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ON THE SCALAR DIFFRACTION BY A CIRCULAR APERTURE IN AN INFINITE PLANE SCREEN

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Summary

If Levine and Schwinger's variational formulation of the diffraction of a plane wave by an aperture in an infinite plane screen is applied to the case where the aperture is a circular hole, the problem of finding the aperture distribution can be reduced to the solution of an infinite system of linear equations. As pointed out by Bouwkamp the most appropriate expansion of the aperture distribution is of the form

$$\Phi_1(\varrho) = \sum_{n=0}^{\infty} b_n P_{2n+1}\,[(1-\varrho^2/a^2)^{\frac{1}{2}}],$$

where a = radius of the aperture. In the present paper the system of equations in b_n is investigated.

§ 1. *Introduction*. By means of their variational principle Levine and Schwinger[1]) reduced the problem of the diffraction of a scalar plane wave by a circular aperture in a perfectly soft infinite plane screen to the solution of an infinite system of linear equations for certain coefficients a_n which determine the field $\Phi_1(\varrho)$ in the aperture, viz.

$$\Phi_1(\varrho) = \sum_{n=1}^{\infty} a_n(1-\varrho^2/a^2)^{n-\frac{1}{2}}, \tag{1.01}$$

where a = radius of the circular aperture.

This system of equations was thoroughly investigated by Magnus[2]). However, as Bouwkamp[3]) has pointed out, an expansion of the aperture distribution of the form

$$\Phi_1(\varrho) = \sum_{n=0}^{\infty} b_n P_{2n+1}\,[(1-\varrho^2/a^2)^{\frac{1}{2}}] \tag{1.02}$$

would have the advantage that the Legendre polynomials under consideration form an orthogonal set in $0 \leq \varrho \leq a$. Application of the variational principle now leads to a system of linear equations for the b_n, the properties of which are simpler than those of the corresponding system for a_n.

If the b_n together with the coefficients of the system of equations are expanded in power series in $\varepsilon = ika$ (k = wave number) and only the first N equations for $b_0, b_1, \ldots, b_N$ are retained, disregarding all powers of ε higher than the $(2N + 1)$th power, the coefficients of the first $2N + 1$ powers of ε in $b_0, \ldots, b_N$ are exact. This property is easily proved when taking into account the nature of the power series expansion of the matrix of the system. Finally, the coefficients $b_0, \ldots, b_3$ are computed up to and including $O(\varepsilon^8)$.

§ 2. *Determination of the system of linear equations.* We consider the diffraction of a scalar plane wave by a finite aperture in a perfectly soft infinite plane screen of zero thickness. The screen coincides with the plane $z=0$ and the incident wave comes from $z=-\infty$.

Now, the amplitude $A_1(\mathbf{n}'', \mathbf{n}')$ of the diffracted wave at large distances behind the screen can be written in a stationary form, viz. [3])

$$A_1(\mathbf{n}'', \mathbf{n}') = \frac{k^2 \cos\theta' \cos\theta'' \int_A \Phi_{\mathbf{n}'}(\boldsymbol{\rho}) \exp(-ik\,\mathbf{n}''.\boldsymbol{\rho})\mathrm{d}S \int_A \Phi_{-\mathbf{n}''}(\boldsymbol{\rho}) \exp(ik\mathbf{n}'.\boldsymbol{\rho})\,\mathrm{d}S}{\int_A\int_A [k^2 \Phi_{\mathbf{n}'}(\boldsymbol{\rho})\Phi_{-\mathbf{n}''}(\boldsymbol{\rho}') - \nabla\Phi_{\mathbf{n}'}(\boldsymbol{\rho}) . \nabla'\Phi_{-\mathbf{n}''}(\boldsymbol{\rho}')]\, G(\boldsymbol{\rho}, \boldsymbol{\rho}')\,\mathrm{d}S\mathrm{d}S'}, \quad (2.01)$$

where

$\mathbf{n}''$ = unit vector in the direction of observation,
$\mathbf{n}'$ = unit vector in the direction of the incident wave,
$\boldsymbol{\rho}$ = radius vector in the plane $z = 0$,
$\Phi_{\mathbf{n}'}(\boldsymbol{\rho})$ = field in the aperture due to an incident wave in the direction of $\mathbf{n}'$,
θ' = angle between $\mathbf{n}'$ and the positive z-direction,
θ'' = angle between $\mathbf{n}''$ and the positive z-direction,
$k = \omega/c$ = wave number corresponding to time-harmonic plane waves with velocity of propagation c,
$G(\boldsymbol{\rho}, \boldsymbol{\rho}')$ = $\exp(ik\,|\boldsymbol{\rho}-\boldsymbol{\rho}'|)/|\boldsymbol{\rho}-\boldsymbol{\rho}'|$, the free space Green's function.

The integrations are extended over the aperture A.

The harmonic time dependence of the form $\exp(-i\omega t)$ is omitted throughout and the gradients are taken in the plane $z = 0$. Only variations of $\Phi(\boldsymbol{\rho})$, which satisfy the condition $\Phi(\boldsymbol{\rho}) = 0$ at the rim of the aperture are admissible. As we shall use (2.01) to obtain the exact value of $\Phi_{\mathbf{n}'}(\boldsymbol{\rho})$ we restrict ourselves to $\mathbf{n}' = \mathbf{n}'' = \mathbf{n}$.

In the case of normal incidence and a circular aperture of radius a, the function $\Phi_{\mathbf{n}}(\boldsymbol{\rho}) = \Phi_1(\varrho)$ will depend on ϱ only. Hence,

$$A_1(\mathbf{n},\mathbf{n}) = \frac{k^2 \int_A \Phi_1(\varrho)\, \mathrm{d}S \int_A \Phi_1(\varrho)\, \mathrm{d}S}{\int\int_{A\,A} [k^2\Phi_1(\varrho)\, \Phi_1(\varrho') - \nabla\Phi_1(\varrho).\nabla'\Phi_1(\varrho')]\, G(\boldsymbol{\rho}, \boldsymbol{\rho}')\, \mathrm{d}S\mathrm{d}S'}, \tag{2.02}$$

where $\mathbf{n}$ is a unit vector in the positive z-direction.

If the correct aperture distribution is assumed in the form [3])

$$\Phi_1(\varrho) = \sum_{n=0}^{\infty} b_n P_{2n+1}\, [(1 - \varrho^2/a^2)^{\frac{1}{2}}], \tag{2.03}$$

we obtain the following stationary expression for $A_1(\mathbf{n}, \mathbf{n})$:

$$A_1 = -\frac{2ab_0^2}{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n}\, b_m\, b_n}, \tag{2.04}$$

where

$$d_{m,n} = (6/ka)^2 \frac{\Gamma(m + \frac{3}{2})\, \Gamma(n + \frac{3}{2})}{\Gamma(m + 1)\, \Gamma(n + 1)}\, .$$

$$.\int_0^{\infty} (v^2 - 1)^{\frac{1}{2}}\, v^{-2}\, J_{2m+3/2}\, (kav)\, J_{2n+3/2}\, (kav)\, \mathrm{d}v. \tag{2.05}$$

On the other hand from the non-stationary expression

$$A_1 = -(ik/2\pi) \int_A \Phi_1(\varrho)\, \mathrm{d}S \tag{2.06}$$

we have

$$A_1 = -\tfrac{1}{3}\, ika^2 b_0. \tag{2.07}$$

Application of the variational principle to (2.04) leads to the following infinite system of linear equations in b_n:

$$\sum_{n=0}^{\infty} d_{m,n}\, b_n = (6/ika)\, \delta_{m,0}, \quad m = 0, 1, 2, 3, \ldots., \tag{2.08}$$

where $\delta_{0,0} = 1$, $\delta_{m,0} = 0$ if $m > 0$. Eq. (2.08) is due to Bouwkamp [3]).

Making the substitutions

$$d_{m,n} = (6/ka)^2\, l_{m,n}, \tag{2.09}$$

$$b_n = -(ika/6) x_n, \tag{2.10}$$

we obtain from (2.08)

$$\sum_{n=0}^{\infty} l_{m,n} x_n = \delta_{m,0}, \quad m = 0, 1, 2, 3, \ldots . \tag{2.11}$$

Introducing the matrix

$$L = (l_{m,n})$$

and the columns

$$\xi = \{x_m\}, \quad \eta = \{1, 0, 0, \ldots .\},$$

we can write the system of linear equations (2.11) in the form

$$L\xi = \eta. \tag{2.12}$$

§ 3. *A power series solution of the linear equations.* The simplest way to obtain a solution of (2.12) is to expand the elements of L and ξ in a power series in

$$\varepsilon = ika. \tag{3.01}$$

As the even and odd powers of ε play a different role in the expansion, we shall write (following Magnus' notation)

$$L = \sum_{p=0}^{\infty} L^{(2p)} \varepsilon^{2p} + \sum_{q=0}^{\infty} L^{(2q+3)} \varepsilon^{2q+3}, \tag{3.02}$$

$$\xi = \sum_{p=0}^{\infty} \xi^{(2p)} \varepsilon^{2p} + \sum_{q=0}^{\infty} \xi^{(2q+3)} \varepsilon^{2q+3}, \tag{3.03}$$

where

$$L^{(2p)} = (l_{m,n}^{(2p)}), \quad L^{(2q+3)} = (l_{m,n}^{(2q+3)}), \quad \xi^{(2p)} = \{x_m^{(2p)}\}, \quad \xi^{(2q+3)} = \{x_m^{(2q+3)}\}$$

and

$$l_{m,n} = \sum_{p=0}^{\infty} l_{m,n}^{(2p)} \varepsilon^{2p} + \sum_{q=0}^{\infty} l_{m,n}^{(2q+3)} \varepsilon^{2q+3}, \tag{3.04}$$

$$x_m = \sum_{p=0}^{\infty} x_m^{(2p)} \varepsilon^{2p} + \sum_{q=0}^{\infty} x_m^{(2q+3)} \varepsilon^{2q+3}. \tag{3.05}$$

As will be shown in the appendix the coefficients of the expansion (3.04) are given by

$$l_{m,n}^{(2p)} = \frac{1}{4} \frac{\Gamma(m+\frac{3}{2})\,\Gamma(n+\frac{3}{2})}{\Gamma(m+1)\,\Gamma(n+1)} \cdot$$

$$\cdot \frac{(-)^{m+n}\,\Gamma(p+\frac{1}{2})\,\Gamma(p-\frac{1}{2})}{\Gamma(-m-n+p-\frac{1}{2})\Gamma(m-n+p+1)\Gamma(-m+n+p+1)\Gamma(m+n+p+\frac{5}{2})}, \tag{3.06}$$

$$l^{(2q+3)}_{m,n} = \frac{1}{4}\,\frac{\Gamma(m+\frac{3}{2})\,\Gamma(n+\frac{3}{2})}{\Gamma(m+1)\,\Gamma(n+1)}\cdot$$
$$\cdot\frac{(-)^{m+n}\,\Gamma(q+2)\,\Gamma(q+1)}{\Gamma(-m-n+q+1)\Gamma(m-n+q+\frac{5}{2})\Gamma(-m+n+q+\frac{5}{2})\Gamma(m+n+q+4)}. \tag{3.07}$$

Substituting the expansions (3.02) and (3.03) in (2.12) we obtain

$$\left(\sum_{p=0}^{\infty} L^{(2p)}\,\varepsilon^{2p} + \sum_{q=0}^{\infty} L^{(2q+3)}\,\varepsilon^{2q+3}\right)\left(\sum_{p=0}^{\infty} \xi^{(2p)}\,\varepsilon^{2p} + \sum_{q=0}^{\infty} \xi^{(2q+3)}\,\varepsilon^{2q+3}\right) = \eta^{(0)}, \tag{3.08}$$

where

$$\eta^{(0)} = \eta = \{1, 0, 0, 0, \ldots.\}. \tag{3.09}$$

Comparing the coefficients of equal powers of ε on both sides of (3.08) we obtain the following sets of linear equations

$$L^{(0)}\,\xi^{(0)} = \eta^{(0)}, \tag{3.10}$$

$$L^{(0)}\,\xi^{(r)} + L^{(2)}\,\xi^{(r-2)} + L^{(3)}\,\xi^{(r-3)} + \ldots.$$
$$\ldots. + L^{(r-3)}\,\xi^{(3)} + L^{(r-2)}\,\xi^{(2)} + L^{(r)}\,\xi^{(0)} = 0, \quad r = 2, 3, 4, \ldots \tag{3.11}$$

The matrices $L^{(2p)}$ and $L^{(2q+3)}$ have the peculiar structure given below. Roughly speaking they contain "many zeros".

$$L^{(2p)} \qquad \begin{array}{c|cccccccc}
 & 0 & 1 & \ldots & p & & & & \\
0 & \times & \times & \times & \times & 0 & 0 & 0 & 0 \\
1 & \times & \times & \times & \times & \times & 0 & 0 & 0 \\
\vdots & \times & \times & \times & \times & \times & \times & 0 & 0 \\
p & \times & \times & \times & \times & \times & \times & \times & 0 \\
 & 0 & \times & \times & \times & \times & \times & \times & \times \\
 & 0 & 0 & \times & \times & \times & \times & \times & \times \\
 & 0 & 0 & 0 & \times & \times & \times & \times & \times \\
 & 0 & 0 & 0 & 0 & \times & \times & \times & \times
\end{array} \qquad \begin{array}{l} l^{(2p)}_{m,n} = 0 \text{ if} \\ n \geq m + p + 1 \\ m \geq n + p + 1 \end{array}$$

$$L^{(2q+3)} \qquad \begin{array}{c|cccccccc}
 & 0 & 1 & \ldots & \ldots & q & & & \\
0 & \times & \times & \times & \times & \times & 0 & 0 & 0 \\
1 & \times & \times & \times & \times & 0 & 0 & 0 & 0 \\
\vdots & \times & \times & \times & 0 & 0 & 0 & 0 & 0 \\
\vdots & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 \\
q & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \qquad \begin{array}{l} l^{(2q+3)}_{m,n} = 0 \text{ if} \\ m + n \geq q + 1 \end{array}$$

In particular $L^{(0)}$ is a diagonal matrix with elements

$$l^{(0)}_{m,m} = \frac{1}{4} \frac{\Gamma(m+\frac{3}{2})\,\Gamma(m+\frac{3}{2})\,\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(m+1)\,\Gamma(m+1)\,\Gamma(-2m-\frac{1}{2})\,\Gamma(2m+\frac{5}{2})}. \quad (3.12)$$

Because of the latter fact we can easily obtain the matrices $S^{(2p)}$ and $S^{(2q+3)}$ which are defined by

$$L^{(2p)} = L^{(0)}\,S^{(2p)}, \quad (3.13)$$

$$L^{(2q+3)} = L^{(0)}\,S^{(2q+3)}, \quad (3.14)$$

where

$$S^{(2p)} = (s^{(2p)}_{m,n}), \qquad S^{(2q+3)} = (s^{(2q+3)}_{m,n}).$$

Their elements are given by

$$s^{(2p)}_{m,n} = \frac{(-)^{m+n}\Gamma(m+1)\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})\Gamma(m+\frac{3}{2})\,\Gamma(n+1)} \cdot$$

$$\cdot \frac{\Gamma(p+\frac{1}{2})\Gamma(p-\frac{1}{2})\Gamma(-2m-\frac{1}{2})\Gamma(2m+\frac{5}{2})}{\Gamma(-m-n+p-\frac{1}{2})\Gamma(m-n+p+1)\Gamma(-m+n+p+1)\Gamma(m+n+p+\frac{5}{2})}, \quad (3.15)$$

$$s^{(2q+3)}_{m,n} = \frac{(-)^{m+n}\Gamma(m+1)\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})\Gamma(m+\frac{3}{2})\Gamma(n+1)} \cdot$$

$$\cdot \frac{\Gamma(q+2)\,\Gamma(q+1)\,\Gamma(-2m-\frac{1}{2})\,\Gamma(2m+\frac{5}{2})}{\Gamma(-m-n+q+1)\Gamma(m-n+q+\frac{5}{2})\Gamma(-m+n+q+\frac{5}{2})\Gamma(m+n+q+4)}. \quad (3.16)$$

The structure of $S^{(2p)}$ and $S^{(2q+3)}$ is similar to that of $L^{(2p)}$ and $L^{(2q+3)}$ respectively.

Introducing $S^{(2p)}$ and $S^{(2q+3)}$ in (3.11) we can determine $\xi^{(r)}$ if $\xi^{(0)}, \xi^{(2)}, \ldots, \xi^{(r-1)}$ are known, viz.

$$L^{(0)}\,\xi^{(0)} = \eta^{(0)} \quad (3.17)$$

$$\xi^{(r)} = -S^{(2)}\,\xi^{(r-2)} - S^{(3)}\,\xi^{(r-3)} - \ldots$$

$$\ldots - S^{(r-3)}\,\xi^{(3)} - S^{(r-2)}\,\xi^{(2)} - S^{(r)}\,\xi^{(0)}, \quad r=2,3,4,\ldots \quad (3.18)$$

From (3.17) it is obvious that in $\xi^{(0)}$ only the element with suffix 0 is different from zero; from (3.18) we see that in $\xi^{(3)}$ too only the element with suffix 0 is different from zero. These results imply that because of the structure of $S^{(2p)}$ and $S^{(2q+3)}$ in $\xi^{(2p)}$ only the elements with suffices $0, 1, \ldots, p$ and in $\xi^{(2q+3)}$ only the elements with suffices $0, 1, \ldots, q$ are different from zero. The proof simply follows by induction from (3.18). Moreover the properties of $\xi^{(2p)}$ and $\xi^{(2q+3)}$

show that the first term in the power series expansion of x_m contains the factor ε^{2m}.

All this proves that an approximate solution of (2.11) is obtained by the following procedure:

(i) Only the equations $m = 0, 1, \ldots, N$ are retained.

(ii) In these equations all powers of ε higher than the $2N$-th power are disregarded.

(iii) The former step implies that we take $x_{N+1} = x_{N+2} = \ldots \ldots = 0$.

In this approximate solution the coefficients of the first $2N$ powers of ε in $x_0, \ldots, x_N$ are exact.

§ 4. *Numerical results.* The columns $\xi^{(0)}, \ldots, \xi^{(7)}$ are determined by means of the recurrence relation (3.18). We found

$$\begin{aligned}
\xi^{(0)} &= \{12/\pi, 0, 0, \ldots\},\\
\xi^{(1)} &= \{0, 0, 0, \ldots\},\\
\xi^{(2)} &= \{-12/5\pi, -4/15\pi, 0, 0, \ldots\},\\
\xi^{(3)} &= \{-8/3\pi^2, 0, 0, \ldots\},\\
\xi^{(4)} &= \{8/35\pi, 4/45\pi, 4/1575\pi, 0, 0, \ldots\},\\
\xi^{(5)} &= \{64/75\pi^2, 8/75\pi^2, 0, 0, \ldots\},\\
\xi^{(6)} &= \{-4/315\pi + 16/27\pi^3, -4/495\pi, -4/4095\pi,\\
&\qquad\qquad -4/315315\pi, 0, 0, \ldots\},\\
\xi^{(7)} &= \{-7464/55125\pi^2, -112/3375\pi^2, -8/6615\pi^2, 0, 0, \ldots\}.
\end{aligned}$$

These results together with (2.10) give

$$\begin{aligned}
b_0 &= \varepsilon\,[-2/\pi + (2/5\pi)\,\varepsilon^2 + (4/9\pi^2)\,\varepsilon^3 - (4/105\pi)\,\varepsilon^4 - (32/225\pi^2)\,\varepsilon^5\\
&\quad + (2/945\pi - 8/81\pi^3)\,\varepsilon^6 + (1244/55125\pi^2)\,\varepsilon^7] + O(\varepsilon^9),\\
b_1 &= \varepsilon^3\,[2/45\pi - (2/135\pi)\,\varepsilon^2 - (4/225\pi^2)\,\varepsilon^3 + (2/1485\pi)\,\varepsilon^4 +\\
&\quad + (56/10125\pi^2)\,\varepsilon^5] + O\,(\varepsilon^9),\\
b_2 &= \varepsilon^5\,[-2/4725\pi + (2/12285\pi)\,\varepsilon^2 + (4/19845\pi^2)\,\varepsilon^3] + O\,(\varepsilon^9),\\
b_3 &= (2/945945\pi)\,\varepsilon^7 + O\,(\varepsilon^9).
\end{aligned}$$

Our values of $b_0, \ldots, b_3$ are in complete agreement with B o u w k a m p's results [4]).

Appendix

Evaluation of $\int_0^\infty (v^2 - 1)^{\frac{1}{2}} v^{-2} J_{2m+3/2}(kav) J_{2n+3/2}(kav)\, dv$.

We shall derive a power series expansion in ka of the integral

$$R_{m,n} - iI_{m,n} = \int_0^\infty (v^2 - 1)^{\frac{1}{2}} v^{-2} J_{2m+3/2}(kav) J_{2n+3/2}(kav)\, dv, \quad \text{(A.01)}$$

where

$$R_{m,n} = \int_1^\infty (v^2 - 1)^{\frac{1}{2}} v^{-2} J_{2m+3/2}(kav) J_{2n+3/2}(kav)\, dv, \quad \text{(A.02)}$$

$$I_{m,n} = \int_0^1 (1 - v^2)^{\frac{1}{2}} v^{-2} J_{2m+3/2}(kav) J_{2n+3/2}(kav)\, dv. \quad \text{(A.03)}$$

The simpler of the two is $I_{m,n}$ which is most easily evaluated if use is made of the power series for the product of two Bessel functions of the first kind [5]):

$$J_\mu(z)J_\nu(z) = \sum_{r=0}^\infty \frac{(-)^r (\frac{1}{2}z)^{\mu+\nu+2r} (\mu + \nu + r + 1)_r}{r!\, \Gamma(\mu + r + 1)\, \Gamma(\nu + r + 1)}. \quad \text{(A.04)}$$

Substituting (A.04) in (A.03) and making use of the Eulerian integral of the first kind [6]), we obtain

$$I_{m,n} = \frac{1}{4} \sum_{r=0}^\infty \frac{(-)^r (ka)^{2m+2n+2r+3} \Gamma(m+n+r+2)\, \Gamma(m+n+r+1)}{\Gamma(r+1)\Gamma(2m+r+\frac{5}{2})\, \Gamma(2n+r+\frac{5}{2})\Gamma(2m+2n+r+4)}. \quad \text{(A.05)}$$

The expression for $R_{m,n}$ is more complicated. To obtain the power series expansion we shall follow a procedure which is analogous to the one given by Bouwkamp [7]). According to Watson [8]) one has, if $-\frac{1}{2} < \mathrm{Re}\,(\mu + \nu) < 0$,

$$J_{2\mu}(kav)J_{2\nu}(kav) =$$

$$= \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\, \Gamma(2s + 2\mu + 2\nu + 1)\, (\frac{1}{2}kav)^{2\mu+2\nu+2s}}{\Gamma(s + 2\mu + 1)\, \Gamma(s + 2\nu + 1)\, \Gamma(s + 2\mu + 2\nu + 1)}\, ds, \quad \text{(A.06)}$$

in which the path of integration coincides with the imaginary axis except for an indentation towards the left at the origin. Transforming the gamma-functions in (A.06) we get

$$J_{2\mu}(kav)J_{2\nu}(kav) =$$

$$= -\frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\frac{1}{2})\Gamma(s + \mu + \nu + \frac{1}{2})\, \Gamma(s + \mu + \nu + 1)\, (kav)^{2s+2\mu+2\nu}\, ds}{\Gamma(s+1)\Gamma(s+2\mu+1)\, \Gamma(s+2\nu+1)\, \Gamma(s+2\mu+2\nu+1) \sin \pi s}$$

and consequently

$$\int_1^\infty (v^2-1)^{\frac12} v^{-2} J_{2\mu}(kav) J_{2\nu}(kav)\, dv =$$

$$-\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(\frac12)\,\Gamma(s+\mu+\nu+\frac12)\,\Gamma(s+\mu+\nu+1)\,(ka)^{2s+2\mu+2\nu}\,ds}{\Gamma(s+1)\,\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)\sin\pi s}\cdot$$

$$\cdot\int_1^\infty (v^2-1)^{\frac12}\, v^{2s+2\mu+2\nu-2} dv.$$

However, for the range of $(\mu+\nu)$ under consideration we have

$$\int_1^\infty (v^2-1)^{\frac12}\, v^{2s+2\mu+2\nu-2}\, dv = \frac{\Gamma(\frac32)\Gamma(s+\mu+\nu-\frac12)}{2\Gamma(s+\mu+\nu+1)}\,\frac{\cos(s+\mu+\nu)\,\pi}{\sin(s+\mu+\nu)\,\pi}.$$

Hence

$$\int_1^\infty (v^2-1)^{\frac12} v^{-2} J_{2\mu}(kav) J_{2\nu}(kav)\, dv =$$

$$-\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(\frac12)\Gamma(\frac32)\Gamma(s+\mu+\nu+\frac12)\,\Gamma(s+\mu+\nu-\frac12)\,(ka)^{2s+2\mu+2\nu}}{2\,\Gamma(s+1)\,\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)\sin\pi s}$$

$$\cdot\frac{\cos(s+\mu+\nu)\,\pi}{\sin(s+\mu+\nu)\,\pi}\,ds. \qquad (A.07)$$

The poles of the integrand to the right of the path of integration are located at $s=r$ and $s=-\mu-\nu+r$ $(r=0,\ 1,\ 2,\ \ldots.)$. Closing the contour to the right, application of the theorem of residues yields (with $\mu=m+\frac34$ and $\nu=n+\frac34$)

$$R_{m,n} =$$

$$= \frac14\sum_{r=0}^{\infty} \frac{(-)^{m+n+r}\,\Gamma(r+\frac12)\,\Gamma(r-\frac12)\,(ka)^{2r}}{\Gamma(-m-n+r-\frac12)\,\Gamma(m-n+r+1)\,\Gamma(-m+n+r+1)\Gamma(m+n+r+\frac52)}. \quad (A.08)$$

By analytic continuation (A.08) holds for all positive values of m and n including zero.

From (A.05) and (A.08) an elegant expression for $R_{m,n}-iI_{m,n}$ can be obtained if in (A.05) the terms corresponding to $r=-1,\ -2,\ \ldots,\ -m-n$ (which are zero) are added. The modified sum then contains all powers of ika except the first power, viz.

$$R_{m,n}-iI_{m,n} = \frac14(-)^{m+n}\sum_{r=0}^{\infty}{}' \frac{\Gamma(\frac12 r+\frac12)\,\Gamma(\frac12 r-\frac12)}{\Gamma(-m-n+\frac12 r-\frac12)\,\Gamma(m-n+\frac12 r+1)}$$

$$\cdot\frac{(ika)^r}{\Gamma(-m+n+\frac12 r+1)\,\Gamma(m+n+\frac12 r+\frac52)}, \qquad (A.09)$$

where the prime denotes that the term corresponding to $r = 1$ has to be omitted.

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