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Theorem on the maximum absorption of electromagnetic radiation by a scattering object of bounded extent

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The absorption of electromagnetic energy by matter distributed in a predefined loading domain of finite extent is investigated in its general aspects. It is shown that under variation of the medium properties in the loading domain, the electromagnetic energy absorbed by the load can, at least in principle, be maximized. The maximization condition serves as constitutive relation, in a nonstandard form, for the relevant medium. The condition is a configurational property and is independent of the distribution of the sources that generate the exciting electromagnetic field.

1. INTRODUCTION

One of the basic theorems of receiving antenna theory is that a load connected to its accessible ports can be optimized such that the electromagnetic energy absorbed by it is maximized. A general proof of this theorem, entirely based on electromagnetic field theory, has been given by *de Hoop and de Jong* [1974] for a one-port antenna and by *de Hoop* [1975] for an arbitrary N-port antenna ($N \geq 1$). In these references it is shown that the impedance matrix of the optimizing load is given by the complex conjugate of the input-impedance matrix of the antenna when used in its transmission mode. Hence, the existence and the uniqueness of the optimizing load in the receiving mode are evident. Moreover, the optimizing condition turns out to be independent of the excitation of the antenna system.

In antenna theory, because of the presence of accessible ports, at which 'voltages' and 'currents' can be defined, the optimization problem is a discrete, algebraic one. In the present paper the problem is generalized to the case where a loading domain whose electromagnetic properties vary continuously with position is present. Fixed sources, located outside the loading domain, generate an electromagnetic field that is partly scattered and partly absorbed by the load. The problem to be investigated is whether there exists a distribution of matter in the loading domain that maximizes the electromagnetic energy absorbed in it. It is shown that this is indeed the case, although the electromagnetic properties of the relevant medium are specified implicitly as a certain relation between the field values in the loading domain and the induced contrast current densities in the load. The optimization condition is unique and turns out to be independent of the excitation of the configuration.

2. DESCRIPTION OF THE CONFIGURATION

In three-dimensional space $\mathbf{R}^3$, impressed sources, occupying a bounded domain $\mathscr{V}_G$, generate an electromagnetic field. A bounded domain $\mathscr{V}_L$, whose intersection with $\mathscr{V}_G$ is empty, is available for loading purposes. The medium in the configuration is linear, time-invariant and, apart from the impressed source distributions in $\mathscr{V}_G$, passive in its electromagnetic behavior. Its properties may vary piecewise continuously with position. No further restrictions as to its properties are made. The boundary surfaces of $\mathscr{V}_G$ and $\mathscr{V}_L$ are denoted by $\partial\mathscr{V}_G$ and $\partial\mathscr{V}_L$, respectively. The complements of $\mathscr{V}_G \cup \partial\mathscr{V}_G$ and $\mathscr{V}_L \cup \partial\mathscr{V}_L$ in $\mathbf{R}^3$ are denoted by $\mathscr{V}'_G$ and $\mathscr{V}'_L$, respectively (Figure 1). The properties of the medium in $\mathscr{V}'_L$ are fixed; those of the medium in $\mathscr{V}_L$ are adjustable. We characterize the latter by their contrast with vacuum.

An observer's position in the configuration is specified by the coordinates x, y, z, with respect to a given, fixed, orthogonal, Cartesian reference frame with origin $\mathscr{O}$ and three mutually perpendicular base vectors of unit length $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$. The vectorial

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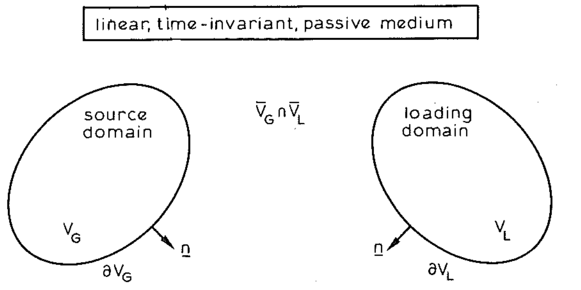

Fig. 1. Source domain $\mathscr{V}_G$ and loading domain $\mathscr{V}_L$ in $\mathbf{R}^3$; $\partial\mathscr{V}_G$ and $\partial\mathscr{V}_L$ are the boundaries of $\mathscr{V}_G$ and $\mathscr{V}_L$, respectively.

position is denoted by $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$. The time coordinate is denoted by t. We prefer to characterize the media by their frequency-domain behavior and therefore we write the electromagnetic field quantities as the superposition of their complex frequency-domain components, the component of angular frequency ω having a time dependence $\exp(-i\omega t)$, where i denotes the imaginary unit.

The electromagnetic state in the configuration is specified by the frequency-domain field vectors **E**, **H**, **D**, **B**; they depend on **r** and ω. Since the corresponding time-domain quantities are real-valued, we have, for any of the field quantities,

$$\text{field quantity}(\mathbf{r}, -\omega) = \text{field quantity}(\mathbf{r}, \omega)^* \quad (1)$$

where the asterisk denotes complex conjugate. The presence of a loading medium in the domain $\mathscr{V}_L$ is accounted for by induced currents of the electric type with volume density $\mathbf{J}_e^s$ (volume density of polarization current) and of the magnetic type with volume density $\mathbf{J}_m^s$ (volume density of magnetization current). These contrast quantities with respect to vacuum are related to the field vectors by

$$\mathbf{J}_e^s = -i\omega(\mathbf{D} - \varepsilon_0\mathbf{E}) \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \quad (2)$$

$$\mathbf{J}_m^s = -i\omega(\mathbf{B} - \mu_0\mathbf{H}) \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \quad (3)$$

Here ε_0 and μ_0 denote the real and positive, permittivity and permeability in vacuo, respectively. The problem is now to derive the relationship between $\{\mathbf{J}_e^s, \mathbf{J}_m^s\}$ on the one hand and $\{\mathbf{E}, \mathbf{H}\}$ on the other that accomplishes maximum energy absorption by the load.

3. FORMULATION OF THE PROBLEM

Field constituents. The total electromagnetic field when a loading medium is present in $\mathscr{V}_L$, is written as the sum of two field constituents: the incident field and the scattered field. The incident field is the field that would be generated by the sources in $\mathscr{V}_G$ if no loading medium were present in $\mathscr{V}_L$; it is distinguished by the superscript i. The scattered field is the field that must be superimposed on the incident field to yield the total field; it will be distinguished by the superscript s. Then we have

$$\text{field quantity} = \text{field quantity}^i + \text{field quantity}^s \quad (4)$$

for any of the field quantities.

Incident field. The incident field is generated by the sources in $\mathscr{V}_G$ and hence it satisfies the electromagnetic field equations (Figure 2):

$$\nabla \times \mathbf{H}^i + i\omega\mathbf{D}^i = \mathbf{J}_e^i \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_G \quad (5)$$

$$\nabla \times \mathbf{E}^i - i\omega\mathbf{B}^i = -\mathbf{J}_m^i \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_G \quad (6)$$

and

$$\nabla \times \mathbf{H}^i + i\omega\mathbf{D}^i = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \bar{\mathscr{V}}_G \quad (7)$$

$$\nabla \times \mathbf{E}^i - i\omega\mathbf{B}^i = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \bar{\mathscr{V}}_G \quad (8)$$

while

$$\nabla \times \mathbf{H}^i + i\omega\varepsilon_0\mathbf{E}^i = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \quad (9)$$

$$\nabla \times \mathbf{E}^i - i\omega\mu_0\mathbf{H}^i = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \quad (10)$$

Across surfaces of discontinuity in electromagnetic properties the usual boundary conditions apply, while causality requires the solution of (5)–(10) to be of an outgoing nature at infinity.

Scattered field. The scattered field is source-free in $\bar{\mathscr{V}}_L$, while the induced volume currents in $\mathscr{V}_L$ act as its sources. The relevant field equations are (Figure 3)

$$\nabla \times \mathbf{H}^s + i\omega\mathbf{D}^s = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \bar{\mathscr{V}}_L \quad (11)$$

$$\nabla \times \mathbf{E}^s - i\omega\mathbf{B}^s = \mathbf{0} \quad \text{when} \quad \mathbf{r} \in \bar{\mathscr{V}}_L \quad (12)$$

while

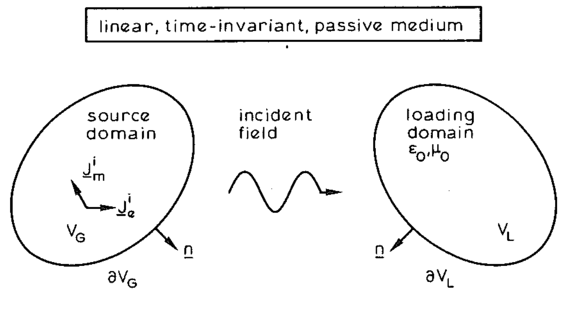

Fig. 2. The incident field and its configuration.

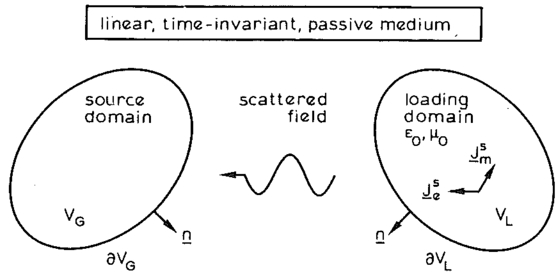

Fig. 3. The scattered field and its configuration.

$$\nabla \times \mathbf{H}^s + i\omega\varepsilon_0\mathbf{E}^s = \mathbf{J}^s_e \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \tag{13}$$

$$\nabla \times \mathbf{E}^s - i\omega\mu_0\mathbf{H}^s = -\mathbf{J}^s_m \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \tag{14}$$

Note that (9)–(10) and (13)–(14) are in accordance with (2)–(3) in view of (4). Across surfaces of discontinuity in electromagnetic properties, again the usual boundary conditions apply, while at infinity the scattered field, too, is of an outgoing nature.

In our analysis we need the formulas that express $\{\mathbf{E}^s, \mathbf{H}^s\}$ in terms of $\{\mathbf{J}^s_e, \mathbf{J}^s_m\}$. These are frequency-domain Green-type representations for an unbounded domain [cf. *Felsen and Marcuvitz,* 1973]. To write them in a form which is manageable for our purpose, we adopt a matrix notation. To this end, we arrange the Cartesian components of the electric and the magnetic field strengths in a column matrix $[F]$ according to the scheme

$$[F] = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} \tag{15}$$

the volume current densities in a column matrix $[J]$ according to the scheme

$$[J] = \begin{bmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{bmatrix} \tag{16}$$

and the tensorial, frequency-domain Green's functions in a square matrix $[G]$ according to the scheme [cf. *de Hoop,* 1977]

$$[G] = \begin{bmatrix} \mathsf{G}_{ee} & \mathsf{G}_{em} \\ \mathsf{G}_{me} & \mathsf{G}_{mm} \end{bmatrix} \tag{17}$$

With the matrix notation thus introduced, we have

$$[F^s(\mathbf{r}, \omega)] = \int_{\mathscr{V}_L} [G(\mathbf{r}, \mathbf{r}', \omega)]\,[J^s(\mathbf{r}', \omega)]\; dV(\mathbf{r}') \tag{18}$$

Energy absorbed by the load. The total energy W^a absorbed by the load is

$$W^a = -(2\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int_{\partial\mathscr{V}_L} \mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\, dA \tag{19}$$

where $\mathbf{n}$ is the unit vector normal to the boundary $\partial\mathscr{V}_L$ of $\mathscr{V}_L$, pointing away from $\mathscr{V}_L$, and $\mathbf{E}\times\mathbf{H}^*$ is the spectral density of the surface density of electromagnetic power flow. We apply Gauss' divergence theorem to the integral over $\partial\mathscr{V}_L$, employ the equations

$$\nabla \times \mathbf{H} + i\omega\varepsilon_0\mathbf{E} = \mathbf{J}^s_e \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \tag{20}$$

$$\nabla \times \mathbf{E} - i\omega\mu_0\mathbf{H} = -\mathbf{J}^s_m \quad \text{when} \quad \mathbf{r} \in \mathscr{V}_L \tag{21}$$

which follow from (9)–(10), (13)–(14), and (4), and use the property

$$(2\pi)^{-1}\int_{-\infty}^{\infty} (i\omega\varepsilon_0\mathbf{E}\cdot\mathbf{E}^* - i\omega\mu_0\mathbf{H}^*\cdot\mathbf{H})\, d\omega = 0 \tag{22}$$

which holds since the integrand is, in view of (1), an odd function of ω. The result is the following expression for the absorbed energy:

$$W^a = (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int_{\mathscr{V}_L} (\mathbf{J}^{s*}_e\cdot\mathbf{E} + \mathbf{J}^s_m\cdot\mathbf{H}^*)\, dV \tag{23}$$

or, employing our matrix notation,

$$W^a = (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int_{\mathscr{V}_L} [J^s]^H [F]\, dV \tag{24}$$

where $[\cdots]^H$ denotes the hermitian conjugate of $[\cdots]$.

Scattered energy. For later convenience, we also introduce the scattered energy W^s, i.e., the energy that the scattered field, if it could exist alone, would carry across $\partial\mathscr{V}_L$ away from $\mathscr{V}_L$:

$$W^s = (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int_{\partial\mathscr{V}_L} \mathbf{n}\cdot(\mathbf{E}^s\times\mathbf{H}^{s*})\, dA \tag{25}$$

A procedure similar to the one employed for rewriting the expression for W^a now leads to

$$W^s = -(2\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int_{\mathscr{V}_L} [J^s]^H [F^s]\, dV \tag{26}$$

Since the scattered field carries energy away from its sources in $\mathscr{V}_L$, we must have $W^s > 0$ if $[J^s] \neq [0]$. This property will be used in section 4.

4. MAXIMIZATION OF THE ABSORBED ENERGY

The relationship that exists in $\mathscr{V}_L$ between the induced currents with volume density $[J^s]$ and the field values $[F]$ is typical for the medium present in $\mathscr{V}_L$. We now look for that particular relationship that maximizes W^a for a given excitation. The latter implies that $[J^i]$, $[F^i]$, and $[G]$ are kept fixed, while $[J^s]$ and $[F^s]$ are varied. Let the subscript opt denote the value of a quantity in the situation where W^a is maximum; then we write

$$[J^s] = [J^s_{\text{opt}}] + [\delta J^s] \tag{27}$$

$$[F] = [F_{\text{opt}}] + [\delta F^s] \tag{28}$$

where in (28) we have taken into account that

$$[\delta F^i] = [0] \tag{29}$$

Substitution of (27)–(28) in (24) leads to

$$W^a = W^a_{\text{opt}} + (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \cdot \int_{\mathscr{V}_L} ([\delta J^s]^H [F_{\text{opt}}] + [J^s_{\text{opt}}]^H [\delta F^s])\, dV + (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \int_{\mathscr{V}_L} [\delta J^s]^H [\delta F^s]\, dV \tag{30}$$

in which

$$W^a_{\text{opt}} = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \int_{\mathscr{V}_L} [J^s_{\text{opt}}]^H [F_{\text{opt}}]\, dV \tag{31}$$

In the first integral on the right-hand side of (30) we substitute the expression (cf. (18))

$$[\delta F^s] = \int_{\mathscr{V}_L} [G][\delta J^s]\, dV \tag{32}$$

On the right-hand side of (30) the first-order terms in $[\delta J^s]$ now cancel provided that the following condition holds:

$$[F_{\text{opt}}(\mathbf{r}, \omega)] + \int_{\mathscr{V}_L} [G(\mathbf{r}', \mathbf{r}, \omega)]^H \cdot [J^s_{\text{opt}}(\mathbf{r}', \omega)]\, dV(\mathbf{r}') = [0]$$

$$\text{when} \quad \mathbf{r} \in \mathscr{V}_L \quad \text{for all} \quad \omega \tag{33}$$

If (33) is satisfied, W^a_{opt} is the maximum value that W^a can attain, since in view of (26) the last term on the right-hand side of (30) is negative for any nonvanishing $[\delta J^s]$. Consequently, the medium that would maximize the energy absorbed in the loading domain $\mathscr{V}_L$ has (33) as its constitutive relation (in a nonstandard form). Mathematically, (33) is an integral equation of the first kind whose solution would yield the constitutive relation for the relevant medium in the standard form. Note that the condition (33) is independent of the excitation of the configuration.

This concludes our general analysis. The implications of (33) for particular situations have not yet been studied.

5. CONCLUSION

The absorption properties of a distributed load of finite extent, under arbitrary electromagnetic excitation, are investigated in their general aspects. It is shown that under variation of the medium properties in the loading domain, the electromagnetic energy absorbed by the load can, at least in principle, be maximized. The maximization condition serves as constitutive relation, in a nonstandard form, for the relevant medium. The condition is a configurational property and is independent of the distribution of the sources that generate the exciting electromagnetic field. The consequences of the theorem for particular cases need further elaboration.

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