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AN ELASTODYNAMIC RECIPROCITY THEOREM FOR LINEAR, VISCOELASTIC MEDIA

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Summary

A reciprocity theorem for impulsive disturbances in a linear, viscoelastic medium is derived. Apart from the condition that the medium be linear and viscoelastic of the Boltzmann type, no restrictions as to the properties of the medium are imposed. Hence, the reciprocity theorem is applicable to disturbances in inhomogeneous and anisotropic media.

As an illustration of its usefulness the reciprocity theorem is used to derive an integral representation of the Kirchhoff type for the displacement vector in a linear, viscoelastic medium.

For reference, also the Laplace transform version of the obtained results is given.

§ 1. *Introduction.* In several branches of mathematical physics reciprocity theorems are used to derive important relations between the physical quantities involved. Restricting ourselves to the field of elastodynamics we mention in particular a recent paper by Payton[1]). In this paper a dynamic analogue of the Betti-Rayleigh reciprocity theorem for (static) elasticity is derived and subsequently applied to some moving-point load problems. The media considered in that paper are linear, homogeneous, isotropic and perfectly elastic. In § 2 of the present paper we derive a similar elastodynamic reciprocity theorem for a more general class of media: the media are only supposed to be linear and viscoelastic of the Boltzmann[2]) type. Hence, the case of inhomogeneous, anisotropic and imperfectly elastic media is included.

As an illustration of its usefulness the reciprocity theorem is employed to obtain an integral representation of the Kirchhoff[3]) type (see also Baker and Copson[4])) for the displacement vector in a linear, viscoelastic medium (§ 3).

For reciprocity relations in seismology we refer to a paper by Knopoff and Gangi[5]), in which the case of inhomogeneous and anisotropic media is considered too, the time dependence of the different quantities being restricted however.

§ 2. *Basic equations governing the motion in a linear, viscoelastic medium.* We consider small-amplitude disturbances in a linear, viscoelastic medium. The Cartesian coordinates of a point in space are denoted by x_1, x_2 and x_3; the time variable is denoted by t. We follow the usual subscript notation for Cartesian vectors and tensors and employ the summation convention; all subscripts are successively assigned the values 1, 2 and 3. Let $u_i = u_i(x_1, x_2, x_3, t) = u_i(\boldsymbol{x}, t)$ be the displacement vector and let $\tau_{ij} = \tau_{ij}(x_1, x_2, x_3, t) = \tau_{ij}(\boldsymbol{x}, t)$ be the stress tensor, then the (linearized) equation of motion is

$$\partial\tau_{ij}/\partial x_j - \rho(\boldsymbol{x})(\partial^2 u_i/\partial t^2) = -f_i, \tag{2.1}$$

where $\rho = \rho(x_1, x_2, x_3) = \rho(\boldsymbol{x})$ is the mass density of the medium and $f_i = f_i(x_1, x_2, x_3, t) = f_i(\boldsymbol{x}, t)$ is the body force density. The viscoelastic properties of the medium are characterized by the (linearized) stress-strain relation

$$\tau_{ij}(\boldsymbol{x}, t) = \int_0^\infty c_{ij;\,pq}(\boldsymbol{x}, \tau)[\partial u_p(\boldsymbol{x}, t - \tau)/\partial x_q]\,\mathrm{d}\tau, \tag{2.2}$$

which is of the type considered by Boltzmann[2]) (see also Chao and Achenbach[6])). Since both the stress and the strain tensor are symmetric tensors the functions $c_{ij;\,pq} = c_{ij;\,pq}(x_1, x_2, x_3, \tau) = c_{ij;\,pq}(\boldsymbol{x}, \tau)$ satisfy the symmetry relations $c_{ij;\,pq} = c_{ji;\,pq} = c_{ji;\,qp} = c_{ij;\,qp}$ for all values of x_1, x_2, x_3 and τ. Further, $c_{ij;\,pq}(\boldsymbol{x}, \tau) = 0$ when $-\infty < \tau < 0$. The medium is called "reciprocal" if, in addition, the symmetry relation

$$c_{ij;\,pq}(\boldsymbol{x}, \tau) = c_{pq;\,ij}(\boldsymbol{x}, \tau) \tag{2.3}$$

holds for all values of x_1, x_2, x_3 and τ.

§ 3. *The reciprocity theorem.* A reciprocity theorem constitutes a relation between the quantities occurring in two physically possible, different situations; the two situations will be called situation A and situation B. The quantities (displacement vector, stress tensor, body force density) characterizing situation A (B) will be denoted by their corresponding symbol to which a superscript "A" ("B") is attached; a similar notation is used to distinguish between the media which are present in the two situations. On the assumption that the properties of the medium present in situation A are related to the properties of the medium present in situation B through the conditions $\rho^{(A)}(\boldsymbol{x}) = \rho^{(B)}(\boldsymbol{x})$ and $c_{ij;\,pq}{}^{(A)}(\boldsymbol{x}, \tau) = c_{pq;\,ij}{}^{(B)}(\boldsymbol{x}, \tau)$ for all values of x_1, x_2, x_3 and τ, the following *reciprocity theorem**) can be formulated. (In view of (2.3) the medium in situation A is identical with the medium in situation B if the media are reciprocal.)

Let $\mathscr{S}$ be a bounded, closed surface with piecewise continuous unit vector n_i in the direction of its outward normal and let $\mathscr{V}$ be the domain inside $\mathscr{S}$. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}\tau \iint_{\mathscr{S}} \{\tau_{ij}^{(A)}(\boldsymbol{x}, \tau)[\partial u_i^{(B)}(\boldsymbol{x}, t-\tau)/\partial(t-\tau)] - \\
&\quad - \tau_{ij}^{(B)}(\boldsymbol{x}, \tau)[\partial u_i^{(A)}(\boldsymbol{x}, t-\tau)/\partial(t-\tau)]\}\, n_j \,\mathrm{d}S = \\
&\quad = -\int_{-\infty}^{\infty} \mathrm{d}\tau \iiint_{\mathscr{V}} \{f_i^{(A)}(\boldsymbol{x}, \tau)[\partial u_i^{(B)}(\boldsymbol{x}, t-\tau)/\partial(t-\tau)] - \\
&\quad - f_i^{(B)}(\boldsymbol{x}, \tau)[\partial u_i^{(A)}(\boldsymbol{x}, t-\tau)/\partial(t-\tau)]\}\,\mathrm{d}V. \qquad (3.1)
\end{aligned}$$

Equation (3.1) is proved by applying the divergence theorem to the left-hand side and employing the equation of motion (2.1) and the stress-strain relation (2.2). Sufficient conditions (to be imposed in situation A as well as in situation B) under which the proof holds are: (i) $f_i(\boldsymbol{x}, t)$ is continuous, $u_i(\boldsymbol{x}, t)$ and $\tau_{ij}(\boldsymbol{x}, t)$ are continuously differentiable with respect to x_1, x_2 and x_3 and $u_i(\boldsymbol{x}, t)$ is twice continuously differentiable with respect to t for all $\boldsymbol{x}$ in $\mathscr{V}$ or on $\mathscr{S}$ and all t in $-\infty < t < \infty$; (ii) $\rho(\boldsymbol{x})$ and $c_{ij;\,pq}(\boldsymbol{x}, \tau)$ are continuous for all $\boldsymbol{x}$ in $\mathscr{V}$ or on $\mathscr{S}$ and all τ in $0 \leqq \tau < \infty$; (iii) as $t \to -\infty$ we have

$$\lim_{t\to-\infty} [\partial u_i(\boldsymbol{x}, t)/\partial t] = 0 \qquad (3.2)$$

and

$$\lim_{t\to-\infty} [\partial u_p(\boldsymbol{x}, t)/\partial x_q] = 0 \qquad (3.3)$$

*) The corresponding reciprocity theorem for electromagnetic fields has recently been given by Ru-Shao Cheo[7]).

for all $\boldsymbol{x}$ in $\mathscr{V}$ or on $\mathscr{S}$. The continuity conditions with respect to the spatial coordinates may be weakened to piecewise continuity and allowance for finite jumps across a sufficiently regular surface, provided that across such a surface the integrand in the left-hand side of (3.1) is continuous.

It is observed that both sides of (3.1) are functions of the time t, while (3.1) is valid for all values of t. Hence, other reciprocity theorems can be derived from (3.1) by applying an arbitrary time operator to both sides of the equation. If, for instance, (3.1) is integrated with respect to t from $-\infty$ to a certain instant which we again call t, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}\tau \iint_{\mathscr{S}} \{\tau_{ij}^{(A)}(\boldsymbol{x}, \tau)\, u_i^{(B)}(\boldsymbol{x}, t-\tau) - \\ - \tau_{ij}^{(B)}(\boldsymbol{x}, \tau)\, u_i^{(A)}(\boldsymbol{x}, t-\tau)\}\, n_j\, \mathrm{d}S = \\ = - \int_{-\infty}^{\infty} \mathrm{d}\tau \iiint_{\mathscr{V}} \{f_i^{(A)}(\boldsymbol{x}, \tau)\, u_i^{(B)}(\boldsymbol{x}, t-\tau) - \\ - f_i^{(B)}(\boldsymbol{x}, \tau)\, u_i^{(A)}(\boldsymbol{x}, t-\tau)\}\, \mathrm{d}V\end{aligned} \tag{3.4}$$

on the assumption that (in situation A as well as in situation B)

$$\lim_{t\to-\infty} u_i(\boldsymbol{x}, t) = 0. \tag{3.5}$$

Equation (3.4) is a reciprocity theorem in which the velocities occurring in (3.1) are replaced by their corresponding displacements. Of course, (3.4) can also be obtained directly by applying the divergence theorem to its left-hand side and employing the equation of motion (2.1) and the stress-strain relation (2.2).

In view of the numerous applications of Laplace transform techniques to elastodynamic problems we also give the reciprocity theorem which is obtained when a Laplace transform with respect to t is applied to either (3.1) or (3.4). We introduce the two-sided Laplace transforms $U_i(\boldsymbol{x}; s)$, $T_{ij}(\boldsymbol{x}; s)$ and $F_i(\boldsymbol{x}; s)$ of $u_i(\boldsymbol{x}, t)$, $\tau_{ij}(\boldsymbol{x}, t)$ and $f_i(\boldsymbol{x}, t)$ with respect to t; for instance,

$$U_i(\boldsymbol{x}; s) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \exp(-st)\, u_i(\boldsymbol{x}, t)\, \mathrm{d}t. \tag{3.6}$$

Similarly, we have, since $c_{ij;pq}(\boldsymbol{x}, \tau) = 0$ when $-\infty < \tau < 0$,

$$C_{ij;pq}(\boldsymbol{x}; s) \stackrel{\text{def}}{=} \int_0^{\infty} \exp(-s\tau)\, c_{ij;pq}(\boldsymbol{x}, \tau)\, \mathrm{d}\tau. \tag{3.7}$$

The equation of motion (2.1) is then transformed into

$$\partial T_{ij}/\partial x_j - \rho(\boldsymbol{x})\, s^2 U_i = -F_i \tag{3.8}$$

and the stress-strain relation (2.2) into

$$T_{ij}(\boldsymbol{x};s) = C_{ij;pq}(\boldsymbol{x};s)[\partial U_p/\partial x_q], \tag{3.9}$$

where the conditions (3.5) and (3.2) have been used.

Both (3.1) and (3.4) are transformed into the following reciprocity theorem for the Laplace transformed quantities:

$$\iint_{\mathscr{S}} \{T_{ij}^{(A)}U_i^{(B)} - T_{ij}^{(B)}U_i^{(A)}\}\, n_j \,\mathrm{d}S = \\ = -\iiint_{\mathscr{V}} \{F_i^{(A)}U_i^{(B)} - F_i^{(B)}U_i^{(A)}\}\,\mathrm{d}V. \tag{3.10}$$

Equations (3.8), (3.9) and (3.10) are valid if all Laplace transforms involved exist and if the necessary continuity conditions are satisfied.

§ 4. *An integral representation for the displacement vector.* As an application of the reciprocity theorem we now proceed to derive from (3.4) a Kirchhoff type[3,4] integral representation for the displacement vector. In the case when the medium is linear, homogeneous, isotropic and perfectly elastic such an integral representation is known (see De Hoop[8]), where also applications to elastodynamic diffraction problems are given).

In (3.4) the situation A is taken to be the actual state of disturbance in the viscoelastic medium; henceforth, the superscript "A" will be dropped. Further, the situation B is taken to be a state of disturbance due to an impulsive body force operative at the point $\mathscr{P}$ (with position vector $\boldsymbol{x}_{\mathscr{P}}$) and at the instant $t = 0$. Using a superscript "G" for this state of disturbance we have

$$f_i^{(G)}(\boldsymbol{x}, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_{\mathscr{P}}, t)\, a_i, \tag{4.1}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_{\mathscr{P}}, t)$ is the four-dimensional delta (generalized) function and a_i is a constant vector. On account of the properties of the delta (generalized) function, (4.1) is equivalent to

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \iiint_{\mathscr{V}} f_i^{(G)}(\boldsymbol{x}, \tau)\, u_i(\boldsymbol{x}, t-\tau)\,\mathrm{d}V = a_i u_i(\boldsymbol{x}_{\mathscr{P}}, t), \tag{4.2}$$

when $\mathscr{P}$ is lying inside $\mathscr{S}$, whereas the left-hand side of (4.2) vanishes when $\mathscr{P}$ is lying outside $\mathscr{S}$. Taking into account (4.2), the reciprocity theorem (3.4) leads to

$$a_i u_i(\boldsymbol{x}_{\mathscr{P}}, t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \{\iiint_{\mathscr{V}} f_i(\boldsymbol{x}, t-\tau)\, u_i^{(G)}(\boldsymbol{x}, \tau)\, \mathrm{d}V + $$
$$+ \iint_{\mathscr{S}} [\tau_{ij}(\boldsymbol{x}, t-\tau)\, u_i^{(G)}(\boldsymbol{x}, \tau) - \tau_{ij}^{(G)}(\boldsymbol{x}, \tau)\, u_i(\boldsymbol{x}, t-\tau)]\, n_j\, \mathrm{d}S\}. \tag{4.3}$$

In this expression $u_i^{(G)}(\boldsymbol{x}, t)$ and $\tau_{ij}^{(G)}(\boldsymbol{x}, t)$ are to be considered as known functions, viz. the displacement vector and the stress tensor of a disturbance due to the body force density (4.1). (It is noted that in the case of a homogeneous medium with special viscoelastic properties, expressions for $u_i^{(G)}(\boldsymbol{x}, t)$ and $\tau_{ij}^{(G)}(\boldsymbol{x}, t)$ have been obtained by Chao and Achenbach[6].) From (4.3) an integral representation for $u_i(\boldsymbol{x}_{\mathscr{P}}, t)$ itself is obtained by expressing the dependence of $u_i^{(G)}(\boldsymbol{x}, t)$ and $\tau_{ij}^{(G)}(\boldsymbol{x}, t)$ on a_i. To this end we write

$$u_i^{(G)}(\boldsymbol{x}, t) = u_{i;k}^{(G)}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_{\mathscr{P}})\, a_k \tag{4.4}$$

and

$$\tau_{ij}^{(G)}(\boldsymbol{x}, t) = \tau_{ij;k}^{(G)}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_{\mathscr{P}})\, a_k. \tag{4.5}$$

Substitution of (4.4) and (4.5) in the right-hand side of (4.3) leads to an equation which is valid for any a_k. Consequently,

$$u_k(\boldsymbol{x}_{\mathscr{P}}, t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \{\iiint_{\mathscr{V}} f_i(\boldsymbol{x}, t-\tau)\, u_{i;k}^{(G)}(\boldsymbol{x}\, \tau \,|\, \boldsymbol{x}_{\mathscr{P}})\, \mathrm{d}V + $$
$$+ \iint_{\mathscr{S}} [\tau_{ij}(\boldsymbol{x}, t-\tau)\, u_{i;k}^{(G)}(\boldsymbol{x}, \tau \,|\, \boldsymbol{x}_{\mathscr{P}}) - $$
$$- \tau_{ij;k}^{(G)}(\boldsymbol{x}, \tau \,|\, \boldsymbol{x}_{\mathscr{P}})\, u_i(\boldsymbol{x}, t-\tau)]\, n_j\, \mathrm{d}S\}. \tag{4.6}$$

This integral representation holds when $\mathscr{P}$ is lying inside $\mathscr{S}$; when $\mathscr{P}$ is lying outside $\mathscr{S}$ the right-hand side of (4.6) vanishes.

It is to be noted that $u_i^{(G)}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_{\mathscr{P}})$ is not uniquely determined by (2.1), (2.2) and (4.1) as no boundary conditions and/or conditions at infinity are imposed. In fact, any solution of (2.1), (2.2) and (4.1) can be used in the integral representation (4.6). Further, it is observed that, with regard to the properties of the medium, the conditions $\rho(\boldsymbol{x}) = \rho^{(G)}(\boldsymbol{x})$ and $c_{ij;pq}(\boldsymbol{x}, \tau) = c_{pq;ij}^{(G)}(\boldsymbol{x}, \tau)$ have to be satisfied for all $\boldsymbol{x}$ in $\mathscr{V}$ or on $\mathscr{S}$ and all τ in $0 \leqq \tau < \infty$.

Finally, the Laplace transform version of (4.6) is found to be

$$U_k(\boldsymbol{x}_{\mathscr{P}}; s) = \iiint_{\mathscr{V}} F_i(\boldsymbol{x}; s)\, U_{i;k}^{(G)}(\boldsymbol{x}; s \,|\, \boldsymbol{x}_{\mathscr{P}})\, \mathrm{d}V + $$
$$+ \iint_{\mathscr{S}} [T_{ij}(\boldsymbol{x}; s)\, U_{i;k}^{(G)}(\boldsymbol{x}; s \,|\, \boldsymbol{x}_{\mathscr{P}}) - T_{ij;k}^{(G)}(\boldsymbol{x}; s \,|\, \boldsymbol{x}_{\mathscr{P}})\, U_i(\boldsymbol{x}; s)]\, n_j\, \mathrm{d}S. \tag{4.7}$$

This result can also be directly obtained from (3.10) by taking $U_{i;k}^{(G)}a_k$ and $T_{ij;k}^{(G)}a_k$ to be a solution of (3.8) and (3.9) with

$$F_i^{(G)}(\boldsymbol{x}; s) = \delta(\boldsymbol{x} - \boldsymbol{x}_{\mathscr{P}})\, a_i, \tag{4.8}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_{\mathscr{P}})$ is the three-dimensional delta (generalized) function.

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