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RADIATION OF PULSES GENERATED BY A VERTICAL ELECTRIC DIPOLE ABOVE A PLANE, NON-CONDUCTING, EARTH

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Summary

At a height h above a plane, non-conducting, earth a vertical electric dipole emits an impulsive electromagnetic wave. The resulting electromagnetic field in the air is determined; it consists of a reflected wave which is superimposed upon the given incident wave. The Hertzian vector corresponding to the reflected wave is expressed in terms of a single integral over a finite interval; this integral is written in such a form that its numerical evaluation can easily be performed.

§ 1. *Introduction.* In the problem of the electromagnetic radiation from a vertical electric dipole situated at a certain height h above a plane earth all field quantities are usually assumed to vary harmonically in time. One of the two well-known methods for solving this steady-state problem is due to Sommerfeld [1]), the other to Weyl [2]). In several recent publications, however, the case is considered where the time dependence of the current in the dipole is impulsive rather than harmonic. We mention the papers by Poritsky [3]), Van der Pol [4]), Pekeris and Alterman [5]), Bremmer [6]) and a report by Levelt [7]). The techniques employed by these authors differ in several respects. Poritsky uses a generalization of Weyl's method to the effect that the total field is written as the superposition of a continuous system of plane pulses. Van der Pol, Pekeris and Alterman, Bremmer and Levelt make use of integral transforms or operational calculus. In this way Van der Pol obtained an elementary result when both the transmitter and the point of observation are located at the ground. Pekeris and

Alterman obtain numerical results for the field above and inside the earth in the case $h = 0$. Their technique originates from Pekeris' studies on impulsive wave propagation in elastic media. It is closely related to a similar method developed by Cagniard [8]) who, too, was concerned with the generation of seismic waves by impulsive sources.

One of the present authors developed a simplified version [9]) of the methods employed by Cagniard and by Pekeris and subsequently applied the simplified procedure to the determination of the surface displacement generated by an interior source in an elastic half-space [10]). In the present paper the latter technique is used to determine the electromagnetic field radiated by a vertical electric dipole located above a plane, non-conducting, earth. The electric moment of the dipole varies in time as a given function $f(t)$, with $f(t) = 0$ when $t < 0$. The attention is confined to the field in the air; unless $h = 0$ (see [5])), the determination of the field inside the earth is much more difficult. Our result is given in the form of a definite integral over a finite interval; this integral can easily be computed numerically.

§ 2. *Statement of the problem and method of solution.* We consider the electromagnetic field in either of two homogeneous, isotropic, semi-infinite media with different electromagnetic properties. A Cartesian coordinate system is introduced such that the upper medium (the air) occupies the half-space $0 < z < \infty$, while the lower medium (the earth) occupies the half-space $-\infty < z < 0$. Their common boundary is the plane $z = 0$. A point in space will be located by either its Cartesian coordinates or its cylindrical coordinates r, φ, z defined through

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z, \tag{2.1}$$

with $0 \leqq r < \infty$, $0 \leqq \varphi < 2\pi$, $-\infty < z < \infty$. The electromagnetic properties of the media are characterized by their permittivity ε and their permeability μ; their conductivity is assumed to be zero. For the upper medium we have $\varepsilon = \varepsilon_1$ and $\mu = \mu_1$, for the lower medium we have $\varepsilon = \varepsilon_2$ and $\mu = \mu_2$.

At $x=0$, $y=0$, $z=h$ ($h>0$) a vertical electric dipole starts to radiate at the instant $t=0$; it is assumed that prior to this instant all field quantities vanish identically. It is well-known (see, e.g., Stratton [11])) that the electromagnetic field generated by this vertical dipole can be derived from a Hertzian vector $\boldsymbol{\Pi}$ of which only the z-component is different from zero. The electric field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$ are expressed in terms of $\boldsymbol{\Pi}$ through the relations

$$\boldsymbol{E} = \operatorname{grad} \operatorname{div} \boldsymbol{\Pi} - \varepsilon\mu \frac{\partial^2 \boldsymbol{\Pi}}{\partial t^2}, \tag{2.2}$$

$$\boldsymbol{H} = \varepsilon \operatorname{curl} \frac{\partial \boldsymbol{\Pi}}{\partial t}. \tag{2.3}$$

In the region $z > 0$ we write

$$\boldsymbol{\Pi} = (u_0 + u_1)\, \boldsymbol{i}_z \qquad (0 < z < \infty), \tag{2.4}$$

where u_0 yields the incident wave, i.e. the field that would exist if the upper medium were unbounded, while u_1 accounts for the reflection of the incident wave against the interface and is defined as the difference between the actual Hertzian vector and $u_0\boldsymbol{i}_z$. Similarly, in the region $z < 0$ we write

$$\boldsymbol{\Pi} = u_2 \boldsymbol{i}_z \qquad (-\infty < z < 0), \tag{2.5}$$

where u_2 yields the refracted field. At any interior point of the appropriate half-spaces $u_1 = u_1(x, y, z, t)$ and $u_2 = u_2(x, y, z, t)$ are assumed to be continuous together with their first and second order partial derivatives. In the region $z > 0$ the function u_1 satisfies the homogeneous wave equation

$$\Delta u_1 - \frac{1}{v_1^2} \frac{\partial^2 u_1}{\partial t^2} = 0; \tag{2.6}$$

in the region $z < 0$ the function u_2 satisfies the differential equation

$$\Delta u_2 - \frac{1}{v_2^2} \frac{\partial^2 u_2}{\partial t^2} = 0. \tag{2.7}$$

In these equations $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ denotes the three-dimensional Laplacian; further,

$$v_1 = (\varepsilon_1\mu_1)^{-\frac{1}{2}} \tag{2.8}$$

and

$$v_2 = (\varepsilon_2\mu_2)^{-\frac{1}{2}} \tag{2.9}$$

are the velocities of propagation in the upper and lower medium, respectively. The function u_0 is given by [12])

$$u_0 = \frac{1}{4\pi\varepsilon_1} \frac{f(t - R_1/v_1)}{R_1}, \tag{2.10}$$

where

$$R_1 = [x^2 + y^2 + (z - h)^2]^{\frac{1}{2}}. \tag{2.11}$$

The function $f(t)$ determines the electric moment of the dipole as a function of time as can be seen from the equation

$$\Delta u_0 - \frac{1}{v_1{}^2} \frac{\partial^2 u_0}{\partial t^2} = -\frac{1}{\varepsilon_1} \delta(x, y, z - h)\, f(t), \tag{2.12}$$

where $\delta(x, y, z - h)$ denotes, in a usual notation, the three-dimensional delta function. According to our assumptions, $f(t) = 0$ when $-\infty < t < 0$. The continuity of E_x, E_y, H_x and H_y at the interface is guaranteed if the following boundary conditions are satisfied:

$$\lim_{z\to+0} \left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} \right) = \lim_{z\to-0} \frac{\partial u_2}{\partial z}, \tag{2.13}$$

$$\varepsilon_1 \lim_{z\to+0} (u_0 + u_1) = \varepsilon_2 \lim_{z\to-0} u_2. \tag{2.14}$$

All field quantities occurring in the problem are now subjected to a one-sided Laplace transform with respect to time; e.g.,

$$F(s) = \int_0^\infty \exp(-st)\, f(t)\, \mathrm{d}t. \tag{2.15}$$

Similarly, U_0, U_1 and U_2 denote the Laplace transforms of u_0, u_1 and u_2, respectively. Following Cagniard [8]), s is restricted to real positive values large enough to ensure the convergence of the integrals of the type (2.15) (it is tacitly assumed that the behaviour of the relevant functions as $t \to \infty$ is such that such a number s can be found). Since, in particular, u_1, $\partial u_1/\partial t$, u_2 and $\partial u_2/\partial t$ are continuous, $U_1 = U_1(x, y, z; s)$ and $U_2 = U_2(x, y, z; s)$ satisfy the differential equations

$$\Delta U_1 - \frac{s^2}{v_1{}^2} U_1 = 0 \tag{2.16}$$

and

$$\Delta U_2 - \frac{s^2}{v_2{}^2} U_2 = 0, \tag{2.17}$$

respectively. The next step is to introduce the two-dimensional Fourier transforms of $U_1(x, y, z; s)$ and $U_2(x, y, z; s)$ with respect to x and y. Let

$$\mathscr{U}_{1,2}(\alpha, \beta; z; s) = \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \exp[is(\alpha x + \beta y)] U_{1,2}(x, y, z; s) \, \mathrm{d}x, \quad (2.18)$$

in which the (real) factor s in the argument of the exponential function has been included for convenience. If $\mathscr{U}_1$ and $\mathscr{U}_2$ were known, U_1 and U_2 could be determined from the inversion integral

$$U_{1,2}(x, y, z; s) = \frac{s^2}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\beta \int_{-\infty}^{\infty} \exp[-is(\alpha x + \beta y)] \mathscr{U}_{1,2}(\alpha, \beta; z; s) \mathrm{d}\alpha. \quad (2.19)$$

The corresponding representation of $U_0(x, y, z; s)$ is known to be [9])

$$U_0(x, y, z; s) = \\ = \frac{sF(s)}{4\pi^2 \varepsilon_1} \int_{-\infty}^{\infty} \mathrm{d}\beta \int_{-\infty}^{\infty} \exp[-s(i\alpha x + i\beta y + \gamma_1 |z - h|)] \frac{1}{2\gamma_1} \mathrm{d}\alpha, \quad (2.20)$$

in which

$$\gamma_1 = \gamma_1(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/v_1^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma_1 \geqq 0). \quad (2.21)$$

Since the boundary conditions are independent of time, they reduce to

$$\lim_{z \to +0} \left(\frac{\partial U_0}{\partial z} + \frac{\partial U_1}{\partial z} \right) = \lim_{z \to -0} \frac{\partial U_2}{\partial z}, \quad (2.22)$$

$$\varepsilon_1 \lim_{z \to +0} (U_0 + U_1) = \varepsilon_2 \lim_{z \to -0} U_2. \quad (2.23)$$

In order to determine $\mathscr{U}_1$ and $\mathscr{U}_2$ we substitute the corresponding representations of U_1 and U_2 (compare (2.19)) in the differential equations (2.16) and (2.17). This procedure leads to two ordinary differential equations for $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively, with z as independent variable. The solutions that remain bounded as $|z| \to \infty$ can be written as

$$\mathscr{U}_1 = \frac{F(s)}{s} \mathscr{A}_1 \exp[-s\gamma_1(z + h)], \quad (2.24)$$

$$\mathscr{U}_2 = \frac{F(s)}{s} \mathscr{A}_2 \exp[s(\gamma_2 z - \gamma_1 h)], \quad (2.25)$$

where γ_1 is given by (2.21) and γ_2 by

$$\gamma_2 = \gamma_2(\alpha, \beta) = (\alpha^2 + \beta^2 + 1/v_2^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma_2 \geqq 0). \quad (2.26)$$

The functions $\mathscr{A}_1 = \mathscr{A}_1(\alpha, \beta)$ and $\mathscr{A}_2 = \mathscr{A}_2(\alpha, \beta)$ follow from the boundary conditions at the interface. It is found that

$$\mathscr{A}_1 = \frac{\varepsilon_2\gamma_1 - \varepsilon_1\gamma_2}{\varepsilon_2\gamma_1 + \varepsilon_1\gamma_2} \frac{1}{2\varepsilon_1\gamma_1}, \tag{2.27}$$

$$\mathscr{A}_2 = \frac{1}{\varepsilon_2\gamma_1 + \varepsilon_1\gamma_2}. \tag{2.28}$$

From these results the Hertzian vector in the upper medium will be determined. Since the incident wave u_0 is already known, we are left with the problem of determining the reflected wave u_1. Equations (2.19) and (2.24) show that $U_1 = U_1(x, y, z; s)$ is of the form

$$U_1(x, y, z; s) = sF(s)\, G_1(x, y, z; s), \tag{2.29}$$

where

$$G_1(x, y, z; s) =$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\beta \int_{-\infty}^{\infty} \exp\{-s[i\alpha x + i\beta y + \gamma_1(z+h)]\} \mathscr{A}_1(\alpha, \beta)\, \mathrm{d}\alpha. \tag{2.30}$$

From now on we restrict the discussion to the case $v_2 < v_1$. In § 3 it will be shown that then the integral at the right-hand side of (2.30) can be transformed into

$$G_1(x, y, z; s) = \int_{R_2/v_1}^{\infty} \exp(-s\tau)\, g_1(x, y, z, \tau)\, \mathrm{d}\tau, \tag{2.31}$$

where only *real* values of τ occur in the integration and where R_2 is given by

$$R_2 = [x^2 + y^2 + (z+h)^2]^{\frac{1}{2}} \tag{2.32}$$

(R_2 = distance from the image of the source to the point of observation). Now we observe that $sF(s) \exp(-s\tau)$ is the Laplace transform of a function of time that vanishes when $t < \tau$ and equals $\mathrm{d}f(t-\tau)/\mathrm{d}t$ when $\tau < t$. Using the notation $\mathrm{d}f/\mathrm{d}t = f'$ we finally obtain for the z-component of the Hertzian vector corresponding to the reflected wave

$$u_1(x, y, z, t) = \begin{cases} 0 & (0 < t < R_2/v_1), \\ \int_{R_2/v_1}^{t} f'(t-\tau)\, g_1(x, y, z, \tau)\, \mathrm{d}\tau & (R_2/v_1 < t < \infty). \end{cases} \tag{2.33}$$

From the foregoing analysis it is clear that $u_1(x, y, z, t)$ reduces to $g_1(x, y, z, t)$ in case $f(t)$ is given by $f(t) = 1$ $(t > 0)$, i.e. $f(t)$ is the Heaviside unit step function.

The electromagnetic field vectors in the air are obtained by using (2.4), (2.10) and (2.33) in the right-hand sides of (2.2) and (2.3).

§ 3. *Determination of the function* $g_1(x, y, z, \tau)$. In the present section it will be shown that the transformations outlined in § 3 of reference [9]) lead to an expression for $g_1(x, y, z, \tau)$ in the form of a single integral over a finite interval. In the integral on the right-hand side of (2.30) we introduce new variables of integration ω and q through

$$\alpha = \omega \cos \varphi - q \sin \varphi, \tag{3.1}$$

$$\beta = \omega \sin \varphi + q \cos \varphi. \tag{3.2}$$

Since $\mathrm{d}\alpha \, \mathrm{d}\beta = \mathrm{d}\omega \, \mathrm{d}q$, we obtain

$$G_1(x, y, z; s) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}q \int_{-\infty}^{\infty} \exp \{- s[i\omega r + \gamma_1(z + h)]\} \mathscr{A}_1 \, \mathrm{d}\omega, \tag{3.3}$$

in which, as $\alpha^2 + \beta^2 = \omega^2 + q^2$,

$$\gamma_{1,2} = (\omega^2 + q^2 + 1/v_{1,2}{}^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma_{1,2} \geqq 0). \tag{3.4}$$

Next we introduce the variable $p = i\omega$ and regard p as a complex variable, while q is kept real. The result is

$$G_1(x, y, z; s) = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} \mathrm{d}q \int_{-i\infty}^{i\infty} \exp\{- s[pr + \gamma_1(z + h)]\} \mathscr{A}_1 \, \mathrm{d}p, \tag{3.5}$$

in which (compare (2.27))

$$\mathscr{A}_1 = \frac{\varepsilon_2\gamma_1 - \varepsilon_1\gamma_2}{\varepsilon_2\gamma_1 + \varepsilon_1\gamma_2} \frac{1}{2\varepsilon_1\gamma_1}, \tag{3.6}$$

with

$$\gamma_{1,2} = (q^2 + 1/v_{1,2}{}^2 - p^2)^{\frac{1}{2}} \quad (\mathrm{Re}\, \gamma_{1,2} \geqq 0). \tag{3.7}$$

In the complex p-plane the integrand in (3.5) has branch points at $p = \pm \Omega_1(q)$ and at $p = \pm \Omega_2(q)$, where

$$\Omega_{1,2}(q) = (q^2 + 1/v_{1,2}{}^2)^{\frac{1}{2}} \quad (\Omega_{1,2} > 0). \tag{3.8}$$

In view of subsequent deformations of the path of integration we take Re $\gamma_1 \geqq 0$ and Re $\gamma_2 \geqq 0$ not only on the imaginary p-axis but everywhere in the p-plane. This implies that branch cuts are introduced along $\mathrm{Im}\, p = 0$, $\Omega_1(q) < |\mathrm{Re}\, p| < \infty$ and along $\mathrm{Im}\, p = 0$, $\Omega_2(q) < |\mathrm{Re}\, p| < \infty$. It can easily be verified that, by virtue of Cauchy's theorem and Jordan's lemma [13]), the integral along the imaginary p-axis in (3.5) can be replaced by an integral along the branch Γ_1 of a hyperbola, where Γ_1 is given through

$$p = (r/R_2^2)\tau \pm i(h/R_2^2)[\tau^2 - R_2^2\Omega_1^2(q)]^{\frac{1}{2}} \quad (R_2\Omega_1(q) < \tau < \infty), \tag{3.9}$$

in which the square root is taken positive or zero. The upper and lower sign in (3.9) refer to the part of Γ_1 located in the upper and lower half of the p-plane, respectively. Along Γ_1 we have

$$\gamma_1 = (h/R_2^2)\tau \mp i(r/R_2^2)[\tau^2 - R_2^2\Omega_1^2(q)]^{\frac{1}{2}} \tag{3.10}$$

and

$$\frac{\partial p}{\partial \tau} = \pm \frac{i\gamma_1}{[\tau^2 - R_2^2\Omega_1^2(q)]^{\frac{1}{2}}}. \tag{3.11}$$

In (3.9), (3.10) and (3.11) the upper and lower signs belong together. Taking into account the symmetry of the path of integration with respect to the real axis and introducing τ as variable of integration we obtain, since q, s and τ are real,

$$G_1(x, y, z; s) =$$
$$= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}q \int_{R_2\Omega_1(q)}^\infty \exp(-s\tau)\, \mathrm{Re}\,\{\mathscr{A}_1\gamma_1\} \frac{1}{[\tau^2 - R_2^2\Omega_1^2(q)]^{\frac{1}{2}}}\, \mathrm{d}\tau. \tag{3.12}$$

Interchanging the order of integration we have

$$G_1(x, y, z; s) =$$
$$= \int_{R_2/v_1}^\infty \exp(-s\tau)\, \mathrm{d}\tau \int_0^{(\tau^2/R_2^2 - 1/v_1^2)^{\frac{1}{2}}} \frac{1}{\pi^2}\, \mathrm{Re}\,\{\mathscr{A}_1\gamma_1\} \frac{1}{[\tau^2 - R_2^2\Omega_1^2(q)]^{\frac{1}{2}}}\, \mathrm{d}q. \tag{3.13}$$

The integral on the right-hand side of (3.13) has the form announced in § 2, eq. (2.31). Consequently, $g_1(x, y, z, \tau)$ is given by

$$g_1(x, y, z, \tau) = \frac{1}{\pi^2 R_2}\int_0^{\frac{1}{2}\pi} \mathrm{Re}\,\{\mathscr{A}_1\gamma_1\}\, \mathrm{d}\psi, \tag{3.14}$$

where a new variable of integration ψ has been introduced through

$$q = (\tau^2/R_2^2 - 1/v_1^2)^{\frac{1}{2}} \sin \psi \quad (0 \leqq \psi \leqq \tfrac{1}{2}\pi). \tag{3.15}$$

In the right-hand side of (3.14) we have to substitute for p and γ_1 the values (compare (3.9) and (3.10))

$$p = (r/R_2^2)\,\tau + i(h/R_2^2)(\tau^2 - R_2^2/v_1^2)^{\frac{1}{2}} \cos \psi, \tag{3.16}$$

$$\gamma_1 = (h/R_2^2)\,\tau - i(r/R_2^2)(\tau^2 - R_2^2/v_1^2)^{\frac{1}{2}} \cos \psi, \tag{3.17}$$

while q is given by (3.15). In all these expressions $R_2/v_1 < \tau < \infty$ and $R_2 = [r^2 + (z+h)^2]^{\frac{1}{2}}$.

§ 4. *Concluding remarks.* The problem of determining the Hertzian vector corresponding to the reflected wave generated by a vertical electric dipole located at a height h above a non-conducting earth with plane boundary has been reduced to the evaluation of the integrals in (2.33) and (3.14). In (2.33) the function $f'(t-\tau)$ takes into account how the electric moment of the dipole varies in time, while $g_1(x, y, z, \tau)$, given by (3.14), depends on the geometry of the boundary value problem and the physical properties of the air and the ground. As was to be expected the function g_1 is independent of φ, i.e. the Hertzian vector is rotationally symmetrical about the z-axis.

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