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# A RECIPROCITY RELATION BETWEEN THE TRANSMITTING AND THE RECEIVING PROPERTIES OF AN ANTENNA

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**Summary**

A reciprocity relation between the transmitting and the receiving properties of an antenna is derived. In the transmitting situation a certain part of the antenna, called "source domain", is capable of carrying external currents, both of the electric and the magnetic type. In the receiving situation a plane electromagnetic wave is incident upon the antenna system.

Whereas the customary way of deriving reciprocity relations in antenna theory starts with considering two separate antennas, the present method assumes a single antenna only.

## § 1. Introduction

One of the basic theorems in the electromagnetic theory of antennas is a reciprocity relation between the transmitting and the receiving properties of an antenna. The customary form of this reciprocity relation (see, for example, [1]) applies to two different antennas a finite distance apart, each of them playing in turn the role of transmitting antenna or receiving antenna. In technical applications, however, one frequently employs the reciprocity theorem to relate the radiation pattern of a transmitting antenna to the pattern of the same antenna when it is receiving an incident plane wave. Clearly, for the latter problem a reciprocity theorem applicable to a single antenna is required. Now, such a reciprocity relation can be obtained from the aforementioned reciprocity theorem applicable to two separate antennas by letting the mutual distance between the antennas become very large and considering the limiting form of the relation between the electromagnetic field quantities involved. This technique has been employed by Stevenson [2] for the special case of a perfectly conducting antenna. In the transmitting situation the antenna is fed by a slice generator and in the receiving situation it is connected to a load (the properties of which can be expressed in terms of a low frequency impedance).

The purpose of the present paper is to derive a reciprocity relation between the transmitting and the receiving properties of a single antenna without having to resort to the corresponding problem for two separate antennas. The employed method has been used before to derive a reciprocity theorem for the electromagnetic scattering of a plane wave by a bounded obstacle [3].

### § 2. Description of the configuration

The antenna system under consideration occupies a bounded domain $\mathscr{V}$ in space; the boundary of $\mathscr{V}$ is a sufficiently regular closed surface $\mathscr{S}$. The Cartesian coordinates of a point in space are denoted by $x_1$, $x_2$, and $x_3$; the time variable is denoted by $t$. The usual subscript notation for Cartesian vectors and tensors is used; all subscripts are to be assigned the values 1, 2, 3. The position vector is denoted by $\boldsymbol{x}$. The electromagnetic fields occurring in the transmitting as well as in the receiving situation are assumed to be harmonic in time with the same angular frequency $\omega$. The complex representation of the field vectors is used; in the formulas the complex time factor $\exp(-\mathrm{i}\omega t)$, common to all field components, is omitted.

The antenna consists of a (lossy) medium whose electromagnetic behaviour is linear; its properties are characterized by the complex tensor permittivity $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{x}; \omega)$ and the complex tensor permeability $\mu_{ij} = \mu_{ij}(\boldsymbol{x}; \omega)$. The permittivity and the permeability are assumed to be continuous functions of position, with the possible exception of a finite number of sufficiently regular bounded surfaces across which they may jump by finite amounts. Across such a discontinuity surface the tangential parts of the electric and the magnetic field are continuous.

In the transmitting situation a subdomain $\mathscr{V}_{\text{source}}$ of $\mathscr{V}$ is capable of carrying "external" currents, both of electric and magnetic type. These external currents represent the sources through which power of nonelectromagnetic origin can be delivered to the system. The boundary of $\mathscr{V}_{\text{source}}$ is a sufficiently regular closed surface

$\mathscr{S}_{\text{source}}$. In the receiving situation no external currents are present.

The medium outside the antenna system is linear, homogeneous, isotropic, and lossless (which includes the case of free space) with real scalar permittivity $\varepsilon_0$ and real scalar permeability $\mu_0$.

In the following $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x}; \omega)$ denotes the electric field and $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{x}; \omega)$ the magnetic field. The superscript 'T' ('R') is used to denote the transmitting (receiving) situation.

## § 3. The antenna as a transmitting system

In the transmitting situation the antenna carries time harmonic external currents. Let $\boldsymbol{J}^{\mathrm{T}} = \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{x}; \omega)$ be the volume density of external electric currents and let $\boldsymbol{K}^{\mathrm{T}} = \boldsymbol{K}^{\mathrm{T}}(\boldsymbol{x}; \omega)$ be the volume density of external magnetic currents; $\boldsymbol{J}^{\mathrm{T}}$ and $\boldsymbol{K}^{\mathrm{T}}$ differ from zero in $\mathscr{V}_{\text{source}}$ only. In $\mathscr{V}$ the electric field $\boldsymbol{E}^{\mathrm{T}}$ and the magnetic field $\boldsymbol{H}^{\mathrm{T}}$ then satisfy the inhomogeneous Maxwell equations

$$(\operatorname{curl} \boldsymbol{H}^{\mathrm{T}})_i + \mathrm{i}\omega \sum_{j=1}^{3} \varepsilon_{ij}^{\mathrm{T}} E_j^{\mathrm{T}} = J_i^{\mathrm{T}} \qquad (i = 1, 2, 3) \qquad (3.1)$$

$$(\operatorname{curl} \boldsymbol{E}^{\mathrm{T}})_i - \mathrm{i}\omega \sum_{j=1}^{3} \mu_{ij}^{\mathrm{T}} H_j^{\mathrm{T}} = -K_i^{\mathrm{T}} \qquad (i = 1, 2, 3). \qquad (3.2)$$

In the domain outside the antenna system $\boldsymbol{E}^{\mathrm{T}}$ and $\boldsymbol{H}^{\mathrm{T}}$ satisfy the homogeneous Maxwell equations

$$\operatorname{curl} \boldsymbol{H}^{\mathrm{T}} + \mathrm{i}\omega\varepsilon_0 \boldsymbol{E}^{\mathrm{T}} = \boldsymbol{0} \qquad (3.3)$$

$$\operatorname{curl} \boldsymbol{E}^{\mathrm{T}} - \mathrm{i}\omega\mu_0 \boldsymbol{H}^{\mathrm{T}} = \boldsymbol{0}. \qquad (3.4)$$

In addition, the transmitted field satisfies the radiation conditions (see [4])

$$\hat{\boldsymbol{r}} \times \boldsymbol{H}^{\mathrm{T}} - (\varepsilon_0/\mu_0)^{\frac{1}{2}} \boldsymbol{E}^{\mathrm{T}} = O(r^{-2}) \quad \text{as} \quad r \to \infty, \qquad (3.5a)$$

$$\hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{T}} + (\mu_0/\varepsilon_0)^{\frac{1}{2}} \boldsymbol{H}^{\mathrm{T}} = O(r^{-2}) \quad \text{as} \quad r \to \infty, \qquad (3.5b)$$

in which $r \stackrel{\text{def}}{=} (\boldsymbol{x} \cdot \boldsymbol{x})^{\frac{1}{2}} \geqq 0$ and $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$. In consequence of (3.3)–(3.5) the following expansion holds (compare [5])

$$\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{x}_{\mathscr{P}}; \omega) = \frac{\exp(\mathrm{i}k_0 r_{\mathscr{P}})}{4\pi r_{\mathscr{P}}} \boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{r}}_{\mathscr{P}}; \omega)[1 + O(r_{\mathscr{P}}^{-1})] \quad \text{as} \quad r_{\mathscr{P}} \to \infty, \qquad (3.6)$$

$$(\mu_0/\varepsilon_0)^{\frac{1}{2}} \boldsymbol{H}^{\mathrm{T}}(\boldsymbol{x}_{\mathscr{P}}; \omega) = \frac{\exp(\mathrm{i}k_0 r_{\mathscr{P}})}{4\pi r_{\mathscr{P}}} [\hat{\boldsymbol{r}}_{\mathscr{P}} \times \boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{r}}_{\mathscr{P}}; \omega)][1 + O(r_{\mathscr{P}}^{-1})] \quad \text{as} \quad r_{\mathscr{P}} \to \infty, \qquad (3.7)$$

where the factor $\boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{r}}_{\mathscr{P}}; \omega)$ is given by

$$\boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{r}}_{\mathscr{P}}; \omega) = \mathrm{i}k_0 \hat{\boldsymbol{r}}_{\mathscr{P}} \times \iint_{\mathscr{S}} [\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{T}}(\boldsymbol{x}; \omega)] \exp(-\mathrm{i}k_0 \hat{\boldsymbol{r}}_{\mathscr{P}} \cdot \boldsymbol{x})\, \mathrm{d}S +$$
$$- \mathrm{i}k_0 (\mu_0/\varepsilon_0)^{\frac{1}{2}} \hat{\boldsymbol{r}}_{\mathscr{P}} \times$$
$$\times \{\hat{\boldsymbol{r}}_{\mathscr{P}} \times \iint_{\mathscr{S}} [\hat{\boldsymbol{n}} \times \boldsymbol{H}^{\mathrm{T}}(\boldsymbol{x}; \omega)] \exp(-\mathrm{i}k_0 \hat{\boldsymbol{r}}_{\mathscr{P}} \cdot \boldsymbol{x})\, \mathrm{d}S\}. \quad (3.8)$$

In these expressions $\boldsymbol{x}_{\mathscr{P}}$ is the position vector of a point $\mathscr{P}$ of observation, $\hat{\boldsymbol{n}}$ is the unit vector in the direction of the outward normal to $\mathscr{S}$ and

$$k_0 \stackrel{\text{def}}{=} \omega(\varepsilon_0 \mu_0)^{\frac{1}{2}} = 2\pi/\lambda_0,$$

$\lambda_0$ being the wavelength in the medium outside the antenna system. It is noted that $\hat{\boldsymbol{r}}_{\mathscr{P}} \cdot \boldsymbol{F}^{\mathrm{T}} = 0$. Although the surface of integration in (3.8) could be any sufficiently regular closed surface completely surrounding the antenna system, it is convenient to choose the boundary $\mathscr{S}$ of the antenna system. In (3.8) the limiting values of $\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{T}}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{H}^{\mathrm{T}}$ when approaching $\mathscr{S}$ from the outside have to be taken. However, due to the continuity of the tangential parts of $\boldsymbol{E}^{\mathrm{T}}$ and $\boldsymbol{H}^{\mathrm{T}}$ across $\mathscr{S}$, the value of the integrands is unambiguous.

## § 4. The antenna as a receiving system

In the receiving situation a time harmonic plane electromagnetic wave is incident upon the antenna system. Since in the receiving situation the antenna carries no external currents, the electric field $\boldsymbol{E}^{\mathrm{R}}$ and the magnetic field $\boldsymbol{H}^{\mathrm{R}}$ satisfy in $\mathscr{V}$ the homogeneous Maxwell equations

$$(\mathrm{curl}\, \boldsymbol{H}^{\mathrm{R}})_i + \mathrm{i}\omega \sum_{j=1}^{3} \varepsilon_{ij}^{\mathrm{R}} E_j^{\mathrm{R}} = 0 \quad (i = 1, 2, 3) \quad (4.1)$$

$$(\mathrm{curl}\, \boldsymbol{E}^{\mathrm{R}})_i - \mathrm{i}\omega \sum_{j=1}^{3} \mu_{ij}^{\mathrm{R}} H_j^{\mathrm{R}} = 0 \quad (i = 1, 2, 3). \quad (4.2)$$

In the domain outside the antenna system the scattered field $\boldsymbol{E}^{\mathrm{s}}$, $\boldsymbol{H}^{\mathrm{s}}$ is defined as the difference between the total electromagnetic field $\boldsymbol{E}^{\mathrm{R}}$, $\boldsymbol{H}^{\mathrm{R}}$ and the incident field $\boldsymbol{E}^{\mathrm{i}}$, $\boldsymbol{H}^{\mathrm{i}}$:

$$\boldsymbol{E}^{\mathrm{s}} \stackrel{\text{def}}{=} \boldsymbol{E}^{\mathrm{R}} - \boldsymbol{E}^{\mathrm{i}} \quad (4.3)$$

$$\boldsymbol{H}^{\mathrm{s}} \stackrel{\text{def}}{=} \boldsymbol{H}^{\mathrm{R}} - \boldsymbol{H}^{\mathrm{i}}. \quad (4.4)$$

The incident field is the plane wave

$$\boldsymbol{E}^{\mathrm{i}} = \boldsymbol{B} \exp(-\mathrm{i}k_0 \hat{\boldsymbol{\beta}} \cdot \boldsymbol{x}) \tag{4.5}$$

$$\boldsymbol{H}^{\mathrm{i}} = (\varepsilon_0/\mu_0)^{\frac{1}{2}} [\boldsymbol{B} \times \hat{\boldsymbol{\beta}}] \exp(-\mathrm{i}k_0 \hat{\boldsymbol{\beta}} \cdot \boldsymbol{x}), \tag{4.6}$$

where $\boldsymbol{B} = \boldsymbol{B}(\omega)$ specifies the amplitude and the state of polarization (in general, elliptic) and $-\hat{\boldsymbol{\beta}}$ is the unit vector in the direction of propagation. Since the wave is transverse, we have $\boldsymbol{B} \cdot \hat{\boldsymbol{\beta}} = 0$. In the domain outside the antenna system both the incident and the scattered field satisfy the homogeneous Maxwell equations

$$\operatorname{curl} \boldsymbol{H}^{\mathrm{i,s}} + \mathrm{i}\omega\varepsilon_0 \boldsymbol{E}^{\mathrm{i,s}} = \boldsymbol{0} \tag{4.7}$$

$$\operatorname{curl} \boldsymbol{E}^{\mathrm{i,s}} - \mathrm{i}\omega\mu_0 \boldsymbol{H}^{\mathrm{i,s}} = \boldsymbol{0}. \tag{4.8}$$

In addition, the scattered field satisfies the radiation condition (compare (3.5))

$$\hat{\boldsymbol{r}} \times \boldsymbol{H}^{\mathrm{s}} - (\varepsilon_0/\mu_0)^{\frac{1}{2}} \boldsymbol{E}^{\mathrm{s}} = O(r^{-2}) \quad \text{as} \quad r \to \infty \tag{4.9}$$

$$\hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{s}} + (\mu_0/\varepsilon_0)^{\frac{1}{2}} \boldsymbol{H}^{\mathrm{s}} = O(r^{-2}) \quad \text{as} \quad r \to \infty. \tag{4.10}$$

In consequence of (4.7)–(4.10) an expansion of $\boldsymbol{E}^{\mathrm{s}}$, $\boldsymbol{H}^{\mathrm{s}}$ similar to (3.6)–(3.8) holds; the detailed form of this expansion is, however, not needed in the course of our investigation.

## § 5. The reciprocity relation

The reciprocity relation is derived with the aid of Lorentz's reciprocity theorem (see Appendix). This theorem can be applied, provided that the medium present in the transmitting situation and the medium present in the receiving situation are related through $\varepsilon_{ij}^{\mathrm{T}} = \varepsilon_{ji}^{\mathrm{R}}$ and $\mu_{ij}^{\mathrm{T}} = \mu_{ji}^{\mathrm{R}}$ at all points in $\mathscr{V}$. (If the media are reciprocal, $\varepsilon_{ij}^{\mathrm{R,T}} = \varepsilon_{ji}^{\mathrm{R,T}}$ and $\mu_{ij}^{\mathrm{R,T}} = \mu_{ji}^{\mathrm{R,T}}$, then the medium in the transmitting situation is identical with the one in the receiving situation. When unidirectional devices are present, the direction of blocking must be reversed when switching from transmission to reception.)

In the first place (A.1) is applied to the domain bounded internally by $\mathscr{S}$ and externally by $\mathscr{S}_r$, where $\mathscr{S}_r$ denotes the surface of the sphere with radius $r$ around the origin ($r > r_0$, if $\mathscr{S}_{r_0}$ is the smallest sphere around the origin, completely surrounding the antenna system). The electromagnetic fields in (A.1) are chosen as follows: $\boldsymbol{E}^{\mathrm{A}} = \boldsymbol{E}^{\mathrm{T}}$, $\boldsymbol{H}^{\mathrm{A}} = \boldsymbol{H}^{\mathrm{T}}$ and $\boldsymbol{E}^{\mathrm{B}} = \boldsymbol{E}^{\mathrm{s}}$, $\boldsymbol{H}^{\mathrm{B}} = \boldsymbol{H}^{\mathrm{s}}$. From (3.6), (3.7), and a similar expansion of $\boldsymbol{E}^{\mathrm{s}}$, $\boldsymbol{H}^{\mathrm{s}}$ it follows that

$$\iint_{\mathscr{S}_r} (\boldsymbol{E}^{\mathrm{T}} \times \boldsymbol{H}^{\mathrm{s}} - \boldsymbol{E}^{\mathrm{s}} \times \boldsymbol{H}^{\mathrm{T}}) \cdot \hat{\boldsymbol{r}} \, \mathrm{d}S = O(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{5.1}$$

As the domain bounded internally by $\mathscr{S}$ and externally by $\mathscr{S}_r$ is free from external currents, both in the transmitting and the receiving situation, we obtain from (A.1) and (5.1) by taking the limit $r \to \infty$

$$\iint_{\mathscr{S}} (\boldsymbol{E}^{\mathrm{T}} \times \boldsymbol{H}^{\mathrm{s}} - \boldsymbol{E}^{\mathrm{s}} \times \boldsymbol{H}^{\mathrm{T}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = 0. \tag{5.2}$$

In (5.2) the limiting value of the integrand when approaching $\mathscr{S}$ from the outside has to be taken. Further we have, by inspection, from (3.8), (4.5), and (4.6)

$$\iint_{\mathscr{S}} (\boldsymbol{E}^{\mathrm{T}} \times \boldsymbol{H}^{\mathrm{i}} - \boldsymbol{E}^{\mathrm{i}} \times \boldsymbol{H}^{\mathrm{T}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{B} \cdot \boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{\beta}}; \omega). \tag{5.3}$$

Addition of (5.2) and (5.3) yields, in view of (4.3) and (4.4),

$$\iint_{\mathscr{S}} (\boldsymbol{E}^{\mathrm{T}} \times \boldsymbol{H}^{\mathrm{R}} - \boldsymbol{E}^{\mathrm{R}} \times \boldsymbol{H}^{\mathrm{T}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{B} \cdot \boldsymbol{F}(\hat{\boldsymbol{\beta}}; \omega). \tag{5.4}$$

In the left hand side of (5.4) the limiting value of the integrand when approaching $\mathscr{S}$ from the outside remains to be taken.

Secondly, (A.1) is applied to the domain $\mathscr{V}$ bounded by $\mathscr{S}$, while the electromagnetic fields are chosen as follows: $\boldsymbol{E}^{\mathrm{A}} = \boldsymbol{E}^{\mathrm{T}}$, $\boldsymbol{H}^{\mathrm{A}} = \boldsymbol{H}^{\mathrm{T}}$ and $\boldsymbol{E}^{\mathrm{B}} = \boldsymbol{E}^{\mathrm{R}}$, $\boldsymbol{H}^{\mathrm{B}} = \boldsymbol{H}^{\mathrm{R}}$. In view of (3.1), (3.2), (4.1), and (4.2) we have $\boldsymbol{J}^{\mathrm{A}} = \boldsymbol{J}^{\mathrm{T}}$, $\boldsymbol{K}^{\mathrm{A}} = \boldsymbol{K}^{\mathrm{T}}$ and $\boldsymbol{J}^{\mathrm{B}} = \boldsymbol{0}$, $\boldsymbol{K}^{\mathrm{B}} = \boldsymbol{0}$, whereas $\boldsymbol{J}^{\mathrm{T}}$ and $\boldsymbol{K}^{\mathrm{T}}$ vanish outside $\mathscr{V}_{\text{source}}$. The result is

$$\iint_{\mathscr{S}} (\boldsymbol{E}^{\mathrm{T}} \times \boldsymbol{H}^{\mathrm{R}} - \boldsymbol{E}^{\mathrm{R}} \times \boldsymbol{H}^{\mathrm{T}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \\ = \iiint_{\mathscr{V}_{\text{source}}} (\boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{E}^{\mathrm{R}} - \boldsymbol{K}^{\mathrm{T}} \cdot \boldsymbol{H}^{\mathrm{R}}) \, \mathrm{d}V. \tag{5.5}$$

In the left hand side of (5.5) the limiting value of the integrand when approaching $\mathscr{S}$ from the inside has to be taken. Combining (5.4) and (5.5) and making use of the continuity of $\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{T,R}}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{H}^{\mathrm{T,R}}$ across $\mathscr{S}$, we obtain the final result

$$\iiint_{\mathscr{V}_{\text{source}}} (\boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{E}^{\mathrm{R}} - \boldsymbol{K}^{\mathrm{T}} \cdot \boldsymbol{H}^{\mathrm{R}}) \, \mathrm{d}V = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{B} \cdot \boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{\beta}}; \omega). \tag{5.6}$$

Equation (5.6) is the reciprocity relation to be proved.

When (5.6) is applied to an antenna consisting of a thin conducting wire, it has to be replaced by (as $\boldsymbol{K}^{\mathrm{T}} \equiv \mathbf{0}$ in this situation)

$$\int_{\mathscr{L}} I^{\mathrm{T}} \boldsymbol{E}^{\mathrm{R}} \cdot \hat{\boldsymbol{\tau}} \, \mathrm{d}s = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{B} \cdot \boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{\beta}}; \omega), \tag{5.7}$$

where $\mathscr{L}$ denotes the antenna, $\hat{\boldsymbol{\tau}}$ is the unit vector along the tangent to $\mathscr{L}$ and $I^{\mathrm{T}}$ is the current distribution in the transmitting situation.

**Appendix**

*Lorentz's theorem.* Let $\boldsymbol{E}^{\mathrm{A,B}}$, $\boldsymbol{H}^{\mathrm{A,B}}$ be time harmonic electromagnetic fields, generated by external currents with volume density $\boldsymbol{J}^{\mathrm{A,B}}$ and $\boldsymbol{K}^{\mathrm{A,B}}$, in a medium with complex tensor permittivity $\varepsilon_{ij}^{\mathrm{A,B}}$ and complex tensor permeability $\mu_{ij}^{\mathrm{A,B}}$ which satisfy the conditions stated in § 2. Let $\mathscr{S}$ be a sufficiently regular bounded closed surface and let $\mathscr{V}$ be the domain inside $\mathscr{S}$. Then

$$\iint_{\mathscr{S}} (\boldsymbol{E}^{\mathrm{A}} \times \boldsymbol{H}^{\mathrm{B}} - \boldsymbol{E}^{\mathrm{B}} \times \boldsymbol{H}^{\mathrm{A}}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S =$$

$$= \iiint_{\mathscr{V}} (-\boldsymbol{E}^{\mathrm{A}} \cdot \boldsymbol{J}^{\mathrm{B}} + \boldsymbol{E}^{\mathrm{B}} \cdot \boldsymbol{J}^{\mathrm{A}} - \boldsymbol{H}^{\mathrm{B}} \cdot \boldsymbol{K}^{\mathrm{A}} + \boldsymbol{H}^{\mathrm{A}} \cdot \boldsymbol{K}^{\mathrm{B}}) \, \mathrm{d}V, \tag{A.1}$$

provided that $\varepsilon_{ij}^{\mathrm{A}} = \varepsilon_{ji}^{\mathrm{B}}$ and $\mu_{ij}^{\mathrm{A}} = \mu_{ji}^{\mathrm{B}}$ at all points in $\mathscr{V}$ or on $\mathscr{S}$. The proof follows from an application of Gauss' divergence theorem and the use of Maxwell's equations (cf. [6]).

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