A NOTE ON THE PROPAGATION OF WAVES IN A CONTINUOUSLY LAYERED MEDIUM

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Summary

The propagation of time harmonic waves in a certain continuously layered medium is considered. The wave number $k = k(\zeta)$ is assumed to vary with the Cartesian coordinate ζ; the law of variation is taken to be the one studied by Epstein. An integral representation for the wave function in this medium is derived. The method by which this is done is considerably simpler than the usual treatment of the problem with the aid of hypergeometric functions.

§ 1. *Introduction.* We consider the propagation of one-dimensional time harmonic waves in a continuously layered medium. In such a medium the wave number $k = k(\zeta)$ is a continuous function of the Cartesian spatial coordinate ζ. The complex wave function $u = u(\zeta)$ describing the one-dimensional wave motion in the medium is assumed to satisfy the differential equation

$$\mathrm{d}^2u/\mathrm{d}\zeta^2 + [k(\zeta)]^2\, u = 0. \tag{1.1}$$

The complex time factor is chosen as $\exp(-i\omega t)$; the dependence of u on ω (ω = radial frequency) is not indicated explicitly.

One method of deriving properties of $u = u(\zeta)$ is to find approximate solutions (usually of an asymptotic character) of (1.1) for a rather general class of profiles $k = k(\zeta)$ (see, e.g., Erdélyi[1]), Heading[2]), Broer[3])).

A different method of gaining insight in the properties of $u = u(\zeta)$ is to choose the function $k = k(\zeta)$ in such a way that exact analytical representations for the solutions of (1.1) can be obtained. In this case the form in which the solutions of (1.1) appear should be not too complicated, whereas the function $k = k(\zeta)$ should contain a number of parameters in order to cover several cases of practical interest (see, e.g., Budden[4]), Wait[5])).

The present paper belongs to the latter class of researches and reconsiders the case that has been studied by Epstein[6]) (see also Brekhovskikh[7]), Rawer[8]) and Budden[9])), where the function $k = k(\zeta)$ is given by (2.4). The usual treatment of the problem makes extensive use of the theory of the hypergeometric function. In the present paper it is shown that a single integral representation for the wave function can be constructed directly from a differential equation closely related to (1.1), by making use of well-known properties of the gamma function.

§ 2. *Formulation of the problem.* Consider the linear differential equation of the second order

$$[a_2 + b_2 \exp(\zeta)]\, \mathrm{d}^2w/\mathrm{d}\zeta^2 + [a_1 + b_1 \exp(\zeta)]\, \mathrm{d}w/\mathrm{d}\zeta + [a_0 + b_0 \exp(\zeta)]\, w = 0, \qquad (2.1)$$

in which a_2, b_2, a_1, b_1, a_0 and b_0 are arbitrary constants and $w = w(\zeta)$. Through the introduction of a new dependent variable, (2.1) can be reduced to a differential equation of the type (1.1). Let

$$w(\zeta) = [1 + (b_2/a_2)\exp(\zeta)]^{\frac{1}{2}(a_1/a_2 - b_1/b_2)}\, [(b_2/a_2)\exp(\zeta)]^{-\frac{1}{2}(a_1/a_2)}\, u(\zeta), \qquad (2.2)$$

then $u = u(\zeta)$ satisfies the differential equation

$$\mathrm{d}^2u/\mathrm{d}\zeta^2 + [k(\zeta)]^2\, u = 0, \qquad (2.3)$$

in which

$$[k(\zeta)]^2 = \frac{a_2^2N_1^2 + a_2b_2(N_1^2 + N_2^2 + M)\exp(\zeta) + b_2^2N_2^2\exp(2\zeta)}{[a_2 + b_2\exp(\zeta)]^2}, \qquad (2.4)$$

with

$$N_1^2 \stackrel{\text{def}}{=} (4a_0a_2 - a_1^2)/4a_2^2, \qquad (2.5)$$

$$N_2^2 \stackrel{\text{def}}{=} (4b_0b_2 - b_1^2)/4b_2^2, \qquad (2.6)$$

$$M \stackrel{\text{def}}{=} \tfrac{1}{2}(a_1/a_2 - b_1/b_2)[1 + \tfrac{1}{2}(a_1/a_2 - b_1/b_2)]. \qquad (2.7)$$

Equation (2.4) shows that the function $k = k(\zeta)$ is essentially the same as the one studied by Epstein[6]). From (2.4) it follows that

$$\lim_{\zeta \to -\infty} [k(\zeta)]^2 = N_1^2 \tag{2.8}$$

and

$$\lim_{\zeta \to \infty} [k(\zeta)]^2 = N_2^2. \tag{2.9}$$

For the sake of convenience we confine ourselves to real values of a_2, b_2, a_1, b_1, a_0 and b_0, and define $k(\zeta)$ to be real and positive when $[k(\zeta)]^2 > 0$ and positive imaginary when $[k(\zeta)]^2 < 0$ (in particular, this convention holds for N_1 and N_2). In order that $k(\zeta)$ be finite everywhere we further suppose $a_2/b_2 > 0$.

As the function $k = k(\zeta)$ approaches a finite limit both as $\zeta \to -\infty$ and as $\zeta \to \infty$, it is expected that (2.3) admits solutions the asymptotic behaviour of which is given by

$$u(\zeta) = [A_1 \exp(iN_1\zeta) + B_1 \exp(-iN_1\zeta)] \times \\ \times [1 + o(1)] \quad \text{as} \quad \zeta \to -\infty \tag{2.10}$$

and

$$u(\zeta) = [A_2 \exp(-iN_2\zeta) + B_2 \exp(iN_2\zeta)] \times \\ \times [1 + o(1)] \quad \text{as} \quad \zeta \to \infty. \tag{2.11}$$

Now, the differential equation (2.3) is of the second order and, consequently, has only two linearly independent solutions. This implies that, between the coefficients in (2.10) and (2.11), there exists a relation of the type

$$B_1 = S_{11}A_1 + S_{12}A_2, \tag{2.12}$$

$$B_2 = S_{21}A_1 + S_{22}A_2, \tag{2.13}$$

where S_{11}, S_{12}, S_{21} and S_{22} constitute the "scattering matrix".

From (2.2), (2.10) and (2.11) we obtain the expected asymptotic behaviour of $w = w(\zeta)$

$$w(\zeta) = (b_2/a_2)^{-\frac{1}{2}(a_1/a_2)} [A_1 \exp(\alpha_1\zeta) + B_1 \exp(\alpha_2\zeta)] \times \\ \times [1 + o(1)] \quad \text{as} \quad \zeta \to -\infty \tag{2.14}$$

and

$$w(\zeta) = (b_2/a_2)^{-\frac{1}{2}(b_1/b_2)} [A_2 \exp(\beta_2\zeta) + B_2 \exp(\beta_1\zeta)] \times \\ \times [1 + o(1)] \quad \text{as} \quad \zeta \to \infty, \tag{2.15}$$

where

$$\alpha_1 \overset{\text{def}}{=\!=} -\tfrac{1}{2}(a_1/a_2) + iN_1, \tag{2.16}$$

$$\alpha_2 \overset{\text{def}}{=\!=} -\tfrac{1}{2}(a_1/a_2) - iN_1, \tag{2.17}$$

$$\beta_1 \overset{\text{def}}{=\!=} -\tfrac{1}{2}(b_1/b_2) + iN_2, \tag{2.18}$$

$$\beta_2 \overset{\text{def}}{=\!=} -\tfrac{1}{2}(b_1/b_2) - iN_2. \tag{2.19}$$

We now consider A_1 and A_2 as given and proceed to construct an integral representation for $w = w(\zeta)$ which reproduces the term containing A_1 in (2.14) as $\zeta \to -\infty$ and the term containing A_2 in (2.15) as $\zeta \to \infty$.

§ 3. *Construction of the solution.* Let $w = w(\zeta)$ be represented by

$$w(\zeta) = (2\pi i)^{-1} \int_L \exp(p\zeta)\, W(p)\, \mathrm{d}p, \tag{3.1}$$

in which the function $W = W(p)$ and the path of integration L in the complex p-plane are to be determined. The right-hand side of (3.1) is now substituted in (2.1). On the assumption that the differentiations with respect to ζ can be carried out under the integral sign, we obtain the following difference equation for $W = W(p)$:

$$(a_2p^2 + a_1p + a_0)\, W(p) + \\ + [b_2(p-1)^2 + b_1(p-1) + b_0]\, W(p-1) = 0. \tag{3.2}$$

In obtaining the second term on the left-hand side of (3.2) it has been assumed that $W = W(p)$ is regular in the domain bounded by L and L', where L' originates from L by shifting it towards the right over a distance 1. On account of (2.16)–(2.19), (3.2) can be rewritten as

$$a_2(p-\alpha_1)(p-\alpha_2)\, W(p) + \\ + b_2(p-1-\beta_1)(p-1-\beta_2)\, W(p-1) = 0. \tag{3.3}$$

Using the difference equation of the gamma function $\Gamma(p+1) - p\Gamma(p) = 0$, solutions of (3.3) can be obtained by inspection. It is found that

$$W(p) = C_1(b_2/a_2)^p \frac{\Gamma(-p+\alpha_1)\,\Gamma(-p+\alpha_2)\,\Gamma(p-\beta_1)}{\Gamma(1-p+\beta_2)} + \\ + C_2(b_2/a_2)^p \frac{\Gamma(p-\beta_2)\,\Gamma(p-\beta_1)\,\Gamma(-p+\alpha_2)}{\Gamma(1+p-\alpha_1)}, \tag{3.4}$$

where C_1 and C_2 are arbitrary constants. Further, L extends to infinity parallel to the imaginary p-axis; the sequences of simple poles $p = \alpha_1 + m$ and $p = \alpha_2 + m$ lie to the right of L, the sequences of simple poles $p = \beta_1 - m$ and $p = \beta_2 - m$ lie to the left of L ($m = 0, 1, 2, \ldots$) (fig. 1). On account of the behaviour of the gamma functions as $|\mathrm{Im}\, p| \to \infty$ substitution of (3.4) in (3.1) leads to an integral along L which is absolutely convergent for all finite real values of ζ.

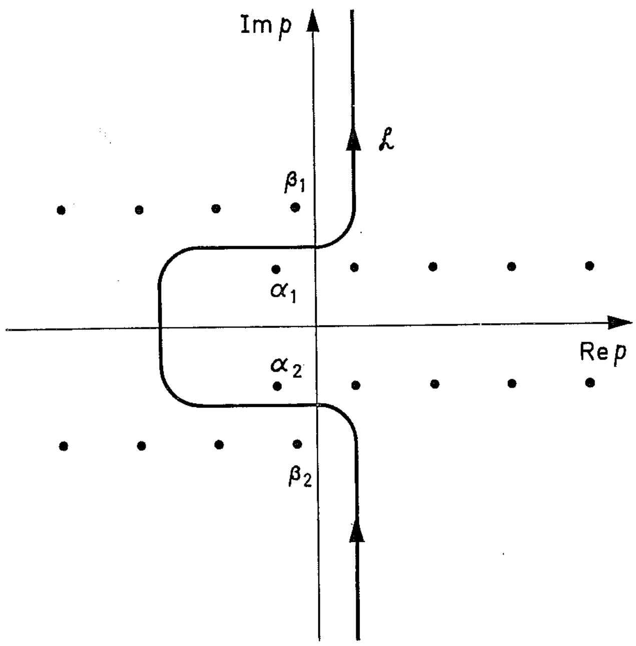

Fig. 1. Path of integration L in the complex p-plane.

The value of the constants C_1 and C_2 will be determined by comparing (2.14) and (2.15) with a series expansion of $w(\zeta)$. This series expansion is derived from (3.1) and (3.4) by using an argument similar to the one employed in Whittaker and Watson[10]) in connection with the hypergeometric function. Moving L away to infinity to the right and applying the theorem of residues we obtain

$$w(\zeta) = -\sum_{m=0}^{\infty} \{\text{Residue of } \exp(p\zeta)\, W(p) \text{ at } p = \alpha_1 + m\} - \\ - \sum_{m=0}^{\infty} \{\text{Residue of } \exp(p\zeta)\, W(p) \text{ at } p = \alpha_2 + m\} \\ \text{when } \zeta + \ln(b_2/a_2) < 0. \tag{3.5}$$

Moving L away to infinity to the left and applying the theorem of residues we obtain

$$w(\zeta) = \sum_{m=0}^{\infty} \{\text{Residue of } \exp(p\zeta)\, W(p) \text{ at } p = \beta_2 - m\} + \\ + \sum_{m=0}^{\infty} \{\text{Residue of } \exp(p\zeta)\, W(p) \text{ at } p = \beta_1 - m\} \\ \text{when } \zeta + \ln(b_2/a_2) > 0. \tag{3.6}$$

The expressions for the residues are not difficult to obtain and will not be reproduced here. Comparison of the first term on the right-hand side of (2.14) with the term in (3.5) resulting from the pole $p = \alpha_1$ gives

$$C_1 = A_1(b_2/a_2)^{-iN_1} \frac{\Gamma(1-\alpha_1+\beta_2)}{\Gamma(-\alpha_1+\alpha_2)\,\Gamma(\alpha_1-\beta_1)}\,. \tag{3.7}$$

Comparison of the first term on the right-hand side of (2.15) with the term in (3.6) resulting from the pole $p = \beta_2$ gives

$$C_2 = A_2(b_2/a_2)^{iN_2} \frac{\Gamma(1+\beta_2-\alpha_1)}{\Gamma(\beta_2-\beta_1)\,\Gamma(-\beta_2+\alpha_2)}\,. \tag{3.8}$$

The other terms corresponding to $m = 0$ in (3.5) and (3.6) give, in connection with (2.12), (2.13), (2.14) and (2.15), the elements of the scattering matrix

$$S_{11} = \left(\frac{b_2}{a_2}\right)^{-2iN_1} \frac{\Gamma(1-\alpha_1+\beta_2)\,\Gamma(-\alpha_2+\alpha_1)\,\Gamma(\alpha_2-\beta_1)}{\Gamma(-\alpha_1+\alpha_2)\,\Gamma(\alpha_1-\beta_1)\,\Gamma(1-\alpha_2+\beta_2)}\,, \tag{3.9}$$

$$S_{12} = \left(\frac{b_2}{a_2}\right)^{-iN_1+iN_2} \frac{\Gamma(1+\beta_2-\alpha_1)\,\Gamma(\alpha_2-\beta_1)}{\Gamma(\beta_2-\beta_1)\,\Gamma(1+\alpha_2-\alpha_1)}\,, \tag{3.10}$$

$$S_{21} = \left(\frac{b_2}{a_2}\right)^{-iN_1+iN_2} \frac{\Gamma(1-\alpha_1+\beta_2)\,\Gamma(-\beta_1+\alpha_2)}{\Gamma(-\alpha_1+\alpha_2)\,\Gamma(1-\beta_1+\beta_2)}\,, \tag{3.11}$$

$$S_{22} = \left(\frac{b_2}{a_2}\right)^{2iN_2} \frac{\Gamma(1+\beta_2-\alpha_1)\,\Gamma(\beta_1-\beta_2)\,\Gamma(-\beta_1+\alpha_2)}{\Gamma(\beta_2-\beta_1)\,\Gamma(-\beta_2+\alpha_2)\,\Gamma(1+\beta_1-\alpha_1)}\,. \tag{3.12}$$

These results are in accordance with those of Epstein[6]). For a detailed discussion of (3.9)–(3.12) we refer to Rawer[8]).

§ 4. *Concluding remarks.* The integral representation (3.1) of $w(\zeta)$, where $W(p)$ is given by (3.4), is easily recognized as the integral representation of a certain hypergeometric function; likewise, (3.5) and (3.6) are hypergeometric series. This shows the connection between the theory developed in § 3 and the usual treatment of the problem in the literature[6–9]. However, the method given in § 3 has the advantage of reducing the necessary mathematical background for calculating the wave motion in an Epstein medium considerably: a knowledge of complex integration and of the basic properties of the gamma function are sufficient.

Finally, it is observed that by the substitution $\zeta = (z - z_0)/D$ an arbitrary reference level $z = z_0$ and a scale factor D (the "thickness" of the layer) can be introduced. From this, and the fact that only ratios of the constants in (2.1) occur in the expression for $w(\zeta)$, it follows that, without loss of generality, we can make $a_2 = 1$ and $b_2 = 1$. Once the latter values have been chosen the Epstein profile is completely determined by the three constants N_1, N_2 and M. However, through (2.5), (2.6) and (2.7), N_1, N_2 and M are expressed in terms of a_1, b_1, a_0 and b_0. Hence, with the restriction that the arbitrariness of N_1, N_2 and M must not be destroyed, one relation between a_1, b_1, a_0 and b_0 can still be chosen. E.g., we can take $a_1 = 0$, which simplifies (2.2) (the choice $a_1 = b_1$ is not permitted, since then $a_1/a_2 = b_1/b_2$ and, hence, $M = 0$).

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