functions for the lossless solid occur, but with the time Laplace transform parameter s replaced by $(s+\alpha)^{1/2}(s+\beta)^{1/2}$. The time-domain counterparts of the latter functions follow from an application of the Schouten-Van der Pol theorem of the time Laplace transformation that relates two time functions whose Laplace transforms are interrelated through the operations of replacing the transform parameter s by a suitable function of s (Schouten 1934, Van der Pol 1934, Van der Pol and Bremmer 1950, Van der Pol 1960, Schouten 1961). For the present case, the consequences of this theorem are worked out in the Appendix. Using the results of this appendix, we obtain

$$G^{\tau f}_{p,q,k}(\mathbf{x}', \mathbf{x}, t; \alpha, \beta) =$$

$$\left[\int_{\tau=0}^{t} U_1(t, \tau; \alpha, \beta) G^{\tau f}_{p,q,k}(\mathbf{x}', \mathbf{x}, \tau; 0, 0) \mathrm{d}\tau\right] H(t) \quad (58)$$

$$G^{\tau h}_{p,q,i,j}(\mathbf{x}', \mathbf{x}, t; \alpha, \beta) = (\alpha + \partial_t)$$

$$\left[\int_{\tau=0}^{t} U_0(t, \tau; \alpha, \beta) G^{\tau h}_{p,q,i,j}(\mathbf{x}', \mathbf{x}, \tau; 0, 0) \mathrm{d}\tau\right] H(t) \quad (59)$$

$$G^{vf}_{r,k}(\mathbf{x}', \mathbf{x}, t; \alpha, \beta) = (\beta + \partial_t)$$

$$\left[\int_{\tau=0}^{t} U_0(t, \tau; \alpha, \beta) G^{vf}_{r,k}(\mathbf{x}', \mathbf{x}, \tau; 0, 0) \mathrm{d}\tau\right] H(t), \quad (60)$$

$$G^{vh}_{r,i,j}(\mathbf{x}', \mathbf{x}, t; \alpha, \beta) =$$

$$\left[\int_{\tau=0}^{t} U_1(t, \tau; \alpha, \beta) G^{vh}_{r,i,j}(\mathbf{x}', \mathbf{x}, \tau; 0, 0) \mathrm{d}\tau\right] H(t), \quad (61)$$

in which

$$U_1(t, \tau; \alpha, \beta) = -\partial_\tau U_0(t, \tau; \alpha, \beta), \quad (62)$$

with

$$U_0(t, \tau; \alpha, \beta) = \exp[-(\alpha+\beta)t/2]$$

$$\times I_0[(|\beta - \alpha|/2)(t^2 - \tau^2)^{1/2}] H(t - \tau), \quad (63)$$

where I_0 denotes the modified Bessel function of the first kind and order zero and H is the Heaviside unit step function. In Eqs (58) - (61) care has to be taken to include in the results the Dirac distribution due to the differentiation of the Heaviside unit step function occurring in Eq (63) and operative at $\tau = t$. Eqs (58) - (61) show how the Green's functions for the dissipative case are related to the Green's functions of the lossless case. For the case considered, the mutual relationship is fairly elementary and involves, after carrying out the differentiations analytically, only single integrations. In this respect it is important to notice that the changes in wave forms that do occur in the presence of losses are, in general, different for the different wave field quantities and for the different types of sources. Only $G^{\tau f}_{p,q,k}$ and $G^{vh}_{r,i,j}$ undergo a common modification as Eqs (58) and (61) indicate.

CONCLUSION

Through the Schouten-Van der Pol theorem of the time Laplace transformation, a relationship has been derived between the Green's functions of the elastic wave generation by known sources in an arbitrarily inhomogeneous and anisotropic, lossless solid and the Green's functions pertaining to the corresponding dissipative solid with two global relaxation parameters related to frictional force/viscosity losses. The two parameters are adjustable and can be used in the explanation of attenuation and dispersion phenomena occurring during the elastic wave propagation in a dissipative solid with an as yet unknown loss mechanism.

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APPENDIX : THE SCHOUTEN - VAN DER POL THEOREM FOR THE REPLACEMENT OF s BY $(s+\alpha)^{1/2}(s+\beta)^{1/2}$

Let $G(t;0,0)$ be a known, causal function of time with support $\{t \in \mathcal{R}; t > 0\}$ and let

$$\hat{G}(s;0,0) = \int_{\tau=0}^{\infty} \exp(-s\tau) G(\tau,0,0)\mathrm{d}\tau \tag{A.1}$$

be its Laplace transform. Let, further,

$$\hat{G}(s;\alpha,\beta) = \hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0], \tag{A.2}$$

then

$$\hat{G}(s;\alpha,\beta) = \int_{\tau=0}^{\infty} \exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau] G(\tau,0,0)\mathrm{d}\tau. \tag{A.3}$$

To arrive at the time-domain counterpart $G(t;\alpha,\beta)$ of $\hat{G}(s;\alpha,\beta)$, we observe that (Abramowitz and Stegun 1964)

$$\exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau]$$

$$= \int_{t=\tau}^{\infty} \exp(-st) U_1(t,\tau;\alpha,\beta)\mathrm{d}t, \tag{A.4}$$

in which

$$U_1(t,\tau;\alpha,\beta) = -\partial_\tau U_0(t,\tau;\alpha,\beta), \tag{A.5}$$

with

$$U_0(t,\tau;\alpha,\beta) = \exp[-(\alpha+\beta)t/2]$$

$$\times I_0[(|\beta-\alpha|/2)(t^2-\tau^2)^{1/2}]H(t-\tau), \tag{A.6}$$

where I_0 denotes the modified Bessel function of the first kind and order zero and H is the Heaviside unit step function. Using Eq (A.4) in Eq (A.3) and employing the uniqueness of the time Laplace transform (Lerch's theorem (Widder 1946)), we end up with

$$G(t;\alpha,\beta) = \left[\int_{\tau=0}^{t} U_1(t,\tau,\alpha,\beta)G(\tau;0,0)\mathrm{d}\tau\right] H(t), \tag{A.7}$$

where care has to be taken to include the Dirac distribution operative at $\tau = t$ due to the differentiation in Eq (A.5) of the Heaviside unit step function occurring in Eq (A.6). Further we use the results that

$$\frac{(s+\alpha)^{1/2}}{(s+\beta)^{1/2}} \hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]$$

$$= (\alpha+s)\frac{\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}}$$

$$\Rightarrow (\alpha+\partial_t)U_0(t,\tau;\alpha,\beta) \tag{A.8}$$

and

$$\frac{(s+\beta)^{1/2}}{(s+\alpha)^{1/2}} \hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]$$

$$= (\beta+s)\frac{\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}}$$

$$\Rightarrow (\beta+\partial_t)U_0(t,\tau;\alpha,\beta), \tag{A.9}$$

where the property

$$\hat{U}_0(t,\tau;\alpha,\beta) = \frac{\exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}} \tag{A.10}$$

has been used. These results are used in the main text.