A time-domain energy theorem for scattering of plane acoustic waves in fluids

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A time-domain energy theorem for the scattering of plane acoustic waves in fluids by an obstacle of bounded extent is derived. It is the counterpart in the time domain of the "optical theorem" or the "extinction cross section theorem" in the frequency domain. No assumptions as to the acoustic behavior of the obstacle need to be made; so, the obstacle may be fluid or solid, acoustically nonlinear, and/or time variant (a kind of behavior that is excluded in the frequency-domain result). As to the wave motion, three different kinds of time behavior are distinguished: (a) transient, (b) periodic, and (c) perpetuating, but with finite mean power flow density. For all three cases the total energy [case (a)] or the time-averaged power [cases (b) and (c)] that is both absorbed and scattered by the obstacle is related to a certain time interaction integral of the incident plane-wave and the spherical-wave amplitude of the scattered wave in the farfield region, when observed in the direction of propagation of the incident wave.

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INTRODUCTION

In the theory of the scattering of acoustic waves in fluids by an obstacle of bounded extent there are several theorems that interrelate the different quantities associated with this scattering. In the frequency-domain analysis of the problem, it must be assumed that the scattering obstacle is linear and time invariant in its acoustic behavior. A time-domain analysis of the scattering problem reveals the more general conditions under which the relevant theorems may also hold in the time domain. In the present paper, the energy theorem for plane-wave scattering is investigated. Its frequency-domain counterpart is known as the "optical theorem" or "extinction cross-section theorem."[1–5] The time-domain derivation shows that the energy theorem holds for fluid or solid obstacles that may be nonlinear and/or time variant in their acoustic behavior. The theorem implies that the total amount of energy that is both absorbed and scattered by the obstacle can, in principle, be determined from a measurement at a single position in the farfield region, provided that the incident plane wave at the position of the obstacle is known from a separate measurement.

I. FORMULATION OF THE SCATTERING PROBLEM

In three-dimensional space $\mathbb{R}^3$ a scattering object is present. It occupies the bounded domain $\mathscr{D}$. The boundary surface of $\mathscr{D}$ is denoted by $\partial\mathscr{D}$ and the complement of the union of $\mathscr{D}$ and $\partial\mathscr{D}$ in $\mathbb{R}^3$ by $\mathscr{D}'$. The unit vector along the normal to $\partial\mathscr{D}$, pointing away from $\mathscr{D}$, is denoted by $\mathbf{n}$ (Fig. 1). It is assumed that $\partial\mathscr{D}$ is piecewise smooth. The acoustic properties of the scattering object remain unspecified; the object may be fluid or solid, and it may show a nonlinear and/or a time-variant behavior. The medium occupying the domain $\mathscr{D}'$ is acoustically characterized by a scalar, positive, constant volume density of mass ρ and a scalar, positive, constant compressibility κ. The speed of acoustic waves in this medium is $c = (\rho\kappa)^{-1/2}$.

Position in space is characterized by the position vector $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$, where x, y, and z are the Cartesian coordinates with respect to the orthogonal Cartesian reference frame with origin O and the three mutually perpendicular base vectors of unit length $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$. In the order indicated, the base vectors form a right-handed system. The time coordinate is denoted by t. Partial differentiation is denoted by ∂, and $\nabla = \mathbf{i}_x\partial_x + \mathbf{i}_y\partial_y + \mathbf{i}_z\partial_z$.

The acoustic field in the configuration is characterized by the acoustic pressure $p = p(\mathbf{r},t)$ and the particle velocity $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$. In $\mathscr{D}'$, where the medium is linear, the total acoustic field is written as the sum of the incident field $\{p^i,\mathbf{v}^i\}$ and the scattered field $\{p^s,\mathbf{v}^s\}$. Note that, in general, the scattered field is not linearly related to the incident field. The incident field is defined everywhere in $\mathbb{R}^3$; the scattered field is defined in $\mathscr{D}'$. The incident field and the scattered field satisfy the linearized, source-free acoustic field equations[6]

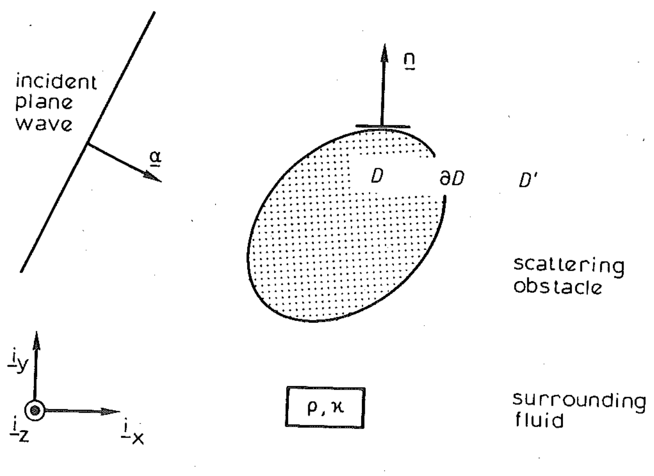

FIG. 1. Scattering configuration with incident plane wave. The speed of acoustic waves in the surrounding fluid is $c = (\rho\kappa)^{-1/2}$.

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$$\nabla p^{i,s} + \rho\,\partial_t \mathbf{v}^{i,s} = \mathbf{0}, \quad \text{when } \mathbf{r} \in \mathscr{D}', \tag{1}$$

$$\nabla\cdot\mathbf{v}^{i,s} + \kappa\,\partial_t p^{i,s} = 0, \quad \text{when } \mathbf{r} \in \mathscr{D}'. \tag{2}$$

At large distances from the scattering object the scattered field admits the representation

$$p^s(\mathbf{r},t) \sim A^p(\mathbf{i}_r,t - |\mathbf{r}|/c)/4\pi|\mathbf{r}|, \quad \text{as } |\mathbf{r}|\to\infty, \tag{3}$$

$$\mathbf{v}^s(\mathbf{r},t) \sim \mathbf{A}^v(\mathbf{i}_r,t - |\mathbf{r}|/c)/4\pi|\mathbf{r}|, \quad \text{as } |\mathbf{r}|\to\infty, \tag{4}$$

where $\mathbf{i}_r = \mathbf{r}/|\mathbf{r}|$ is the unit vector in the direction of observation. The right-hand sides of Eqs. (3) and (4) are the expressions for the field intensities in the farfield region. Between the acoustic-pressure and the particle-velocity farfield amplitude radiation characteristics A^p and $\mathbf{A}^v$ the following relations exist:

$$A^p = Z\mathbf{i}_r\cdot\mathbf{A}^v, \tag{5}$$

$$\mathbf{A}^v = Y A^p\,\mathbf{i}_r, \tag{6}$$

where $Z = (\rho/\kappa)^{1/2}$ is the acoustic plane-wave impedance and $Y = (\kappa/\rho)^{1/2}$ is the acoustic plane-wave admittance in the medium surrounding the obstacle. Note that the time argument of A^p and $\mathbf{A}^v$ is delayed by the travel time from the origin (which is located in the neighborhood of the obstacle) to the point of observation.

In the analysis we need further the instantaneous power flow P^i of the incident wave across $\partial\mathscr{D}$ and towards $\mathscr{D}$, i.e.,

$$P^i = -\int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n}\cdot(p^i\mathbf{v}^i)\,dA, \tag{7}$$

the instantaneous power flow P^s that the scattered wave carries away from $\partial\mathscr{D}$ towards $\mathscr{D}'$, i.e.,

$$P^s = \int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n}\cdot(p^s\mathbf{v}^s)\,dA, \tag{8}$$

and the instantaneous flow P^a of power that is absorbed by the obstacle, i.e.,

$$P^a = -\int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n}\cdot(p\mathbf{v})\,dA. \tag{9}$$

For the incident wave we now take the uniform plane wave propagating in the direction of the unit vector $\boldsymbol{\alpha}$:

$$\{p^i,\mathbf{v}^i\} = \{p^i(t - \boldsymbol{\alpha}\cdot\mathbf{r}/c),\mathbf{v}^i(t - \boldsymbol{\alpha}\cdot\mathbf{r}/c)\}. \tag{10}$$

Between p^i and $\mathbf{v}^i$ the following relations exist:

$$p^i = Z\boldsymbol{\alpha}\cdot\mathbf{v}^i, \tag{11}$$

$$\mathbf{v}^i = Y p^i\boldsymbol{\alpha}, \tag{12}$$

where Z and Y are the same as in Eqs. (5) and (6).

II. SURFACE-SOURCE REPRESENTATION OF THE SCATTERED FIELD

The basic tool in the derivation of the energy theorem is the time-domain surface-source representation of the scattered field. This representation is the acoustic analog of the Kirchhoff representation for scalar wave fields. Let

$$\mathbf{f}_S = p^s\mathbf{n}, \quad \text{when } \mathbf{r}\in\partial\mathscr{D}, \tag{13}$$

and

$$q_S = \mathbf{n}\cdot\mathbf{v}^s, \quad \text{when } \mathbf{r}\in\partial\mathscr{D}, \tag{14}$$

denote the scattered-field surface densities of force and cubic dilatation rate, respectively, and let

$$\Phi(\mathbf{r},t) = \int_{t_0}^{\infty} dt'\int_{\mathbf{r}'\in\partial\mathscr{D}} G(\mathbf{r}-\mathbf{r}',t-t')q_S(\mathbf{r}',t')dA \tag{15}$$

and

$$\Psi(\mathbf{r},t) = \int_{t_0}^{\infty} dt'\int_{\mathbf{r}'\in\partial\mathscr{D}} G(\mathbf{r}-\mathbf{r}',t-t')\mathbf{f}_S(\mathbf{r}',t')\,dA, \tag{16}$$

denote the corresponding scalar and vector potentials. In Eqs. (15) and (16)

$$G(\mathbf{r},t) = (4\pi|\mathbf{r}|)^{-1}\delta(t - |\mathbf{r}|/c) \tag{17}$$

denotes the free-space Green's function of the three-dimensional scalar wave equation. Then, the following integral relation for the scattered field holds:

$$\rho\partial_t\Phi - \nabla\cdot\Psi = \{1,\tfrac{1}{2},0\}p^s(\mathbf{r},t),$$

$$\text{when } \mathbf{r}\in\{\mathscr{D}',\partial\mathscr{D},\mathscr{D}\}, \quad t\in(t_0,\infty), \tag{18}$$

$$-\nabla\Phi + \kappa\partial_t\Psi + \rho^{-1}\nabla\times\left(\nabla\times\int_{t_0}^{t}\Psi\,dt'\right) = \{1,\tfrac{1}{2},0\}\mathbf{v}^s(\mathbf{r},t),$$

$$\text{when } \mathbf{r}\in\{\mathscr{D}',\partial\mathscr{D},\mathscr{D}\}, \quad t\in(t_0,\infty). \tag{19}$$

In Eqs. (18) and (19) we have taken into account the condition of causality, i.e., we have assumed that the scattered field vanishes everywhere in $\mathscr{D}'$ prior to t_0, where t_0 is the instant at which the incident wave hits the obstacle. A concise derivation of Eqs. (18) and (19) can be obtained with the aid of a Laplace transform with respect to time and a Fourier transform over $\mathscr{D}'$. From the derivation it follows that in the right-hand sides of Eqs. (13) and (14) the limiting values upon approaching $\partial\mathscr{D}$ via $\mathscr{D}'$ have to be taken.

By letting $|\mathbf{r}|\to\infty$ in Eqs. (15)–(19), we arrive at integral representations for the farfield amplitude radiation characteristics of the scattered wave. In the expression for $G(\mathbf{r}-\mathbf{r}',t-t')$ [cf. Eq. (17)] we employ the relation

$$|\mathbf{r}-\mathbf{r}'| = |\mathbf{r}| - \mathbf{i}_r\cdot\mathbf{r}' + \text{vanishing terms}, \quad \text{as } |\mathbf{r}|\to\infty. \tag{20}$$

The use of Eq. (20) in Eqs. (15) and (16) leads to

$$\{\Phi,\Psi\} \sim \{A^{\Phi},\mathbf{A}^{\Psi}\}(\mathbf{i}_r,t - |\mathbf{r}|/c)/4\pi|\mathbf{r}|, \quad \text{as } |\mathbf{r}|\to\infty, \tag{21}$$

where

$$A^{\Phi}(\mathbf{i}_r,t) = \int_{\mathbf{r}'\in\partial\mathscr{D}} q_S(\mathbf{r}',t + \mathbf{i}_r\cdot\mathbf{r}'/c)\,dA, \tag{22}$$

$$\mathbf{A}^{\Psi}(\mathbf{i}_r,t) = \int_{\mathbf{r}'\in\partial\mathscr{D}} \mathbf{f}_S(\mathbf{r}',t + \mathbf{i}_r\cdot\mathbf{r}'/c)\,dA. \tag{23}$$

The use of Eq. (21) in Eqs. (18) and (19) leads to the asymptotic expressions Eqs. (3) and (4) with

$$A^p = \rho\partial_t A^{\Phi} + c^{-1}\mathbf{i}_r\cdot\partial_t\mathbf{A}^{\Psi}, \tag{24}$$

$$\mathbf{A}^v = c^{-1}\mathbf{i}_r\partial_t A^{\Phi} + \kappa\mathbf{i}_r(\mathbf{i}_r\cdot\partial_t\mathbf{A}^{\Psi}). \tag{25}$$

It can easily be verified that the right-hand sides of Eqs. (24) and (25) satisfy Eqs. (5) and (6).

III. THE ENERGY THEOREM

The time-domain energy theorem takes on different shapes, depending on the type of time behavior of the acoustic field. Three cases are considered: (a) transient fields, (b) time-periodic fields, and (c) perpetuating fields with bounded mean value. The three cases will be dealt with separately.

A. Transient fields

Transient fields vanish prior to a certain instant and go to zero as $t \to \infty$, and these properties hold at any point in space. In our scattering problem the instant t_0 at which the incident wave hits the obstacle marks the onset of the scattering phenomenon. By applying Gauss' divergence theorem to the domain $\mathscr{D}$ and to the vector $p^i \mathbf{v}^i$, and using Eqs. (1) and (2), it follows that

$$\int_{t_0}^{\infty} P^i \, dt = 0, \tag{26}$$

where P^i is given by Eq. (7). This result expresses that the fluid with constitutive coefficients ρ and κ is lossless. Further, the total energy W^a that is absorbed by the obstacle is

$$W^a = \int_{t_0}^{\infty} P^a \, dt, \tag{27}$$

where P^a is given by Eq. (9), while the total energy W^s carried by the scattered wave is

$$W^s = \int_{t_0}^{\infty} P^s \, dt, \tag{28}$$

where P^s is given by Eq. (8). Let us consider now the expression for the sum of the absorbed energy and the scattered energy. With the aid of $p = p^i + p^s$ and $\mathbf{v} = \mathbf{v}^i + \mathbf{v}^s$ the relevant expression can be rewritten as

$$W^a + W^s = -\int_{t_0}^{\infty} dt \int_{\mathbf{r} \in \partial\mathscr{D}} \mathbf{n} \cdot (p^i \mathbf{v}^s + p^s \mathbf{v}^i) \, dA. \tag{29}$$

The substitution of Eq. (10) and the use of Eqs. (13) and (14) yield

$$W^a + W^s = -\int_{t_0}^{\infty} dt \int_{\mathbf{r} \in \partial\mathscr{D}} [\, p^i(t - \boldsymbol{\alpha} \cdot \mathbf{r}/c) \, q_S(\mathbf{r},t) + \mathbf{v}^i(t - \boldsymbol{\alpha} \cdot \mathbf{r}/c) \cdot \mathbf{f}_S(\mathbf{r},t)] \, dA. \tag{30}$$

We now introduce the instant t^i at which the incident wave reaches the origin of our chosen coordinate system. Then, we have $p^i(t) = 0$ and $\mathbf{v}^i(t) = \mathbf{0}$ when $-\infty < t < t^i$, and the right-hand side of Eq. (30) can, upon shifting the time integration, be rewritten as

$$W^a + W^s = -\int_{t^i}^{\infty} dt \int_{\mathbf{r} \in \partial\mathscr{D}} [\, p^i(t) \, q_S(\mathbf{r},t + \boldsymbol{\alpha} \cdot \mathbf{r}/c) + \mathbf{v}^i(t) \cdot \mathbf{f}_S(\mathbf{r},t + \boldsymbol{\alpha} \cdot \mathbf{r}/c)] \, dA. \tag{31}$$

Obviously, the relation between t_0 and t^i is given by

$$t_0 = t^i + \min_{\mathbf{r} \in \partial\mathscr{D}} (\boldsymbol{\alpha} \cdot \mathbf{r}/c). \tag{32}$$

After comparing the right-hand side of Eq. (31) with the expressions for $A^p(\boldsymbol{\alpha},t)$ and $\mathbf{A}^v(\boldsymbol{\alpha},t)$ that result from Eqs. (22)–(25), and taking into account the relations Eqs. (11) and (12), we arrive at

$$W^a + W^s = -\rho^{-1} \int_{t^i}^{\infty} p^i(t) \left(\int_{t^i}^{t} A^p(\boldsymbol{\alpha},t') \, dt' \right) dt \tag{33}$$

and

$$W^a + W^s = -\kappa^{-1} \int_{t^i}^{\infty} \mathbf{v}^i(t) \cdot \left(\int_{t^i}^{t} \mathbf{A}^v(\boldsymbol{\alpha},t') \, dt' \right) dt. \tag{34}$$

Equations (33) and (34) constitute the energy theorem for transient plane-wave scattering. In the right-hand sides the scattered-field spherical-wave amplitudes in the farfield region occur in the direction of observation $\boldsymbol{\alpha}$, i.e., in the direction of propagation of the incident wave.

B. Time-periodic fields

For time-periodic fields, with period T, we introduce the time-averaged values, over a period, of the different power flows. Let $\langle \cdots \rangle_T$ denote the time average over a period, i.e.,

$$\langle \cdots \rangle_T = T^{-1} \int_{t_0}^{t_0 + T} \cdots \, dt. \tag{35}$$

Then, the sum of the (time averaged) absorbed power and the (time averaged) scattered power can be written as [cf. Eq. (29)]

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\left\langle \int_{\mathbf{r} \in \partial\mathscr{D}} \mathbf{n} \cdot (p^i \mathbf{v}^s + p^s \mathbf{v}^i) \, dA \right\rangle_T. \tag{36}$$

The substitution of Eq. (10) and the use of Eqs. (13) and (14), followed by an interchange of the time integration with the one over $\partial\mathscr{D}$ and a shift of the variable in the resulting time integration lead to

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\int_{\mathbf{r} \in \partial\mathscr{D}} \langle p^i(t) \, q_S(\mathbf{r},t + \boldsymbol{\alpha} \cdot \mathbf{r}/c) + \mathbf{v}^i(t) \cdot \mathbf{f}_S(\mathbf{r},t + \boldsymbol{\alpha} \cdot \mathbf{r}/c) \rangle_T \, dA. \tag{37}$$

From this we arrive at

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\rho^{-1} \left\langle p^i(t) \int_{t_0}^{t} A^p(\boldsymbol{\alpha},t') dt' \right\rangle_T \tag{38}$$

and

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\kappa^{-1} \left\langle \mathbf{v}^i(t) \cdot \int_{t_0}^{t} \mathbf{A}^v(\boldsymbol{\alpha},t') \, dt' \right\rangle_T. \tag{39}$$

Equations (38) and (39) constitute the energy theorem for time-periodic plane-wave scattering. Note that in the time integration of the scattered-field spherical-wave amplitudes the properties

$$\langle A^p(\boldsymbol{\alpha},t') \rangle_T = 0 \tag{40}$$

and

$$\langle \mathbf{A}^v(\boldsymbol{\alpha},t') \rangle_T = \mathbf{0} \tag{41}$$

hold in view of Eqs. (24) and (25).

Obviously, it has been assumed here that the incident field and the scattered field are both time periodic with the same period T. Now, with regard to the scattering object this implies that a possible time-varying behavior has to comply with this assumption, i.e., the acoustic properties of the scattering object must at most be time periodic with the same period T, too.

C. Perpetuating fields

For perpetuating fields we assume that the time-averaged values of the different power flow densities exist. Let $\langle \cdots \rangle_\infty$ denote the relevant time averages, i.e.,

$$\langle \cdots \rangle_\infty = \lim_{T_1 \to -\infty, T_2 \to \infty} (T_2 - T_1)^{-1} \int_{T_1}^{T_2} \cdots dt. \quad (42)$$

In accordance with this, the fields are assumed to have bounded values as $t \to -\infty$ and as $t \to \infty$. Then, the sum of the time-averaged absorbed power and the time-averaged scattered power can be written as

$$\langle P^a \rangle_\infty + \langle P^s \rangle_\infty = -\left\langle \int_{\mathbf{r} \in \partial \mathscr{D}} \mathbf{n} \cdot (p^i \mathbf{v}^s + p^s \mathbf{v}^i)\, dA \right\rangle_\infty . \quad (43)$$

The substitution of Eq. (10) and the use of Eqs. (13) and (14), followed by an interchange of the time integration with the one over $\partial\mathscr{D}$ and a shift of the variable in the resulting time integration lead to

$$\langle P^a \rangle_\infty + \langle P^s \rangle_\infty = -\int_{\mathbf{r} \in \partial \mathscr{D}} \langle p^i(t)\, q_S(\mathbf{r}, t + \boldsymbol{\alpha} \cdot \mathbf{r}/c) + \mathbf{v}^i(t) \cdot \mathbf{f}_S(\mathbf{r}, t + \boldsymbol{\alpha} \cdot \mathbf{r}/c) \rangle_\infty\, dA. \quad (44)$$

From this we arrive at

$$\langle P^a \rangle_\infty + \langle P^s \rangle_\infty = -\rho^{-1} \left\langle p^i(t) \int_{-\infty}^{t} A^p(\boldsymbol{\alpha}, t')\, dt' \right\rangle_\infty \quad (45)$$

and

$$\langle P^a \rangle_\infty + \langle P^s \rangle_\infty = -\kappa^{-1} \left\langle \mathbf{v}^i(t) \cdot \int_{-\infty}^{t} \mathbf{A}^v(\boldsymbol{\alpha}, t')\, dt' \right\rangle_\infty . \quad (46)$$

Equations (45) and (46) constitute the energy theorem for the scattering of perpetuating plane waves. Note that in the time integration of the scattered-field spherical-wave amplitudes the properties

$$\langle A^p(\boldsymbol{\alpha}, t') \rangle_\infty = 0 \quad (47)$$

and

$$\langle \mathbf{A}^v(\boldsymbol{\alpha}, t') \rangle_\infty = \mathbf{0} \quad (48)$$

hold. In comparison with the case of time periodic fields no restrictions are, in this case, laid upon the possible time behavior of the acoustic properties of the scattering object.

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