# Power reciprocity in antenna theory

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*Indexing term: Antenna theory*


## Abstract

A rigorous derivation of the power-reciprocity theorem in antenna theory is presented. It relates the absorption cross-section of the load in the receiving situation to the power gain of the antenna in the transmitting situation. The most general case is considered, namely (*a*) the material of which the antenna is made may be lossy, anisotropic and inhomogeneous, if only linear in its electromagnetic behaviour, (*b*) the vectorial amplitude radiation characteristic of the antenna in the transmitting situation and the incident plane wave in the receiving situation may be arbitrarily elliptically polarised, and (*c*) the impedance of the load in the receiving situation may be arbitrarily mismatched to the input impedance of the antenna in the transmitting situation. Therefore the known, restricted form of the theorem is generalised; also some notions are made more precise. As a corollary, a generalised theorem is presented as to the average value of the absorption cross-section of the load taken over all directions of incidence.


## List of symbols

$\boldsymbol{E}, \boldsymbol{H}$ = electric, magnetic field strength
$\boldsymbol{D}, \boldsymbol{B}$ = electric, magnetic flux density
$\boldsymbol{e}, \boldsymbol{h}$ = electric, magnetic vectorial amplitude radiation characteristic
$\omega$ = angular frequency
$\epsilon_0, \mu_0, k_0, \lambda_0$ = permittivity, permeability, wavenumber, wavelength in free space outside the antenna system
$V, I$ = voltage across, current flowing into or from accessible port
$Z$ = impedance at accessible port
$P$ = time-averaged electromagnetic power
$G$ = power gain of antenna
$D$ = directive gain of antenna
$\sigma$ = effective area of antenna
$\eta$ = efficiency
$\hat{\boldsymbol{\alpha}}, A, S^i$ = direction of incidence, complex electric-field amplitude and radiation intensity of incident plane wave
$\mathcal{E}$ = electromotive force in equivalent Thévenin network

*Superscripts*

$T$ transmitting situation
$R$ receiving situation
$i$ incident field
$s$ scattered field

*Subscripts*

$L$ load
*in* input (at accessible port)
*pol* state of polarisation

## 1 Introduction

One of the basic theorems in the electromagnetic theory of antennas is a reciprocity relation between the transmitting and the receiving properties of an antenna. In its customary form, this reciprocity relation applies to two different antennas a finite distance apart, each of which plays in turn the rôle of transmitting and receiving antenna (see, for example, References 1, 2). In technical applications, however, the reciprocity theorem is usually employed to relate the radiation characteristic of a certain antenna in the transmitting situation to the properties of the same antenna when it is receiving an incident plane wave. Clearly, this problem requires a reciprocity theorem applicable to a single antenna. Now, such a reciprocity relation can be obtained from the aforementioned reciprocity theorem pertaining to two separate antennas by letting the mutual distance between the two antennas become very large, and considering the limiting form of the relation between the electromagnetic-field quantities involved (see, for example, References 3, 4, 5,*) A more elegant procedure, however, is to derive directly a reciprocity theorem applicable to a single antenna, without having to resort, as an intermediate step, to the corresponding theorem pertaining to two separate antennas. How this is done has been shown by one of the present authors in a previous paper,[6] where Lorentz's reciprocity theorem for electromagnetic fields serves as the starting point.

The results mentioned thus far deal with reciprocity theorems that relate to each other the *complex amplitudes* of the time-harmonic oscillations of the different electromagnetic-field quantities involved. The purpose of the present paper is to obtain a reciprocity relation in which certain quantities related to the time-averaged electromagnetic *power flows* in the transmitting and in the receiving situation occur, again for a single antenna. It will be shown that such a relation can be obtained provided that, in the transmitting situation, the antenna system is fed at a single accessible port. In the receiving situation, a load is connected to this port. In its final form, the theorem can be written as

$$\sigma_L(\hat{\boldsymbol{\alpha}}) = (\lambda_0^2/4\pi)\,\eta_L\,\eta_{pol}(\hat{\boldsymbol{\alpha}})\,G(\hat{\boldsymbol{\alpha}}) \qquad (1)$$

where

$\sigma_L$ = absorption cross-section of the load
$\eta_L$ = efficiency of the load
$\eta_{pol}$ = polarisation efficiency
$G$ = power gain of antenna
$\hat{\boldsymbol{\alpha}}$ = unit vector in direction of incidence of plane wave

*COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 98

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In a less general form, a theorem of this kind has been in use in antenna theory for a long time already. One usually takes either $\eta_L = 1$ (matched load) or $\eta_{pol} = 1$ (matched states of polarisation), or $\eta_L = 1$ and $\eta_{pol} = 1$. In addition, in most cases only the state of linear polarisation is considered. The 'proofs' usually contain a few intuitive arguments based on electrical network concepts†‡. In the present paper, the most general case (under the restriction of a single accessible port) is discussed, and the proof is a rigorous one, based throughout on electromagnetic-field concepts and Lorentz's reciprocity theorem for electromagnetic fields.

As a corollary of eqn. 1, a generalised theorem is obtained as to the average value of the absorption cross-section of the load taken over all directions of incidence.

## 2 Description of the configuration

The antenna system under consideration occupies a bounded domain $V$ in space. Externally, $V$ is bounded by a sufficiently regular closed surface $S_0$, and internally, $V$ is bounded by a sufficiently regular closed surface $S_1$. The surface $S_1$ is considered as the termination of the antenna system, and on it the description in terms of a single accessible port holds (Fig. 1). Parts of $S_0$ and $S_1$ may coincide. The region $V$ thus introduced allows us to distinguish the antenna system from the environment into which it radiates or scatters, as well as from the terminals at which it is accessible. The cartesian co-ordinates of a point in space are denoted by

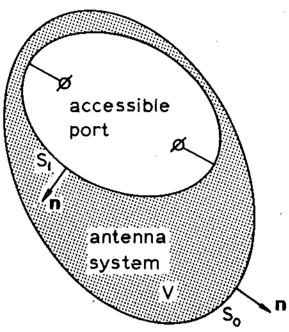


**Fig. 1**
*Antenna configuration*

$x$, $y$ and $z$; the time variable is denoted by $t$. The position vector is denoted by $\boldsymbol{r}$. The electromagnetic fields occurring in the transmitting situation, as well as those occurring in the receiving situation, are assumed to vary sinusoidally in time with the same angular frequency $\omega$. The complex representation of the field vectors is used, and in the formulas, the complex time factor $\exp(-i\omega t)$, common to all field components, is omitted.

The antenna consists of a medium, the electromagnetic behaviour of which is linear and passive, no further restrictions as to its electromagnetic properties being imposed. The properties of the medium may change abruptly when crossing a (bounded) surface, but, across such a surface of discontinuity in properties, the tangential parts of both the electric- and the magnetic-field vector are continuous. Other parts of the antenna system may consist of conducting surfaces. These surfaces are assumed to be electrically perfectly conducting, and on them the tangential part of the electric-field vector vanishes.

The medium outside $S_0$ is assumed to be linear, homogeneous, isotropic and lossless, with real scalar permittivity $\epsilon_0$ and real scalar permeability $\mu_0$; this includes the case of free space.

In the following Sections, $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$ and $\boldsymbol{B}$ denote the space- and frequency-dependent complex representations of the electric-field vector, the magnetic-field vector, the electric-flux density and the magnetic-flux density, respectively. All quantities are expressed in terms of SI units. The superscripts $T$ and $R$ are used to denote the transmitting and the receiving situation, respectively.

## 3 The antenna in the transmitting situation

In the transmitting situation (Fig. 2) the accessible port of the antenna is fed by a source. Let $I^T$ denote the electric current fed into the port, and let $V^T$ denote the voltage across the port. (In the single-mode waveguide description, $I^T$ and $V^T$ denote the complex amplitudes, in a chosen transverse reference plane, of the transverse parts of the magnetic and the electric field pertaining to the wave-

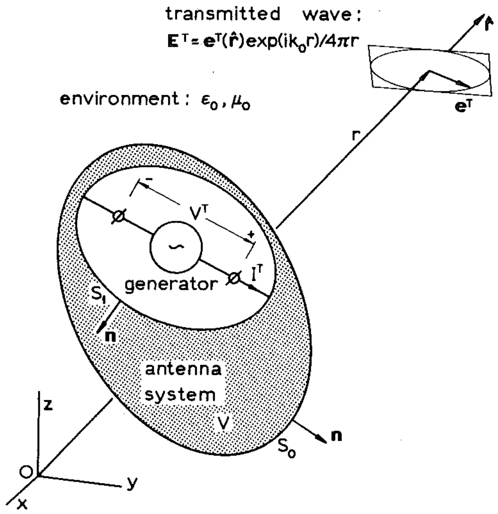


**Fig. 2**
*Antenna in transmitting situation*

Accessible port fed by a source

guide mode; the relevant transverse modal functions should be properly normalised.) As a consequence of the uniqueness theorem of electromagnetic fields, the voltage $V^T$ is linearly related to the current $I^T$ through the relation

$$V^T = Z_{in} I^T \tag{2}$$

where $Z_{in}$ is the input impedance of the radiating antenna system in the transmitting situation. The time-averaged electromagnetic power $P_{in}$ fed into the antenna system in the transmitting situation is then given by

$$P_{in} = \tfrac{1}{2}\,\mathrm{Re}[V^T I^{T*}] \tag{3}$$

where $^*$ denotes the complex conjugate.

In the domain $V$, between $S_0$ and $S_1$, the electromagnetic-field vectors satisfy the sourcefree electromagnetic-field equations

$$\mathrm{curl}\, \boldsymbol{H}^T + i\omega \boldsymbol{D}^T = \boldsymbol{0} \tag{4}$$

$$\mathrm{curl}\, \boldsymbol{E}^T - i\omega \boldsymbol{B}^T = \boldsymbol{0} \tag{5}$$

and the constitutive equations which express $\{\boldsymbol{D}^T, \boldsymbol{B}^T\}$ linearly in terms of $\{\boldsymbol{E}^T, \boldsymbol{H}^T\}$. Owing to the continuity of the tangential parts of $\boldsymbol{E}^T$ and $\boldsymbol{H}^T$ across $S_1$ we can rewrite the expression for $P_{in}$ as

$$P_{in} = \tfrac{1}{2}\,\mathrm{Re}\left[\iint_{S_1} (\boldsymbol{E}^T \times \boldsymbol{H}^{T*}) \cdot \boldsymbol{n}\, dA\right] \tag{6}$$

where $\boldsymbol{n}$ denotes the unit vector along the outward normal.

†COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 101
‡SILVER, S.: *op. cit.*, p. 49

In the domain outside $S_0$, $\boldsymbol{E}^T$ and $\boldsymbol{H}^T$ satisfy the source-free electromagnetic-field equations

$$\operatorname{curl} \boldsymbol{H}^T + \mathrm{i}\omega\epsilon_0 \boldsymbol{E}^T = \boldsymbol{0} \tag{7}$$

$$\operatorname{curl} \boldsymbol{E}^T - \mathrm{i}\omega\mu_0 \boldsymbol{H}^T = \boldsymbol{0} \tag{8}$$

In addition, the transmitted field satisfies the radiation conditions[7,8] †

$$\hat{\boldsymbol{r}} \times \boldsymbol{H}^T + (\epsilon_0/\mu_0)^{\frac{1}{2}} \boldsymbol{E}^T = O(r^{-2}) \quad \text{as } r \to \infty \tag{9}$$

$$\hat{\boldsymbol{r}} \times \boldsymbol{E}^T - (\mu_0/\epsilon_0)^{\frac{1}{2}} \boldsymbol{H}^T = O(r^{-2}) \quad \text{as } r \to \infty \tag{10}$$

in which $\hat{\boldsymbol{r}}\, (= \boldsymbol{r}/r)$ denotes the unit vector in the radial direction and $r$ is the distance from the origin to a point in space. As a consequence of eqns. 7–10, the following representation holds:

$$\{\boldsymbol{E}^T(\boldsymbol{r}_P), \boldsymbol{H}^T(\boldsymbol{r}_P)\} \simeq \{\boldsymbol{e}^T(\hat{\boldsymbol{r}}_P), \boldsymbol{h}^T(\hat{\boldsymbol{r}}_P)\} \exp(\mathrm{i}k_0 r_P)/4\pi r_P \quad \text{as } r_P \to \infty \tag{11}$$

in which

$$k_0 = \omega(\epsilon_0\mu_0)^{\frac{1}{2}} = 2\pi/\lambda_0 \tag{12}$$

$\lambda_0$ being the wavelength in the medium outside $S_0$. $P$ denotes the point of observation with position vector $\boldsymbol{r}_P$. Between $\boldsymbol{e}^T$ and $\boldsymbol{h}^T$ the following relations exist:

$$\boldsymbol{e}^T = (\mu_0/\epsilon_0)^{\frac{1}{2}} (\boldsymbol{h}^T \times \hat{\boldsymbol{r}}_P)$$

and

$$\boldsymbol{h}^T = (\epsilon_0/\mu_0)^{\frac{1}{2}} (\hat{\boldsymbol{r}}_P \times \boldsymbol{e}^T) \tag{13}$$

These relations, together with eqn. 11, are in accordance with eqns. 9 and 10. We shall denote $\boldsymbol{e}^T = \boldsymbol{e}^T(\hat{\boldsymbol{r}}_P)$ as the vectorial amplitude radiation characteristic of the antenna system. For a given antenna, it is only dependent on the direction of observation and is transverse with respect to the direction of propagation of the expanding spherical wave generated by the antenna, i.e. $\hat{\boldsymbol{r}}_P \cdot \boldsymbol{e}^T = 0$. The time-averaged electromagnetic power $P^T$ radiated by the antenna is given by

$$P^T = \tfrac{1}{2} \operatorname{Re}\{\textstyle\iint_{S_0} (\boldsymbol{E}^T \times \boldsymbol{H}^{T*}) \cdot \boldsymbol{n}\, \mathrm{d}A\} \tag{14}$$

where $\boldsymbol{n}$ denotes the unit vector along the direction of the outward normal. Since the medium outside $S_0$ is lossless, we can replace $S_0$ in eqn. 14 by a sphere whose radius is taken to be so large that eqn. 11 holds. Then, we can rewrite eqn. 14 as

$$P^T = \frac{1}{32\pi^2} \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} \iint_\Omega \boldsymbol{e}^T \cdot \boldsymbol{e}^{T*}\, \mathrm{d}\Omega \tag{15}$$

where $\Omega$ denotes the sphere of unit radius. Incidentally, eqn. 15 proves that $P^T > 0$ for any nonidentically vanishing $\boldsymbol{e}^T$.

The directive gain $D = D(\hat{\boldsymbol{r}}_P)$ of the antenna is introduced as‡

$$D \,\Delta\, \frac{4\pi \boldsymbol{e}^T \cdot \boldsymbol{e}^{T*}}{\iint_\Omega \boldsymbol{e}^T \cdot \boldsymbol{e}^{T*}\, \mathrm{d}\Omega} \tag{16}$$

and the power gain $G = G(\hat{\boldsymbol{r}}_P)$ of the antenna as§

$$G \,\Delta\, \eta^T D \tag{17}$$

where

$$\eta^T \,\Delta\, P^T/P_{in} \tag{18}$$

is the efficiency of the antenna in the transmitting situation. For a passive antenna system we have $P^T \leq P_{in}$, and hence $\eta^T \leq 1$. The maximum value, $\eta^T = 1$, occurs if $P^T = P_{in}$, i.e. if the antenna system is lossless. In the latter case, we also have $G = D$. On account of eqn. 15, we can write the expression for $G$ as

$$G = \frac{1}{8\pi} \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} (\boldsymbol{e}^T \cdot \boldsymbol{e}^{T*})/P_{in} \tag{19}$$

It is known that $\boldsymbol{e}^T = \boldsymbol{e}^T(\hat{\boldsymbol{r}}_P)$ can be expressed in terms of the values that the tangential parts of $\boldsymbol{E}^T$ and $\boldsymbol{H}^T$ admit on $S_0$; the relevant expression is,[6] †

$$\begin{aligned} \boldsymbol{e}^T = {} & \mathrm{i}k_0 \hat{\boldsymbol{r}}_P \times \textstyle\iint_{S_0} \{\boldsymbol{n} \times \boldsymbol{E}^T(\boldsymbol{r})\} \exp(-\mathrm{i}k_0 \hat{\boldsymbol{r}}_P \cdot \boldsymbol{r})\, \mathrm{d}A \\ & -\mathrm{i}k_0 (\mu_0/\epsilon_0)^{\frac{1}{2}} \hat{\boldsymbol{r}}_P \\ & \times [\hat{\boldsymbol{r}}_P \times \textstyle\iint_{S_0} \{\boldsymbol{n} \times \boldsymbol{H}^T(\boldsymbol{r})\} \exp(-\mathrm{i}k_0 \hat{\boldsymbol{r}}_P \cdot \boldsymbol{r})\, \mathrm{d}A] \end{aligned} \tag{20}$$

Our proof of the reciprocity relation is in part based on this expression.

## 4 The antenna in the receiving situation

In the receiving situation a time-harmonic, uniform, plane electromagnetic wave is incident upon the antenna system, while the accessible port is connected to the load (Fig. 3). As incident field $\{\boldsymbol{E}^i, \boldsymbol{H}^i\}$ we take

$$\boldsymbol{E}^i = \boldsymbol{A} \exp(-\mathrm{i}k_0 \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{r}) \tag{21}$$

and

$$\boldsymbol{H}^i = (\epsilon_0/\mu_0)^{\frac{1}{2}} (\boldsymbol{A} \times \hat{\boldsymbol{\alpha}}) \exp(-\mathrm{i}k_0 \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{r}) \tag{22}$$

where $\boldsymbol{A}$ is a complex vector that specifies the amplitude and the phase of the plane wave at the origin, as well as its state of polarisation, and $-\hat{\boldsymbol{\alpha}}$ denotes the unit vector in the direction of propagation. (We call $\hat{\boldsymbol{\alpha}}$ the direction of incidence.) The state of polarisation is, in general, elliptic, but is linear

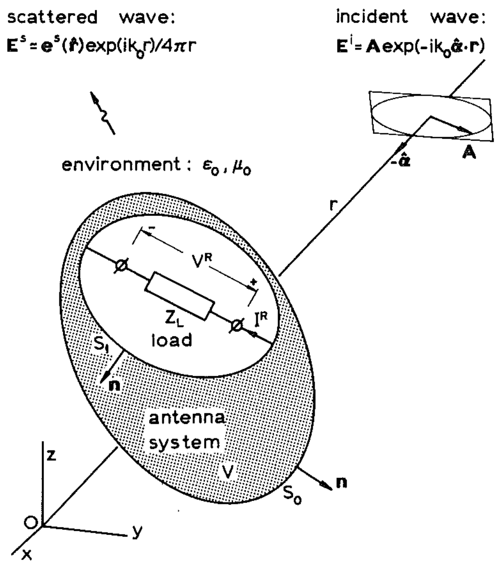


**Fig. 3**
*Antenna in receiving situation*

Plane electromagnetic wave incident on it and accessible port connected to load

if $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{0}$ and circular if $\boldsymbol{A} \cdot \boldsymbol{A} = 0$. Since the wave is transverse, we have $\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{A} = 0$. In the domain outside $S_0$, the scattered field $\{\boldsymbol{E}^s, \boldsymbol{H}^s\}$ is introduced as the difference between the actual (total) field $\{\boldsymbol{E}^R, \boldsymbol{H}^R\}$ and the field of the incident wave $\{\boldsymbol{E}^i, \boldsymbol{H}^i\}$:

$$\boldsymbol{E}^s \,\Delta\, \boldsymbol{E}^R - \boldsymbol{E}^i$$

and

$$\boldsymbol{H}^s \,\Delta\, \boldsymbol{H}^R - \boldsymbol{H}^i \tag{23}$$

In the domain outside $S_0$, both the incident and the total field, and hence the scattered field, satisfy the sourcefree electromagnetic-field equations

† SILVER, S.: *op. cit.*, p. 85
‡ COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 33
§ COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 34

† VAN BLADEL, J.: *op. cit.*, pp. 256–257

$$\mathrm{curl}\, \boldsymbol{H}^{i,s} + \mathrm{i}\omega\epsilon_0 \boldsymbol{E}^{i,s} = \mathbf{0} \tag{24}$$

$$\mathrm{curl}\, \boldsymbol{E}^{i,s} - \mathrm{i}\omega\mu_0 \boldsymbol{H}^{i,s} = \mathbf{0} \tag{25}$$

In addition, the scattered field satisfies the radiation conditions

$$\hat{r} \times \boldsymbol{H}^s + (\epsilon_0/\mu_0)^{\frac{1}{2}} \boldsymbol{E}^s = O(r^{-2}) \quad \text{as } r \to \infty \tag{26}$$

$$\hat{r} \times \boldsymbol{E}^s - (\mu_0/\epsilon_0)^{\frac{1}{2}} \boldsymbol{H}^s = O(r^{-2}) \quad \text{as } r \to \infty \tag{27}$$

The time-averaged power-flow density $\boldsymbol{S}^i$ in the incident wave is given by $\boldsymbol{S}^i = -S^i\hat{\boldsymbol{\alpha}}$, where $S^i$ is given by

$$S^i = \tfrac{1}{2}\, \mathrm{Re}\{(\boldsymbol{E}^i \times \boldsymbol{H}^{i*})\cdot(-\hat{\boldsymbol{\alpha}})\} \tag{28}$$

On account of eqns. 21 and 22, this reduces to

$$S^i = \tfrac{1}{2}(\epsilon_0/\mu_0)^{\frac{1}{2}} \boldsymbol{A}\cdot\boldsymbol{A}^* \tag{29}$$

The time-averaged power $P^R = P^R(\hat{\boldsymbol{\alpha}})$, received by the antenna system, is given by

$$P^R = -\tfrac{1}{2}\, \mathrm{Re}\{\textstyle\iint_{S_0} (\boldsymbol{E}^R \times \boldsymbol{H}^{R*})\cdot\boldsymbol{n}\, \mathrm{d}A\} \tag{30}$$

With the aid of eqns. 29 and 30, the receiving cross-section (also called the effective area) $\sigma^R = \sigma^R(\hat{\boldsymbol{\alpha}})$ of the antenna is introduced as

$$\sigma^R \,\Delta\, P^R/S^i \tag{31}$$

Further, the scattered power $P^s = P^s(\hat{\boldsymbol{\alpha}})$ is defined as

$$P^s \,\Delta\, \tfrac{1}{2}\, \mathrm{Re}\{\textstyle\iint_{S_0} (\boldsymbol{E}^s \times \boldsymbol{H}^{s*})\cdot\boldsymbol{n}\, \mathrm{d}A\} \tag{32}$$

and the scattering cross-section $\sigma^s = \sigma^s(\hat{\boldsymbol{\alpha}})$ of the antenna system as

$$\sigma^s \,\Delta\, P^s/S^i \tag{33}$$

As a consequence of eqns. 24–27, the following representation holds:

$$\{\boldsymbol{E}^s(\boldsymbol{r}_P), \boldsymbol{H}^s(\boldsymbol{r}_P)\} \simeq \{\boldsymbol{e}^s(\hat{\boldsymbol{r}}_P), \boldsymbol{h}^s(\hat{\boldsymbol{r}}_P)\} \exp(\mathrm{i}k_0 r_P)/4\pi r_P \quad \text{as } r_P \to \infty \tag{34}$$

Between $\boldsymbol{e}^s$ and $\boldsymbol{h}^s$, relations of the type in eqn. 13 exist, and for $\boldsymbol{e}^s$, a representation similar to eqn. 20 can be obtained. Using the latter, the cross-section theorem for scattering can be proved.[9]† The result

$$\sigma^R(\hat{\boldsymbol{\alpha}}) + \sigma^s(\hat{\boldsymbol{\alpha}}) = \frac{\lambda_0}{2\pi} \frac{\mathrm{Im}\{\boldsymbol{A}^*\cdot\boldsymbol{e}^s(-\hat{\boldsymbol{\alpha}})\}}{\boldsymbol{A}\cdot\boldsymbol{A}^*} \tag{35}$$

is not needed in our further discussion.

In the domain $V$, between $S_0$ and $S_1$, the electromagnetic-field vectors satisfy the sourcefree electromagnetic-field equations

$$\mathrm{curl}\, \boldsymbol{H}^R + \mathrm{i}\omega \boldsymbol{D}^R = \mathbf{0} \tag{36}$$

$$\mathrm{curl}\, \boldsymbol{E}^R - \mathrm{i}\omega \boldsymbol{B}^R = \mathbf{0} \tag{37}$$

and the constitutive equations which express $\{\boldsymbol{D}^R, \boldsymbol{B}^R\}$ linearly in terms of $\{\boldsymbol{E}^R, \boldsymbol{H}^R\}$. The time-averaged electromagnetic power $P_L = P_L(\hat{\boldsymbol{\alpha}})$ dissipated in the load is given by

$$P_L = -\tfrac{1}{2}\, \mathrm{Re}\{\textstyle\iint_{S_1} (\boldsymbol{E}^R \times \boldsymbol{H}^{R*})\cdot\boldsymbol{n}\, \mathrm{d}A\} \tag{38}$$

Owing to the continuity of the tangential parts of $\boldsymbol{E}^R$ and $\boldsymbol{H}^R$ across $S_1$, we can express $P_L$ also in terms of the electric current $I_L$ flowing into the load and the voltage $V_L$ across it as

$$P_L = \tfrac{1}{2}\, \mathrm{Re}[V_L I_L^*] \tag{39}$$

Further, the absorption cross-section $\sigma_L = \sigma_L(\hat{\boldsymbol{\alpha}})$ of the load is introduced as

$$\sigma_L \,\Delta\, P_L/S^i \tag{40}$$

where $S^i$ is given by eqn. 29. For a passive antenna system, the relation $P_L \leq P^R$ holds, and hence $\sigma_L \leq \sigma^R$. If the antenna system is lossless, we have $P_L = P^R$, and hence $\sigma_L = \sigma^R$. Since the electromagnetic properties of the load are assumed to be linear, $V_L$ is linearly related to $I_L$. We write this relation as

$$V_L = Z_L I_L \tag{41}$$

where $Z_L$ is the impedance of the load.

## 5 The reciprocity relation

The starting point for the derivation of the reciprocity relation is Lorentz's reciprocity theorem for electromagnetic fields.† This theorem can be applied, provided that the electromagnetic properties of the medium present in the transmitting situation and those of the medium present in the receiving situation are interrelated in such a way that, at all points in space, the relation

$$\boldsymbol{E}^T\cdot\boldsymbol{D}^R - \boldsymbol{E}^R\cdot\boldsymbol{D}^T - \boldsymbol{H}^T\cdot\boldsymbol{B}^R + \boldsymbol{H}^R\cdot\boldsymbol{B}^T = 0 \tag{42}$$

holds. In the domain outside $S_0$ this is obviously the case as the constitutive equations here are simply $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$ and $\boldsymbol{B} = \mu_0\boldsymbol{H}$, both in the transmitting and in the receiving situation. In the domain $V$, between $S_0$ and $S_1$, the situation may be more complicated. Here, eqn. 42 holds without change of properties of the medium when the medium is reciprocal. In all other cases, the medium is nonreciprocal, and the appropriate change in properties has to be made when switching from transmission to reception and vice versa. It is noted that, in the general condition (eqn. 42), nonreciprocal media, including those showing the magnetoelectric effect,[10] are included. If eqn. 42 is satisfied, Lorentz's theorem states that

$$\textstyle\iint_S (\boldsymbol{E}^T \times \boldsymbol{H}^R - \boldsymbol{E}^R \times \boldsymbol{H}^T)\cdot\boldsymbol{n}\, \mathrm{d}A = 0 \tag{43}$$

for any sufficiently regular, bounded, closed surface $S$, provided that the domain bounded by $S$ is free from electromagnetic sources. If eqn. 43 is applied to a domain outside $S_0$, we may, on account of eqns. 23–25, also replace $\{\boldsymbol{E}^R, \boldsymbol{H}^R\}$ in eqns. 42 and 43 by either $\{\boldsymbol{E}^s, \boldsymbol{H}^s\}$ or $\{\boldsymbol{E}^i, \boldsymbol{H}^i\}$.

Let $S_r$ denote the sphere with radius $r$ and centre at the origin, where $r$ is chosen so large that $S_r$ completely surrounds $S_0$. Since the fields $\{\boldsymbol{E}^T, \boldsymbol{H}^T\}$ and $\{\boldsymbol{E}^s, \boldsymbol{H}^s\}$ both satisfy the radiation conditions (eqns. 9 and 10 and 26 and 27, respectively) we have

$$\lim_{r\to\infty} \textstyle\iint_{S_r} (\boldsymbol{E}^T \times \boldsymbol{H}^s - \boldsymbol{E}^s \times \boldsymbol{H}^T)\cdot\boldsymbol{n}\, \mathrm{d}A = 0 \tag{44}$$

Consequently, the application of Lorentz's theorem (eqn. 43) to the domain bounded internally by $S_0$ and externally by $S_r$, and to the fields $\{\boldsymbol{E}^T, \boldsymbol{H}^T\}$ and $\{\boldsymbol{E}^s, \boldsymbol{H}^s\}$ leads in the limit $r \to \infty$ to

$$\textstyle\iint_{S_0} (\boldsymbol{E}^T \times \boldsymbol{H}^s - \boldsymbol{E}^s \times \boldsymbol{H}^T)\cdot\boldsymbol{n}\, \mathrm{d}A = 0 \tag{45}$$

Next, we observe that, from eqns. 20, 21 and 22, it follows that

$$\textstyle\iint_{S_0} (\boldsymbol{E}^T \times \boldsymbol{H}^i - \boldsymbol{E}^i \times \boldsymbol{H}^T)\cdot\boldsymbol{n}\, \mathrm{d}A = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{A}\cdot\boldsymbol{e}^T(\hat{\boldsymbol{\alpha}}) \tag{46}$$

On adding eqns. 45 and 46, and using eqn. 23, we arrive at

$$\textstyle\iint_{S_0} (\boldsymbol{E}^T \times \boldsymbol{H}^R - \boldsymbol{E}^R \times \boldsymbol{H}^T)\cdot\boldsymbol{n}\, \mathrm{d}A = (\mathrm{i}\omega\mu_0)^{-1} \boldsymbol{A}\cdot\boldsymbol{e}^T(\hat{\boldsymbol{\alpha}}) \tag{47}$$

We proceed with the application of Lorentz's theorem (eqn. 43) to the domain $V$, bounded internally by $S_1$ and

† VAN BLADEL, J.: *op. cit.*, p. 258

† COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 24

externally by $S_0$ and to the fields $\{E^T, H^T\}$ and $\{E^R, H^R\}$, which gives

$$\iint_{S_1} (E^T \times H^R - E^R \times H^T)\cdot n \, dA = \iint_{S_0} (E^T \times H^R - E^R \times H^T)\cdot n \, dA \qquad (48)$$

However, on $S_1$ the field description in terms of the single accessible port holds. Taking into account the direction of $n$ and the chosen direction of the currents $I^T$ and $I^R$, we can rewrite the left-hand side of eqn. 48 as

$$\iint_{S_1} (E^T \times H^R - E^R \times H^T)\cdot n \, dA = -(V^T I^R + V^R I^T) \qquad (49)$$

Combining eqns. 47, 48 and 49, we arrive at the amplitude reciprocity relation

$$V^T I^R + V^R I^T = -(\mathrm{i}\omega\mu_0)^{-1} A \cdot e^T(\hat{\alpha}) \qquad (50)$$

A first corollary of eqn. 50 is that it justifies rigorously the network representation of an antenna in the receiving situation when the properties of the latter as a transmitting antenna are known. Dividing both sides of eqn. 50 by $I^T$ and using eqn. 2 we obtain

$$I^R Z_{in} + V^R = \mathcal{E}^R \qquad (51)$$

where the 'equivalent electromotive force' $\mathcal{E}^R = \mathcal{E}^R(\hat{\alpha})$ is given by

$$\mathcal{E}^R = -(\mathrm{i}\omega\mu_0)^{-1} A \cdot e^T(\hat{\alpha})/I^T \qquad (52)$$

Eqn. 51 describes the properties of a 1-port electrical network with an internal voltage source (Thévenin representation). Eqn. 52 shows how $\mathcal{E}^R$ depends on the amplitude, the phase and the state of polarisation of the incident wave and, through the geometrical factor $e^T(\hat{\alpha})/I^T$, on the direction of incidence. Also, eqns. 51 and 52 form the basis of the experimental technique for determining the vectorial amplitude radiation characteristic of a given antenna, in the transmitting situation, by using the antenna as a receiving antenna and measuring its reaction on an incident plane wave. In this procedure, the following steps are to be carried out:

(i) $Z_{in}$ is measured in the transmitting situation
(ii) specific values of $A$ and $\hat{\alpha}$ are chosen
(iii) $V^R$ and $I^R$ are measured (or one of them if $Z_L$ is known)
(iv) $A \cdot e^T(\hat{\alpha})/I^T$ is calculated from eqns. 51 and 52
(v) new values of $A$ and $\hat{\alpha}$ are selected, and from step (ii) onward, the procedure is repeated until enough values of $e^T(\hat{\alpha})/I^T$ are known. It is noted that both $A$ and $e^T(\hat{\alpha})$ are transverse with respect to $\hat{\alpha}$ and that, by varying the complex vector $A$ the state of polarisation of the incident plane wave may be varied.

As a second corollary, we shall derive from eqn. 50 the power-reciprocity relation. To this aim, we multiply each side of eqn. 50 by its complex conjugate and obtain

$$|V^T I^R + V^R I^T|^2 = (\omega\mu_0)^{-2} |A \cdot e^T(\hat{\alpha})|^2 \qquad (53)$$

In this result, we introduce the power gain $G(\hat{\alpha})$ of the antenna in the transmitting situation and the absorption cross-section $\sigma_L(\hat{\alpha})$ of the load in the receiving situation; these quantities are given by eqn. 19 and by eqns. 40 and 29, respectively. The result is arranged as follows:

$$\frac{|V^T I^R + V^R I^T|^2}{16 P_{in} P_L} \sigma_L(\hat{\alpha}) = \frac{\lambda_0^2}{4\pi} \frac{|A \cdot e^T(\hat{\alpha})|^2}{[A \cdot A^*][e^T(\hat{\alpha}) \cdot e^{T*}(\hat{\alpha})]} G(\hat{\alpha}) \qquad (54)$$

In eqn. 54, we introduce the polarisation efficiency $\eta_{pol} = \eta_{pol}(\hat{\alpha})$

$$\eta_{pol} \,\Delta\, \frac{|A \cdot e^T(\hat{\alpha})|^2}{[A \cdot A^*][e^T(\hat{\alpha}) \cdot e^{T*}(\hat{\alpha})]} \qquad (55)$$

and the efficiency of the load $\eta_L$

$$\eta_L \,\Delta\, \frac{16 P_{in} P_L}{|V^T I^R + V^R I^T|^2} \qquad (56)$$

It is observed that $\eta_{pol} > 0$, provided that $A \neq \mathbf{0}$ and $e^T(\hat{\alpha}) \neq \mathbf{0}$, and, on account of Schwarz's inequality, $\eta_{pol} \leq 1$. The maximum value $\eta_{pol} = 1$ is obtained if $A = \chi e^{T*}(\hat{\alpha})$, where $\chi$ is some complex, scalar constant of proportionality. In this case, we call the state of polarisation of the incident wave 'matched' to the state of polarisation of the vectorial amplitude radiation characteristic of the antenna. It is observed that, for a passive antenna system and a dissipative load, we have $P_{in} > 0$, $P_L > 0$ and $V^T I^R + V^R I^T \neq 0$, and hence $\eta_L > 0$. Further, we shall show that $\eta_L \leq 1$. To prove this we introduce in eqn. 56 the expressions $P_{in} = \frac{1}{4}(V^T I^{T*} + V^{T*} I^T)$ and $P_L = \frac{1}{4}(V^R I^{R*} + V^{R*} I^R)$ and rearrange the result as

$$\eta_L = 1 - \frac{|V^{T*} I^R - V^R I^{T*}|^2}{|V^T I^R + V^R I^T|^2} \qquad (57)$$

This shows that $\eta_L \leq 1$. On using eqns. 2 and 41 we can rewrite eqn. 57 as

$$\eta_L = 1 - \frac{|Z_{in}^* - Z_L|^2}{|Z_{in} + Z_L|^2} \qquad (58)$$

The maximum value $\eta_L = 1$ is obtained if $V^{T*} I^R = V^R I^{T*}$ or $Z_L = Z_{in}^*$. In this case, the load is 'matched' to the antenna. Incidentally, it is observed that eqn. 58 shows that $\eta_L$ is independent of the direction of incidence $\hat{\alpha}$. Substitution of eqns. 55 and 56 in eqn. 54 leads to the final result

$$\sigma_L(\hat{\alpha}) = (\lambda_0^2/4\pi)\, \eta_L \eta_{pol}(\hat{\alpha}) G(\hat{\alpha}) \qquad (59)$$

If, for a given antenna system and for a given incident wave, the impedance of the load is taken to be adjustable, eqn. 59 shows that maximum power is dissipated in the load if the load is matched to the antenna. The matching condition $Z_L = Z_{in}^*$ is apparently independent of the direction of incidence $\hat{\alpha}$ of the incident wave. The matching condition for the states of polarisation, on the contrary, is dependent on $\hat{\alpha}$.

A special case of eqn. 59 arises if all three following conditions are satisfied:

(*a*) the load is matched to the antenna ($\eta_L = 1$)
(*b*) the antenna system is lossless ($G(\hat{\alpha}) = D(\hat{\alpha})$ and $\sigma_L(\hat{\alpha}) = \sigma^R(\hat{\alpha})$)
(*c*) $\eta_{pol}(\hat{\alpha}) = 1$ for all $\hat{\alpha}$. Then, eqn. 59 reduces to

$$\sigma^R(\hat{\alpha}) = (\lambda_0^2/4\pi)\, D(\hat{\alpha}) \quad \text{(special case)} \qquad (60)$$

which is a well known result.†‡

One of the domains of application of eqn. 59 is the experimental technique for determining the power gain of a given antenna in the transmitting situation, by using the antenna as a receiving antenna and measuring the absorption cross-section of the load, when a plane wave is incident. In this procedure, the following steps are to be carried out:

(i) $Z_{in}$ is measured in the transmitting situation
(ii) $Z_L$ is selected
(iii) $\eta_L$ is calculated from eqn. 58
(iv) specific values of $A$ and $\hat{\alpha}$ are chosen
(v) $S^i$ is calculated from eqn. 29
(vi) $P_L$ is measured

† SILVER, S.: *op. cit.*, p. 51, eqn. 81
‡ COLLIN, R.E., and ZUCKER, F.J.: *op. cit.*, p. 100

(vii) keeping $A \cdot A^*$ and $\hat{\alpha}$ fixed, the state of polarisation of the incident plane wave is varied until $P_L$ reaches its maximum value; then $\eta_{pol} = 1$ (note that during this process $S^i$ does not change in value)
(viii) the corresponding value of $\sigma_L(\hat{\alpha})$ is calculated from eqn. 40
(ix) $G(\hat{\alpha})$ is calculated from eqn. 59 with $\eta_{pol} = 1$
(x) new values of $A$ and $\hat{\alpha}$ are chosen, and from step (iv) onward the procedure is repeated until enough values of $G(\hat{\alpha})$ are known. It is noted that step (vii) is necessary because the state of polarisation of $e^T(\hat{\alpha})$ is unknown.

## 6 Average absorption cross-section of the load

If in eqn. 59 we introduce the directive gain $D$, we obtain

$$\sigma_L(\hat{\alpha}) = (\lambda_0^2/4\pi)\, \eta_L \eta_{pol}(\hat{\alpha})\, \eta^T D(\hat{\alpha}) \tag{61}$$

By integrating this equation over all directions of incidence, and by dividing the result by $4\pi$, since $\hat{\alpha} \cdot \hat{\alpha} = 1$, we obtain

$$(4\pi)^{-1} \iint_\Omega \sigma_L(\hat{\alpha})\, d\hat{\alpha} = (\lambda_0/4\pi)^2 \eta_L \eta^T \iint_\Omega \eta_{pol}(\hat{\alpha})\, D(\hat{\alpha})\, d\hat{\alpha} \tag{62}$$

The left-hand side is the average of the absorption cross-section of the load, taken over all directions of incidence. In the right-hand side, we take into account that $0 \leq \eta_{pol} \leq 1$ for all $\hat{\alpha}$ and that

$$\iint_\Omega D(\hat{\alpha})\, d\hat{\alpha} = 4\pi \tag{63}$$

This leads to

$$(4\pi)^{-1} \iint_\Omega \sigma_L(\hat{\alpha})\, d\hat{\alpha} \leq (\lambda_0^2/4\pi)\, \eta_L \eta^T \tag{64}$$

The equality sign holds if $\eta_{pol}(\hat{\alpha}) = 1$, for all $\hat{\alpha}$, i.e. if, at all values of $\hat{\alpha}$, the state of polarisation of the incident plane wave is matched to the state of polarisation of the vectorial amplitude radiation characteristic of the antenna.

A special case arises if all three following conditions are satisfied:
(*a*) the load is matched to the antenna ($\eta_L = 1$)
(*b*) the antenna system is lossless ($\eta^T = 1$ and $\sigma_L(\hat{\alpha}) = \sigma^R(\hat{\alpha})$)
(*c*) $\eta_{pol}(\hat{\alpha}) = 1$ for all $\hat{\alpha}$.
Then, eqn. 64 reduces to

$$(4\pi)^{-1} \iint_\Omega \sigma^R(\hat{\alpha})\, d\hat{\alpha} = \lambda_0^2/4\pi \quad \text{(special case)} \tag{65}$$

which is a well known result.†

†SILVER, S.: *op. cit.*, p. 51, eqn. 80

## 7 Conclusion

A rigorous derivation is presented of the power reciprocity theorem for a single antenna. It relates the absorption cross-section of the load, in the receiving situation, to the power gain of the antenna, in the transmitting situation. The result is given for the most general conditions, which include:
(*a*) The material of which the antenna system is made may be lossy, anisotropic and inhomogeneous as far as its electromagnetic properties are concerned.
(*b*) The vectorial amplitude radiation characteristic of the antenna in the transmitting situation and the incident plane wave in the receiving situation may be arbitrarily elliptically polarised.
(*c*) The impedance of the load in the receiving situation may be arbitrarily mismatched to the input impedance of the antenna in the transmitting situation.

The only restrictions are:
(i) The material of which the antenna is made and the load behave linearly as far as their electromagnetic properties are concerned.
(ii) The antenna system comprises a single accessible port at which it is fed in the transmitting situation and loaded in the receiving situation.

As a corollary, a generalised theorem is presented for the average value of the absorption cross-section of the load, taken over all directions of incidence.

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