Appl. sci. Res. Section B, Vol. 8

A MODIFICATION OF CAGNIARD'S METHOD FOR SOLVING SEISMIC PULSE PROBLEMS

by A. T. DE HOOP

Laboratorium voor Theoretische Elektrotechniek, Technische Hogeschool, Delft, Netherlands

Summary

A modification of Cagniard's method for solving seismic pulse problems is given. In order to give a clear picture of our method, two simple problems are solved, viz. the determination of the scalar cylindrical wave generated by an impulsive line source and the scalar spherical wave generated by an impulsive point source.

§ 1. *Introduction.* The application of Cagniard's [1]) method in obtaining exact solutions of three-dimensional seismic pulse problems leads to complicated expressions for the components of the displacement vector in the elastic solid. This is partly due to the fact that in a homogeneous, isotropic, elastic solid two types of waves, travelling with different velocities, occur. In order to give a clear picture of Cagniard's method, Dix [2]) applied it to a simple problem in scalar wave propagation, viz. the determination of the spherical wave generated by an impulsive point source located in a homogeneous, isotropic, unbounded medium. Even in this simple problem (the solution of which can also be obtained by less complicated methods) quite a number of transformations of complex contour integrals are involved.

In the present paper it is shown that Cagniard's method can be simplified considerably if the corresponding modification for two-dimensional problems as developed by the present author [3,4]) is taken as a guidance. Again, the aforementioned point source problem will be considered; for reference, also the solution of the corresponding line source problem will be given.

It is remarked that the resulting method is also simpler than

the technique employed by Pekeris [5–10]), which is slightly different from the one due to Cagniard.

§ 2. *The scalar wave generated by an impulsive line source.* Let x, y, z be Cartesian coordinates in three-dimensional space. A point in space will be located by either its Cartesian coordinates, its cylindrical coordinates r, φ, z defined through

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z, \tag{2.1}$$

with $0 \leqq r < \infty$, $0 \leqq \varphi < 2\pi$, $-\infty < z < \infty$, or its spherical polar coordinates defined through

$$x = R \sin \theta \cos \varphi, \quad y = R \sin \theta \sin \varphi, \quad z = R \cos \theta, \tag{2.2}$$

with $0 \leqq R < \infty$, $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi < 2\pi$.

The two-dimensional wave function $u = u(x, y, t)$ due to the presence of a two-dimensional line source acting at $x = 0$, $y = 0$, satisfies the two-dimensional scalar wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = - \delta(x, y) f(t), \tag{2.3}$$

where $\delta(x, y)$ denotes the two-dimensional delta function and v is the wave front velocity. The function $f(t)$ determines the strength of the line source as a function of time; it is assumed that $f(t) = 0$ when $t < 0$. Further, it is assumed that the medium is at rest prior to the instant $t = 0$ and that everywhere outside the source $u = u(x, y, t)$ is continuous and has continuous partial derivatives of the first and second order.

Following Cagniard, all functions of time are subjected to a one-sided Laplace transform with respect to time. Let

$$F(s) = \int_0^\infty \exp(-st) f(t) \, dt \tag{2.4}$$

and

$$U(x, y; s) = \int_0^\infty \exp(-st) u(x, y, t) \, dt, \tag{2.5}$$

where s is a real, positive, number large enough to ensure the convergence of the integrals (2.4) and (2.5) (it is assumed that the behaviour of $f(t)$ and $u(x, y, t)$ as $t \to \infty$ is such that such a number s can be found). Since u and $\partial u/\partial t$ are continuous, $U(x, y; s)$ satisfies

the differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{s^2}{v^2} U = -\delta(x, y) F(s). \tag{2.6}$$

In order to solve (2.6) we introduce the Fourier transform of $U(x, y; s)$ with respect to x. Let

$$\mathscr{U}(\alpha; y; s) = \int_{-\infty}^{\infty} \exp(is\alpha x) U(x, y; s) \, dx, \tag{2.7}$$

where the factor s in the argument of the exponential function has been included for convenience. With (2.7) the following equation for $\mathscr{U}(\alpha; y; s)$ is obtained

$$\frac{d^2\mathscr{U}}{dy^2} - s^2\gamma^2\mathscr{U} = -\delta(y) F(s), \tag{2.8}$$

where

$$\gamma = \gamma(\alpha) = (\alpha^2 + 1/v^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma \geqq 0). \tag{2.9}$$

As indicated in (2.9), γ is defined as that branch of the square root at the right-hand side of (2.9) for which $\mathrm{Re}\, \gamma \geqq 0$. The solution of (2.8) that is bounded as $|y| \to \infty$ is given by

$$\mathscr{U}(\alpha; y; s) = \frac{F(s)}{2s\gamma} \exp(-s\gamma\, |y|). \tag{2.10}$$

With the aid of Fourier's inversion theorem we then obtain for $U(x, y; s)$ the expression

$$U(x, y; s) = \frac{F(s)}{2\pi} \int_{-\infty}^{\infty} \exp(-is\alpha x - s\gamma\, |y|) \frac{1}{2\gamma} \, d\alpha. \tag{2.11}$$

In the right-hand side of (2.11) we write $\alpha = -ip$ and consider p as a complex variable in the p-plane. This leads to

$$U(x, y; s) = \frac{F(s)}{2\pi i} \int_{-i\infty}^{i\infty} \exp[-s(px + \gamma\, |y|)] \frac{1}{2\gamma} \, dp, \tag{2.12}$$

in which

$$\gamma = (1/v^2 - p^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma \geqq 0). \tag{2.13}$$

The only singularities of the integrand in (2.12) are branch points at $p = 1/v$ and $p = -1/v$. In view of subsequent deformations

of the path of integration we take $\operatorname{Re}\gamma \geqq 0$ everywhere in the p-plane. This implies that branch cuts are introduced along $\operatorname{Im} p = 0$, $1/v < |\operatorname{Re} p| < \infty$.

The next step towards the solution of the transient problem is to perform the integration in the p-plane along such a path that the right-hand side of (2.12) can be recognized as the Laplace transform of a certain function of time. The analysis which follows will show that the path has to be selected such that

$$px + \gamma|y| = \tau, \tag{2.14}$$

where τ is real and positive. If $r/v < \tau < \infty$, eq. (2.14) represents the branch Γ of a hyperbola, where Γ is given through

$$p = \frac{x}{r^2}\tau \pm i\frac{|y|}{r^2}(\tau^2 - r^2/v^2)^{\frac{1}{2}} \quad (r/v < \tau < \infty), \tag{2.15}$$

in which the square root is taken positive. It is easily verified that, by virtue of Cauchy's theorem and Jordan's lemma [11]), the integral along the imaginary p-axis is equal to the integral along Γ. Along Γ we have

$$\gamma = \frac{|y|}{r^2}\tau \mp i\frac{x}{r^2}(\tau^2 - r^2/v^2)^{\frac{1}{2}} \tag{2.16}$$

and

$$\frac{\partial p}{\partial \tau} = \pm\frac{i\gamma}{(\tau^2 - r^2/v^2)^{\frac{1}{2}}}. \tag{2.17}$$

In (2.15), (2.16) and (2.17) the upper and lower signs belong together. Taking into account the symmetry of the path of integration with respect to the real axis and introducing τ as variable of integration we obtain

$$U(x, y; s) = \frac{F(s)}{2\pi}\int_{r/v}^{\infty} \exp(-s\tau)(\tau^2 - r^2/v^2)^{-\frac{1}{2}}\,\mathrm{d}\tau. \tag{2.18}$$

This expression is of the general form

$$U(x, y; s) = F(s)\int_0^{\infty} \exp(-s\tau)\, g(x, y, \tau)\,\mathrm{d}\tau, \tag{2.19}$$

where, in our case,

$$g(x, y, \tau) = \begin{cases} 0 & (0 < \tau < r/v), \\ \dfrac{1}{2\pi}(\tau^2 - r^2/v^2)^{-\frac{1}{2}} & (r/v < \tau < \infty). \end{cases} \tag{2.20}$$

Application of the shift rule for Laplace transforms to the function $F(s)\exp(-s\tau)$ directly yields the function $u(x, y, t)$. We obtain

$$u(x, y, t) = \int_0^t f(t - \tau)\, g(x, y, \tau)\, \mathrm{d}\tau \qquad (t > 0), \tag{2.21}$$

while, from our assumptions, $u(x, y, t) = 0$ when $t < 0$. In our case we have

$$u(x, y, t) = \begin{cases} 0 & (0 < t < r/v), \\ \dfrac{1}{2\pi}\displaystyle\int_{r/v}^{t} f(t - \tau)(\tau^2 - r^2/v^2)^{-\frac{1}{2}}\, \mathrm{d}\tau & (r/v < t < \infty). \end{cases} \tag{2.22}$$

From the final result (2.22) it is clear that $g(x, y, t)$ can be regarded as the wave function corresponding to a delta function time dependence of the source.

§ 3. *The scalar wave generated by an impulsive point source.* The three-dimensional wave function $u = u(x, y, z, t)$ due to the presence of a point source acting at $x = 0$, $y = 0$, $z = 0$ satisfies the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = -\delta(x, y, z)\, f(t), \tag{3.1}$$

where $\delta(x, y, z)$ denotes the three-dimensional delta function. Again, we assume that, outside the source, u is continuous and has continuous partial derivatives of the first and second order. Further, $f(t) = 0$ when $t < 0$ and $u \equiv 0$ when $t < 0$. The following one-sided Laplace transforms with respect to time are introduced

$$F(s) = \int_0^\infty \exp(-st)\, f(t)\, \mathrm{d}t \tag{3.2}$$

and

$$U(x, y, z; s) = \int_0^\infty \exp(-st)\, u(x, y, z, t)\, \mathrm{d}t. \tag{3.3}$$

Since u and $\partial u/\partial t$ are continuous, $U(x, y, z; s)$ satisfies the differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{s^2}{v^2} U = -\delta(x, y, z) F(s). \tag{3.4}$$

In order to solve (3.4) we introduce the two-dimensional Fourier transform of $U(x, y, z; s)$ with respect to x and y. Let

$$\mathscr{U}(\alpha, \beta; z; s) = \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \exp[is(\alpha x + \beta y)]\, U(x, y, z; s)\, \mathrm{d}x. \tag{3.5}$$

Then $\mathscr{U}(\alpha, \beta; z; s)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 \mathscr{U}}{\mathrm{d}z^2} - s^2\gamma^2 \mathscr{U} = -\delta(z) F(s), \tag{3.6}$$

where

$$\gamma = \gamma(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/v^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma \geqq 0). \tag{3.7}$$

The solution of (3.7) that is bounded as $|z| \to \infty$ is given by

$$\mathscr{U}(\alpha, \beta; z; s) = \frac{F(s)}{2s\gamma} \exp(-s\gamma\, |z|). \tag{3.8}$$

With the aid of Fourier's inversion theorem we obtain the following expression for $U(x, y, z; s)$:

$$U(x, y, z; s) = \frac{sF(s)}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\beta \int_{-\infty}^{\infty} \exp[-is(\alpha x + \beta y) - s\gamma\, |z|] \frac{1}{2\gamma}\, \mathrm{d}\alpha. \tag{3.9}$$

Again, we shall try to cast the integral on the right-hand side of (3.9) in such a form that $u(x, y, z, t)$ can be found more or less by inspection. It will be advantageous to transform the exponential function into a form which resembles the one occurring in the two-dimensional problem. This is accomplished by introducing new variables of integration ω and q through

$$\alpha = \omega \cos \varphi - q \sin \varphi, \quad \beta = \omega \sin \varphi + q \cos \varphi. \tag{3.10}$$

Since $\mathrm{d}\alpha\, \mathrm{d}\beta = \mathrm{d}\omega\, \mathrm{d}q$, we obtain

$$U(x, y, z; s) = \frac{sF(s)}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \exp[-is\omega r - s\gamma\, |z|] \frac{1}{2\gamma}\, \mathrm{d}\omega, \tag{3.11}$$

in which, as $\alpha^2 + \beta^2 = \omega^2 + q^2$,

$$\gamma = (\omega^2 + q^2 + 1/v^2)^{\frac{1}{2}} \quad (\mathrm{Re}\,\gamma \geqq 0). \tag{3.12}$$

In order to bring the right-hand side of (3.11) in a form which is analogous to the two-dimensional case, we introduce the variable $p = i\omega$ and regard p as a complex variable in the p-plane, while q is kept real. The result is

$$U(x, y, z; s) = \frac{sF(s)}{4\pi^2 i} \int_{-\infty}^{\infty} \mathrm{d}q \int_{-i\infty}^{i\infty} \exp[-s(pr + \gamma|z|)] \frac{1}{2\gamma}\, \mathrm{d}p, \tag{3.13}$$

in which

$$\gamma = (q^2 + 1/v^2 - p^2)^{\frac{1}{2}} \quad (\mathrm{Re}\,\gamma \geqq 0). \tag{3.14}$$

From now on, the procedure is similar to the one outlined in § 2. By virtue of Cauchy's theorem and Jordan's lemma the integration along the imaginary p-axis can be replaced by an integration along the branch Γ of a hyperbola, where Γ is given through

$$p = \frac{r}{R^2}\tau \pm i\,\frac{|z|}{R^2}\,[\tau^2 - R^2(q^2 + 1/v^2)]^{\frac{1}{2}}$$

$$(R(q^2 + 1/v^2)^{\frac{1}{2}} < \tau < \infty). \tag{3.15}$$

Along Γ we have

$$\gamma = \frac{|z|}{R^2}\tau \mp i\,\frac{r}{R^2}\,[\tau^2 - R^2(q^2 + 1/v^2)]^{\frac{1}{2}} \tag{3.16}$$

and

$$\frac{\partial p}{\partial \tau} = \pm \frac{i\gamma}{[\tau^2 - R^2(q^2 + 1/v^2)]^{\frac{1}{2}}}. \tag{3.17}$$

In (3.15), (3.16) and (3.17) the upper and lower signs belong together. Taking into account the symmetry of the path of integration with respect to the real axis and introducing τ as variable of integration we obtain

$$U(x, y, z; s) =$$

$$= \frac{sF(s)}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}q \int_{R(q^2 + 1/v^2)^{\frac{1}{2}}}^{\infty} \exp(-s\tau)\,[\tau^2 - R^2(q^2 + 1/v^2)]^{-\frac{1}{2}}\, \mathrm{d}\tau. \tag{3.18}$$

Now we interchange the order of integration, which leads to

$$U(x, y, z; s) =$$

$$= \frac{sF(s)}{4\pi^2} \int_{R/v}^{\infty} \exp(-s\tau)\, d\tau \int_{-(\tau^2/R^2 - 1/v^2)^{\frac{1}{2}}}^{(\tau^2/R^2 - 1/v^2)^{\frac{1}{2}}} [\tau^2 - R^2 (q^2 + 1/v^2)]^{-\frac{1}{2}}\, dq$$

$$= \frac{sF(s)}{4\pi R} \int_{R/v}^{\infty} \exp(-s\tau)\, d\tau,$$

or

$$U(x, y, z; s) = F(s) \frac{\exp(-sR/v)}{4\pi R}. \tag{3.19}$$

Application of the shift rule yields the well-known result

$$u(x, y, z, t) = \frac{f(t - R/v)}{4\pi R}. \tag{3.20}$$

§ 4. *Conclusion.* The procedure outlined in the present paper provides a method by means of which several mixed initial-boundary value problems can be solved. For *two-dimensional problems* also other methods are available, in particular the "method of conical flow". This method has been applied by Maue [12]) and, more recently, by Miles [13]) to several two-dimensional problems in elastodynamics.

Received 19th May, 1960.

REFERENCES

1) Cagniard, L., Réflexion et réfraction des ondes séismiques progressives, Gauthier-Villars, Paris 1939.
2) Dix, C. H., Geophysics **19** (1954) 722.
3) Hoop, A. T. de, The surface line source problem, Second Annual Report Seismic Scattering Project, Institute of Geophysics, University of California, Los Angeles, California, U.S.A., 1957.
4) Hoop, A. T. de, Representation theorems for the displacement in an elastic solid and their application to elastodynamic diffraction theory, Thesis, Delft, Netherlands, 1958; Ch. IV.
5) Pekeris, C. L., Proc. Nat. Acad. Sci. **41** (1955) 469.
6) Pekeris, C. L., Proc. Nat. Acad. Sci. **41** (1955) 629.
7) Pekeris, C. L., Proc. Nat. Acad. Sci. **42** (1956) 439.
8) Pekéris, C. L. and Z. Alterman, J. Appl. Phys. **28** (1957) 1317.
9) Pekeris, C. L. and H. Lifson, J. Acoust. Soc. Amer. **29** (1957) 1233.
10) Pekeris, C. L. and I. M. Longman, J. Acoust. Soc. Amer. **30** (1958) 323.
11) Whittaker, E. T. and G. N. Watson, A course of modern analysis, 4th ed., p. 115, Cambridge 1950.
12) Maue, A. —W., Z. angew. Math. Mech. **34** (1954) 1.
13) Miles, J. W., Homogeneous solutions in elastic wave propagation. Report GM–TR–0165–00350 of Space Technology Laboratories, Ramo-Wooldridge Corporation, Los Angeles, California, U.S.A., 1958; also Quart. Appl. Math. **18** (1960) 37.