Reprinted from JOURNAL OF APPLIED PHYSICS, Vol. 24, No. 10, 1325–1327, October, 1953

On the Propagation Constant in Gentle Circular Bends in Rectangular Wave Guides—Matrix Theory

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(Received May 18, 1953)

The propagation constant γ_{mn} of the m,nth mode in a gentle circular bend in a rectangular wave guide is derived with the use of matrix theory.

INTRODUCTION

IN his publication "Reflections from circular bends in rectangular wave guides—Matrix theory"[1] Rice has obtained a general expression for the reflection coefficients due to a gentle bend in a rectangular wave guide. Special attention has been paid to the dominant mode reflection coefficients g_{10}^- and d_{01}^- corresponding to H-bends (magnetic intensity in plane of the bend) and E-bends (electric intensity in plane of the bend), respectively.

However, to obtain the reflection coefficients of the m,nth mode the propagation constant γ_{mn} of this mode must be known. For the latter Rice uses the results obtained by Buchholz[2] and Marshak[3] while using the matrix theory for the aforementioned special cases.

In the present paper it will be shown that extending the work of Rice in both cases (H-bends and E-bends) the propagation constant of the m,nth mode γ_{mn} can be derived with the aid of matrix theory. The results are in accordance with those in reference 1.

1. PROPAGATION OF THE m,nTH MODE IN A GENTLE BEND. H IN PLANE OF THE BEND

In the case of H in plane of the bend we deal with the following form of the vector potential A (see Eq. (1.3–1), reference 1) being $B=0$[4]

$$A=\cos(\pi ny/b)\sum_{l=1}^{\infty}\alpha_{ln}(z)\sin(\pi lx/a), \qquad (1.1)$$

where n has a fixed value and l runs through the values 1, 2, 3, $\cdots$ (Fig. 1). The notations of reference 1 will be followed closely.

In order to determine the propagation constant γ_{mn} we have to find the mth characteristic root of the matrix Γ_α^2 defined by Eq. (1.3–5) of reference 1,

$$\Gamma_\alpha^2=P^{-1}(\Gamma_0^2+S), \qquad (1.2)$$

where

$$P_{rs}=(2/a)\int_0^a(\rho_1^2/\rho^2)\sin(\pi rx/a)\sin(\pi sx/a)dx, \qquad (1.3)$$

$$S_{rs}=-2\pi sa^{-2}\int_0^a\sin(\pi rx/a)\cos(\pi sx/a)dx/\rho, \qquad (1.4)$$

and the elements of the diagonal matrix Γ_0^2 are

$$\delta_l^2=\Gamma_{ln}^2=\sigma^2+(\pi l/a)^2+(\pi n/b)^2,\quad \sigma=i2\pi/\lambda_0, \qquad (1.5)$$

λ_0= wavelength in free space.

With

$$F=\Gamma_\alpha^2-\Gamma_0^2 \qquad (1.6)$$

Eq. (1.2) becomes

$$F=(P^{-1}-I)\Gamma_0^2+P^{-1}S, \qquad (1.7)$$

where I denotes the unit matrix.

The mth characteristic root of Γ_α^2 is (see Eq. (3.2–2), reference 1)

$$\gamma_m^2=\delta_m^2+F_{mm}+\sum_{s=1}^{\infty}{}'F_{ms}F_{sm}/(\delta_m^2-\delta_s^2),\quad s\neq m. \qquad (1.8)$$

Substituting (1.5) in (1.8) gives

$$\gamma_m^2=\delta_m^2+F_{mm}-\sum_{s=1}^{\infty}{}'F_{ms}F_{sm}a^2/\pi^2(s^2-m^2). \qquad (1.9)$$

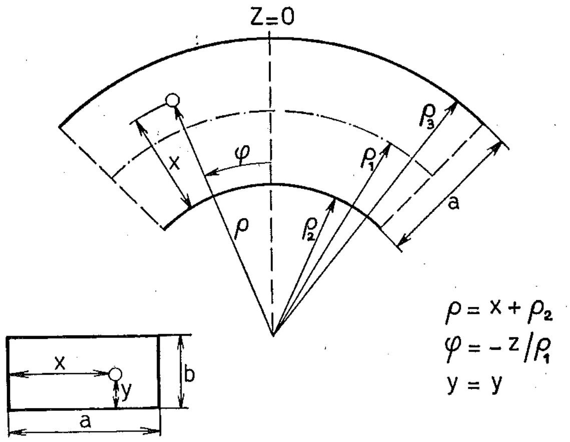

FIG. 1. Coordinate system used in circular bend in a rectangular wave guide.

[1] S. O. Rice, Bell System Tech. J. **27**, 305–349 (1948).
[2] H. Buchholz, Elek. Nachr. Tech. **16**, 73–85 (1939).
[3] R. E. Marshak, "Theory of circular bends in rectangular waveguides," Radiation Laboratory Report (June 24, 1943), pp. 43–45.
[4] S. A. Schelkunoff, *Electromagnetic Waves* (D. Van Nostrand and Company, Inc., New York, 1943), p. 127.

Our first task is to determine the elements of the matrix F by means of the matrix Eq. (1.7). In the case of a gentle bend we have

$$P=I+R, \tag{1.10}$$

where R is a square matrix whose elements are small compared to unity. As the nondiagonal elements must be accurate to within $0(\xi)$, $\xi=a/\rho_1$, and the elements of the principal diagonal to within $0(\xi^2)$, we make use of the asymptotic expansions of R_{ij} and S_{ij} as mentioned in Appendix I, reference 1. When the matrix multiplication is carried out, we find (see Eq. (4.1–4), reference 1)

$$F_{ij}=-R_{ij}\Gamma_{jn}^2+S_{ij},$$
$$F_{ii}=\left(-R_{ii}+\sum_{s=1}^{\infty}R_{is}R_{si}\right)\Gamma_{in}^2+S_{ii}-\sum_{s=1}^{\infty}R_{is}S_{si}. \tag{1.11}$$

In our case we need the elements

$$F_{ms}=-R_{ms}\Gamma_{sn}^2+S_{ms},$$
$$F_{sm}=-R_{sm}\Gamma_{mn}^2+S_{sm},$$
$$F_{mm}=\left(-R_{mm}+\sum_{s=1}^{\infty}R_{ms}R_{sm}\right)\Gamma_{mn}^2+S_{mm}-\sum_{s=1}^{\infty}R_{ms}S_{sm}. \tag{1.12}$$

Because the nondiagonal elements must be accurate to within $0(\xi)$ we have to take the odd values of $s-m$. The value of m is fixed, hence, when m is odd s has one of the values 2, 4, 6, $\cdots$ when m is even s has one of the values 1, 3, 5, $\cdots$. The result is

$$R_{ms}\sim 16\xi ms/\pi^2(s^2-m^2)^2, \tag{1.13}$$
$$S_{ms}\sim 4\xi ms/a^2(s^2-m^2), \tag{1.14}$$
$$R_{sm}\sim 16\xi ms/\pi^2(s^2-m^2)^2, \tag{1.15}$$
$$S_{sm}\sim -4\xi ms/a^2(s^2-m^2). \tag{1.16}$$

The elements of the principal diagonal must be accurate to within $0(\xi^2)$. Hence

$$R_{mm}\sim(\xi^2/4)(1-6/\pi^2m^2), \tag{1.17}$$
$$S_{mm}\sim-\xi^2/2a^2. \tag{1.18}$$

With the values of (1.13)–(1.18) the elements of the matrix F turn out to be

$$F_{ms}=-4\xi ms[4\Gamma_{mn}^2\pi^{-2}(s^2-m^2)^{-2}+3a^{-2}(s^2-m^2)^{-1}],$$
$$F_{sm}=-4\xi ms[4\Gamma_{mn}^2\pi^{-2}(s^2-m^2)^{-2}+a^{-2}(s^2-m^2)^{-1}],$$
$$F_{mm}=(\xi^2/12)[\Gamma_{mn}^2(1-6\pi^{-2}m^{-2})+6a^{-2}], \tag{1.19}$$

where in the expressions for F_{ms} and F_{sm} the value of $s-m$ is supposed to be odd.

The summations which arise in the evaluation of F_{mm} are of the type

$$\sum_s s^2(s^2-m^2)^{-p}, \tag{1.20}$$

where the value of $s-m$ must be odd. These summations will be discussed in the appendix. Substitution of the values of (1.19) in (1.9) and use of the sums (1.20) gives the propagation constant γ_{mn} in the bend

$$\gamma_m^2=\gamma_{mn}^2=\Gamma_{mn}^2-(\xi^2/4a^2)[1+\Gamma_{mn}^2a^2(1-6\pi^{-2}m^{-2}) +(\Gamma_{mn}a/\pi m)^4(5-\pi^2m^2/3)], \tag{1.21}$$

which is in accordance with Eq. (4.1–10), reference 1.

2. PROPAGATION OF THE m,nTH MODE IN A GENTLE BEND. E IN PLANE OF THE BEND

In the case of E in plane of the bend we deal with the following form of the vector potential B (see Eq. (1.3–11), reference 1), being $A=0$

$$B=\sin(\pi ny/b)\sum_{l=0}^{\infty}\beta_{ln}(z)\cos(\pi lx/a), \tag{2.1}$$

where n has a fixed value and l runs through the values 0, 1, 2, $\cdots$.

In order to determine the propagation constant γ_{mn} we have to find the mth characteristic root of the matrix Γ_β^2 defined by Eq. (1.3–13), reference 1,

$$\Gamma_\beta^2=Q^{-1}(\Gamma_0^2+U), \tag{2.2}$$

where

$$Q_{rs}=(\epsilon_r/a)\int_0^a(\rho_1^2/\rho^2)\cos(\pi rx/a)\cos(\pi sx/a)dx, \tag{2.3}$$

$$U_{rs}=\pi s\epsilon_r a^{-2}\int_0^a\cos(\pi rx/a)\sin(\pi sx/a)dx/\rho, \tag{2.4}$$

where $\epsilon_0=1$ and $\epsilon_r=2$ for $r>0$. The elements of the diagonal matrix Γ_0^2 are

$$\delta_l^2=\Gamma_{ln}^2=\sigma^2+(\pi l/a)^2+(\pi n/b)^2. \tag{2.5}$$

With

$$F=\Gamma_\beta^2-\Gamma_0^2, \tag{2.6}$$

Eq. (2.2) becomes

$$F=(Q^{-1}-I)\Gamma_0^2+Q^{-1}U. \tag{2.7}$$

The mth characteristic root of Γ_β^2 is (see Eq. (3.2–2), reference 1)

$$\gamma_m^2=\delta_m^2+F_{mm}+\sum_{s=0}^{\infty}{}'F_{ms}F_{sm}/(\delta_m^2-\delta_s^2),\quad s\neq m. \tag{2.8}$$

Substituting (2.5) in (2.8) gives

$$\gamma_m^2=\delta_m^2+F_{mm}-\sum_{s=0}^{\infty}{}'F_{ms}F_{sm}a^2/\pi^2(s^2-m^2). \tag{2.9}$$

With the restrictions under consideration and substituting $Q=I+T$ where T is a square matrix whose elements are small compared to unity the elements of the matrix F turn out to be

$$F_{ms}=-T_{ms}\Gamma_{sn}^2+U_{ms},$$
$$F_{sm}=-T_{sm}\Gamma_{mn}^2+U_{m},$$
$$F_{mm}=\left(-T_{mm}+\sum_{s=0}^{\infty}T_{ms}T_{sm}\right)\Gamma_{mn}^2+U_{mm}-\sum_{s=0}^{\infty}T_{ms}U_{sm}. \tag{2.10}$$

Using the asymptotic expansions of Appendix I, reference 1, one obtains

$$T_{ms} \sim (\epsilon_m/2)8\xi(m^2+s^2)/\pi^2(s^2-m^2)^2, \quad (2.11)$$

$$U_{ms} \sim (\epsilon_m/2)4\xi s^2/a^2(s^2-m^2), \quad (2.12)$$

$$T_{sm} \sim (\epsilon_s/2)8\xi(m^2+s^2)/\pi^2(s^2-m^2)^2, \quad (2.13)$$

$$U_{sm} \sim -(\epsilon_s/2)4\xi m^2/a^2(s^2-m^2), \quad (2.14)$$

with $m>0$.

The elements of the principal diagonal must be accurate to within $0(\xi^2)$, they turn out to be

$$T_{mm} \sim (\epsilon_m/2)(\xi^2/4)(1+6\pi^{-2}m^{-2}), \quad (2.15)$$

$$m>0$$

$$U_{mm} \sim (\epsilon_m/2)(\xi^2/2a^2). \quad (2.16)$$

With the values of (2.11)–(2.16) we obtain the elements of the matrix F

$$\begin{aligned} F_{ms} &= -4\xi[2(m^2+s^2)\pi^{-2}(s^2-m^2)^{-2}\Gamma_{mn}^2 \\ &\qquad +(2m^2+s^2)a^{-2}(s^2-m^2)^{-1}], \\ F_{sm} &= -4\xi[2(m^2+s^2)\pi^{-2}(s^2-m^2)^{-2}\Gamma_{mn}^2 \\ &\qquad +a^{-2}m^2(s^2-m^2)^{-1}], \\ F_{mm} &= (\xi^2/12)[\Gamma_{mn}^2(1+6\pi^{-2}m^{-2})-6a^{-2}], \end{aligned} \quad (2.17)$$

with $m>0$.

The two series arising in the evaluation of F_{mm} reduce to

$$\sum_s (\epsilon_s/2)(m^2+s^2)^2(s^2-m^2)^{-4}$$

and

$$\sum_s (\epsilon_s/2)m^2(m^2+s^2)(s^2-m^2)^{-3},$$

where the value of $s-m$ must be odd. These series will be discussed in the appendix. Substituting the values of (2.17) in (2.9) and using the results of the appendix gives the propagation constant γ_{mn} in the bend

$$\begin{aligned} \gamma_m^2 = \gamma_{mn}^2 = \Gamma_{mn}^2 &+ (\xi^2/4a^2)[3-(\Gamma_{mn}a/\pi m)^2(10+\pi^2m^2) \\ &+ (\Gamma_{mn})^4(7+\pi^2m^2/3)], \quad m>0 \end{aligned} \quad (2.18)$$

which is in accordance with Eq. (4.3–7), reference 1.

Finally, we have to consider the case that $m=0$ and n is arbitrary. However, the value of n has no influence on the determination of the propagation constant γ_{on}, hence we only have to replace Γ_{01} in Eq. (4.3–6), reference 1, by Γ_{on}. The result is

$$\gamma_{on}^2 = \Gamma_{on}^2 - (\xi^2\Gamma_{on}^2/60)(5+2a^2\Gamma_{on}^2). \quad (2.19)$$

Equations (1.21), (2.18), and (2.19) give the propagation constants for an arbitrary mode generated in a gentle circular bend (either H-bend or E-bend) in a rectangular wave guide. They are correct to within $0(\xi^2)$, where $\xi = a/\rho_1$ and ρ_1 is the radius of curvature of the bend.

APPENDIX

The summations of Sec. 1 reduce to

$$\sigma_p = \sum_s s^2(s^2-m^2)^{-p}, \quad (A.1)$$

where m has a fixed value and $s-m$ must be odd.

At first we consider the case that m is odd, $s=2, 4, 6, \cdots$. As the typical term of σ_p can be expanded in partial fractions,

$$\sigma_p = \sum_s (s^2-m^2)^{-p+1} + m^2 \sum_s (s^2-m^2)^{-p}, \quad (A.2)$$

the determination of the summations

$$\tau_p = \sum_s (s^2-m^2)^{-p} \quad (A.3)$$

will be sufficient. Now τ_p can be expanded in the following way:

$$\tau_p = (2m)^{-p} \sum_s [(s-m)^{-1}-(s+m)^{-1}]^p. \quad (A.4)$$

By use of (A.4) we obtain

$$\begin{aligned} \tau_1 = (2m)^{-1}\{[1/(2-m)+1/(4-m)+\cdots+1/(m-4) \\ +1/(m-2)+1/m+1/(m+2)+\cdots] \\ -[1/(m+2)+1/(m+4)+\cdots]\}, \end{aligned}$$

or

$$\tau_1 = 1/2m^2. \quad (A.5)$$

In the same way we obtain τ_2 up to and including τ_5.

$$\tau_2 = \pi^2/16m^2 - 1/2m^4, \quad (A.6)$$

$$\tau_3 = -3\pi^2/64m^4 + 1/2m^6, \quad (A.7)$$

$$\tau_4 = \pi^4/768m^4 + 5\pi^2/128m^6 - 1/2m^8, \quad (A.8)$$

$$\tau_5 = -5\pi^4/3072m^6 - 35\pi^2/1024m^8 + 1/2m^{10}, \quad (A.9)$$

By making use of these results we obtain

$$\begin{aligned} \sigma_3 &= \pi^2/64m^2, \\ \sigma_4 &= \pi^4/768m^2 - \pi^2/128m^4, \\ \sigma_5 &= -\pi^4/3072m^4 + 5\pi^2/1024m^6. \end{aligned} \quad (A.10)$$

When m is even, we obtain for the summations τ_p

$$\tau_1 = 0, \quad (A.11)$$

$$\tau_2 = \pi^2/16m^2, \quad (A.12)$$

$$\tau_3 = -3\pi^2/64m^4, \quad (A.13)$$

$$\tau_4 = \pi^4/768m^4 + 5\pi^2/128m^6, \quad (A.14)$$

$$\tau_5 = -35\pi^2/1024m^8 - 5\pi^4/3072m^6. \quad (A.15)$$

By making use of these results we obtain for σ_3, σ_4, and σ_5 the same values as given by (A.10), where m was odd.

The summations arising from Sec. 2 are somewhat more difficult than those of Sec. 1 because of the factor ϵ_s ($\epsilon_s=2$ when $s>0$ and $\epsilon_0=1$). However, carrying out the procedure outlined here, one is led to the same values of σ_3, σ_4 and σ_5 as given in (A.10).

ACKNOWLEDGMENT

The author is indebted to Professor J. P. Schouten of the Technological University Delft for many helpful discussions.