# ON THE PLANE-WAVE EXTINCTION CROSS-SECTION OF AN OBSTACLE

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**Summary**

A time-harmonic plane electromagnetic wave is incident upon an obstacle of finite dimensions. The properties of the obstacle are such that electromagnetic power is both absorbed and scattered. A close relation exists between the extinction cross-section of the obstacle and the amplitude and phase of the scattered wave in the direction of propagation of the incident wave. The exact form of this relation, the "cross-section theorem", is proved by making use of an explicit representation of the scattered field. The result is valid for a plane wave with arbitrary elliptic polarization. Finally, a similar relation for the scattering of sound waves is given.

§ 1. *Introduction.* In the study of the scattering and diffraction of a time-harmonic plane electromagnetic wave by an obstacle of finite dimensions, one of the major problems is the evaluation of the average power absorbed and the average power scattered by the obstacle. It has been argued by Van de Hulst [1]) [2]) that, on physical grounds, the sum of absorbed and scattered power is expected to be closely related to the amplitude and phase of the scattered wave in the direction in which the incident wave propagates. When expressed in terms of the extinction cross-section (for its definition, see § 3), the indicated relation is called the "cross-section theorem".

The special case of perfectly conducting obstacles has been investigated by Levine and Schwinger [3]) (scattering by plane obstacles of vanishing thickness) and by Storer and Sevick [4]) (scattering by obstacles of arbitrary shape). The proof given by these authors is based on the integral equation to be satisfied by the surface-current density at the boundary of the obstacle. The generalization to the case of an obstacle with arbitrary electromagnetic properties is due to Jones [5]) who, in order to prove the theorem, employed the method of stationary phase for two-dimensional integrals. However, in studying Jones' paper, the present author observed that this complicated method can be avoided by using an explicit representation of the far-zone scattered field. The latter method leads to the proof given in § 3, where the general case of an elliptically polarized incident plane wave is considered.

At this point *) it may be remarked that cross-section theorems similar to the ones given in § 3 and § 4 hold in any type of scattering problem [6]). In the field of scattering by atomic systems a relation of this kind has been known at least since 1932 [7]). For details the reader is referred to the literature on the subject [6]) [7]) [8]).

§ 2. *The far-zone scattered field.* A time-harmonic, elliptically polarized, plane electromagnetic wave is incident upon an obstacle of finite dimensions. The boundary of the obstacle is a sufficiently regular closed surface $S$. The electric and magnetic properties of the obstacle are assumed to be such that electromagnetic power is both absorbed and scattered. The medium in the domain outside $S$ is assumed to be homogeneous, isotropic and non-conducting (which includes the case of free space), with inductive capacities $\varepsilon_0$ and $\mu_0$. In the exterior domain, the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{H}$ are written as the sum of the incident field $\mathbf{E}_i$, $\mathbf{H}_i$ and the scattered field $\mathbf{E}_s$, $\mathbf{H}_s$:

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s, \tag{2.1}$$

$$\mathbf{H} = \mathbf{H}_i + \mathbf{H}_s. \tag{2.2}$$

Both the incident and the scattered field satisfy Maxwell's equations

$$\operatorname{curl} \mathbf{H} = - i\omega\varepsilon_0 \mathbf{E}, \tag{2.3}$$

$$\operatorname{curl} \mathbf{E} = i\omega\mu_0 \mathbf{H}, \tag{2.4}$$

where $\omega$ is the angular frequency of the exponential time dependence of the form $\exp(- i\omega t)$. This factor, which has been omitted throughout, is common to all field components.

*) For this remark the author is indebted to Professor R. Kronig, Technische Hogeschool, Delft, Netherlands.

Let $\mathbf{r}$ be the radius vector drawn from a fixed origin located somewhere in the domain occupied by the obstacle. The incident field is then given by

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{A} \exp(-\, ik\boldsymbol{\alpha}\cdot\mathbf{r}), \tag{2.5}$$

$$\mathbf{H}_i(\mathbf{r}) = (\varepsilon_0/\mu_0)^{\frac{1}{2}}(\mathbf{A} \times \boldsymbol{\alpha}) \exp(-\, ik\boldsymbol{\alpha}\cdot\mathbf{r}), \tag{2.6}$$

where $\mathbf{A}$ specifies the polarization of the incident wave (in general, elliptic) and $\boldsymbol{\alpha}$ is the unit vector pointing *toward* the source at infinity. Further,

$$k = \omega(\varepsilon_0\mu_0)^{\frac{1}{2}} = 2\pi/\lambda, \tag{2.7}$$

$\lambda$ being the wave length.

The next problem is to obtain an expression for the scattered field. It has been stated already that $\mathbf{E}_s$ and $\mathbf{H}_s$ satisfy Maxwell's equations (2.3) and (2.4). In addition, the field vectors shall satisfy the radiation condition [9] *)

$$\int_{S_R} |\mathbf{E}_s - (\mu_0/\varepsilon_0)^{\frac{1}{2}}(\mathbf{H}_s \times \boldsymbol{i}_R)|^2 \, \mathrm{d}S = o(1) \qquad (R \to \infty), \tag{2.8}$$

where $S_R$ is a sphere of radius $R$ around some point of observation and $\boldsymbol{i}_R$ is the unit vector in the direction of the outward normal to $S_R$. Under these conditions the following representation holds [10]

$$4\pi\mathbf{E}_s(\mathbf{r}) = \operatorname{curl}\int_S [\mathbf{n} \times \mathbf{E}_s(\boldsymbol{\rho})] \frac{e^{ikR}}{R} \, \mathrm{d}S - \frac{1}{i\omega\varepsilon_0} \operatorname{curl}\operatorname{curl}\int_S [\mathbf{n} \times \mathbf{H}_s(\boldsymbol{\rho})] \frac{e^{ikR}}{R} \, \mathrm{d}S, \tag{2.9}$$

$$4\pi\mathbf{H}_s(\mathbf{r}) = \operatorname{curl}\int_S [\mathbf{n} \times \mathbf{H}_s(\boldsymbol{\rho})] \frac{e^{ikR}}{R} \, \mathrm{d}S + \frac{1}{i\omega\mu_0} \operatorname{curl}\operatorname{curl}\int_S [\mathbf{n} \times \mathbf{E}_s(\boldsymbol{\rho})] \frac{e^{ikR}}{R} \, \mathrm{d}S. \tag{2.10}$$

In (2.9) and (2.10) $\mathbf{n}$ is the unit vector in the direction of the outward normal to $S$ and $R = |\mathbf{r} - \boldsymbol{\rho}|$ is the distance from the point of observation $\mathbf{r} = (x, y, z)$ to the point of integration $\boldsymbol{\rho} = (\xi, \eta, \zeta)$. Although $S$ in (2.9) and (2.10) could be any suffi-

*) For a vector $\mathbf{A}$ whose components are complex numbers, we have $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}^*$, where $\mathbf{A}^*$ denotes the complex conjugate to $\mathbf{A}$.

ciently regular bounded closed surface completely surrounding the obstacle, we take, for the sake of simplicity, $S$ to be the surface of the obstacle.

Let $\boldsymbol{\beta}$ denote the unit vector in the direction of observation; then, $\mathbf{r} = r\boldsymbol{\beta}$. At large distances from the obstacle we have

$$R^{-1}\exp(ikR) = r^{-1}\exp(ikr - ik\boldsymbol{\beta}\cdot\boldsymbol{\rho}) + O(r^{-2}) \quad (r \to \infty). \quad (2.11)$$

Using this type of expansion we obtain from (2.9) and (2.10)

$$\mathbf{E}_s(\mathbf{r}) = \mathbf{F}(\boldsymbol{\beta})\,\frac{e^{ikr}}{ikr} + O(r^{-2}) \quad (r \to \infty), \quad (2.12)$$

$$(\mu_0/\varepsilon_0)^{\frac{1}{2}}\mathbf{H}_s(\mathbf{r}) = [\boldsymbol{\beta} \times \mathbf{F}(\boldsymbol{\beta})]\,\frac{e^{ikr}}{ikr} + O(r^{-2}) \quad (r \to \infty), \quad (2.13)$$

where the (complex) factor $\mathbf{F}(\boldsymbol{\beta})$ is given by

$$4\pi\mathbf{F}(\boldsymbol{\beta}) = -\,k^2\boldsymbol{\beta} \times \int_S [\mathbf{n} \times \mathbf{E}_s(\boldsymbol{\rho})]\exp(-\,ik\boldsymbol{\beta}\cdot\boldsymbol{\rho})\,\mathrm{d}S +$$
$$+\,(\mu_0/\varepsilon_0)^{\frac{1}{2}}k^2\boldsymbol{\beta} \times \{\boldsymbol{\beta} \times \int_S [\mathbf{n} \times \mathbf{H}_s(\boldsymbol{\rho})]\exp(-\,ik\boldsymbol{\beta}\cdot\boldsymbol{\rho})\,\mathrm{d}S\}. \quad (2.14)$$

The first term of the right-hand side of (2.12) and (2.13) is called the "far-zone approximation". The special case of forward scattering is obtained by taking $\boldsymbol{\beta} = -\,\boldsymbol{\alpha}$.

In obtaining (2.12), (2.13) and (2.14) we have used the fact that each Cartesian component of the integrals on the right-hand side of (2.9) and (2.10) admits a representation of the form

$$u(\mathbf{r}) = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{a_n(\boldsymbol{\beta})}{r^n}, \quad (2.15)$$

where $u(\mathbf{r})$ denotes any of these Cartesian components. This expansion converges absolutely and uniformly in the domain $r \geqq r_0 + \delta > r_0$, where $r = r_0$ is the smallest sphere around the origin, completely surrounding the obstacle. The series (2.15) can be differentiated term by term with respect to the coordinates any number of times and the resulting series is absolutely and uniformly convergent in $r \geqq r_0 + \delta > r_0$. Performing the necessary differentiations we then obtain for each Cartesian component of $\mathbf{E}_s(\mathbf{r})$ and $\mathbf{H}_s(\mathbf{r})$ a representation of the form (2.15). In fact, the first term of the right-hand side of (2.12) and (2.13) is the first term of the relevant series expansion. Further, it can be shown that all coefficients of this expansion are determined by $\mathbf{F}(\boldsymbol{\beta})$.

Finally, the result satisfies the radiation condition (2.8). For a proof of these statements we refer to a paper by Wilcox [9]).

§ 3. *Proof of the cross-section theorem.* From the complex form of Poynting's theorem [11]) it follows that the average power *absorbed* by the obstacle is given by

$$P_a = -\tfrac{1}{2}\,\mathrm{Re}\int_S (\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{n}\,\mathrm{d}S. \tag{3.1}$$

The average *scattered* power is *defined* as

$$P_s = \tfrac{1}{2}\,\mathrm{Re}\int_S (\mathbf{E}_s\times\mathbf{H}_s^*)\cdot\mathbf{n}\,\mathrm{d}S. \tag{3.2}$$

Substituting (2.1) and (2.2) in the right-hand side of (3.1) and taking into account that

$$\tfrac{1}{2}\,\mathrm{Re}\int_S (\mathbf{E}_i\times\mathbf{H}_i^*)\cdot\mathbf{n}\,\mathrm{d}S = 0, \tag{3.3}$$

we get

$$P_a + P_s = -\tfrac{1}{2}\,\mathrm{Re}\int_S (\mathbf{E}_i^*\times\mathbf{H}_s + \mathbf{E}_s\times\mathbf{H}_i^*)\cdot\mathbf{n}\,\mathrm{d}S. \tag{3.4}$$

When (2.5) and (2.6) are used in the integral on the right-hand side of (3.4), the result is closely related to the expression for $\mathbf{F}(-\boldsymbol{\alpha})$ following from (2.14). By inspection we have

$$\int_S (\mathbf{E}_i^*\times\mathbf{H}_s + \mathbf{E}_s\times\mathbf{H}_i^*)\cdot\mathbf{n}\,\mathrm{d}S = 4\pi k^{-2}(\varepsilon_0/\mu_0)^{\frac{1}{2}}\mathbf{A}^*\cdot\mathbf{F}(-\boldsymbol{\alpha}). \tag{3.5}$$

Substitution of (3.5) in (3.4) yields an expression that relates $P_a + P_s$ to the far-zone scattered field in the forward direction.

Finally, the absorption cross-section $\sigma_a$ and the scattering cross-section $\sigma_s$ of the obstacle are introduced. These quantities are defined as follows: the absorption (scattering) cross-section is the ratio of the mean power absorbed (scattered) by the obstacle to the mean intensity of power flow in the incident field. The latter quantity is given by

$$P_i^{(1)} = -\tfrac{1}{2}\,\mathrm{Re}\,[(\mathbf{E}_i\times\mathbf{H}_i^*)\cdot\boldsymbol{\alpha}],$$

which reduces to

$$P_i^{(1)} = \tfrac{1}{2}(\varepsilon_0/\mu_0)^{\frac{1}{2}}\,|\mathbf{A}|^2. \tag{3.6}$$

Equations (3.4), (3.5) and (3.6) lead to the result

$$\sigma_a + \sigma_s = -\frac{4\pi}{k^2}\,\frac{\mathrm{Re}\,[\mathbf{A}^*\cdot\mathbf{F}(-\boldsymbol{\alpha})]}{|\mathbf{A}|^2}\,. \tag{3.7}$$

Equation (3.7) is known as the "cross-section theorem". The sum of the absorption cross-section and the scattering cross-section is often called the extinction cross-section [12]).

§ 4. *The extinction cross-section for scattering of sound waves.* In concluding this paper we want to remark that a similar cross-section theorem holds for the scattering of a plane sound wave by an obstacle of finite dimensions. The corresponding problem can be formulated in terms of a scalar function, viz. the velocity potential. Since the proof of the theorem is analogous to the one outlined in the preceding sections, we confine ourselves to stating the result. Again, the complex time factor $\exp(-i\omega t)$ is omitted.

If $\mathbf{v}$ is the (irrotational) particle velocity, a velocity potential $\Phi$ is introduced such that

$$\mathbf{v} = -\operatorname{grad}\Phi. \tag{4.1}$$

The excess pressure $p$ is related to the velocity potential through

$$p = -i\omega\rho_0\Phi, \tag{4.2}$$

where $\rho_0$ is the density of the medium.

Let $\Phi_i(\mathbf{r})$ and $\Phi_s(\mathbf{r})$ be the velocity potential of the incident and the scattered sound field, respectively; then, $\Phi = \Phi_i + \Phi_s$. Further, let $\Phi_i(\mathbf{r})$ be given by

$$\Phi_i(\mathbf{r}) = A\exp(-ik\boldsymbol{\alpha}\cdot\mathbf{r}), \tag{4.3}$$

in which $k = \omega/c$, $c$ being the velocity of sound. At large distances from the obstacle, $\Phi_s(\mathbf{r})$ is written in the form [13])

$$\Phi_s(\mathbf{r}) = F(\boldsymbol{\beta})\frac{e^{ikr}}{ikr} + O(r^{-2}) \quad (r \to \infty). \tag{4.4}$$

The average power absorbed by the obstacle is given by

$$P_a = -\tfrac{1}{2}\operatorname{Re}\int_S p\,\mathbf{n}\cdot\mathbf{v}^*\,\mathrm{d}S, \tag{4.5}$$

or, expressed in terms of the velocity potential,

$$P_a = -\tfrac{1}{2}\operatorname{Re}\{i\omega\rho_0\int_S \Phi\,(\mathbf{n}\cdot\operatorname{grad}\Phi^*)\,\mathrm{d}S\}, \tag{4.6}$$

where $\mathbf{n}$ is the unit vector in the direction of the outward normal to the boundary $S$ of the obstacle. The average scattered power is defined as

$$P_s = \tfrac{1}{2} \operatorname{Re} \{i\omega\rho_0 \int_S \Phi_s(\mathbf{n}\cdot\operatorname{grad} \Phi_s^*) \, dS\}. \quad (4.7)$$

The mean intensity of power flow in the incident wave is

$$P_i^{(1)} = \tfrac{1}{2}\omega\rho_0 k \, |A|^2. \quad (4.8)$$

When the expression for $P_a + P_s$ is compared with the representation of $F(\boldsymbol{\beta})$ it follows by inspection that $P_a + P_s$ can be expressed in terms of $F(-\boldsymbol{\alpha})$. Introduction of the absorption cross-section $\sigma_a = P_a/P_i^{(1)}$ and the scattering cross-section $\sigma_s = P_s/P_i^{(1)}$ then yields

$$\sigma_a + \sigma_s = -\frac{4\pi}{k^2} \frac{\operatorname{Re}[A^*F(-\boldsymbol{\alpha})]}{|A|^2} = -\frac{4\pi}{k^2} \operatorname{Re}\left[\frac{F(-\boldsymbol{\alpha})}{A}\right]. \quad (4.9)$$

Equation (4.9) is known as the cross-section theorem for sound waves. The special case of scattering by a plane obstacle of vanishing thickness has been investigated by Levine and Schwinger [14]).

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