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A TIME-DOMAIN ENERGY THEOREM FOR THE SCATTERING OF PLANE ELASTIC WAVES

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A time-domain energy theorem for the scattering of plane elastic waves by an obstacle of bounded extent is derived. The obstacle is embedded in a homogeneous, isotropic, perfectly elastic medium. As to the elastodynamic behavior of the obstacle no assumptions have to be made; so, lossy, non-linear and time-variant behavior is included. As to the wave motion, three different kinds of time behavior are distinguished: (a) transient, (b) periodic, and (c) perpetuating, but with finite mean power flow density. For these cases, the total energy (case (a)) or the time-averaged power (cases (b) and (c)) that is both absorbed and scattered by the obstacle is related to a certain time interaction integral of the incident plane wave (P or S) and the spherical-wave amplitude of the scattered wave of the same type (P or S) in the far-field region, when observed in the direction of propagation of the incident wave.

1. Introduction

In the theory of the scattering of elastic waves by an obstacle of bounded extent embedded in a homogeneous, isotropic, perfectly elastic medium, there are several theorems that interrelate the different quantities associated with this scattering. In the frequency-domain analysis of the scattering problem, it must be assumed that the scattering obstacle is linear and time invariant in its elastodynamic behavior. A time-domain analysis may reveal whether the conditions under which the corresponding theorems in the time domain hold, are more general. For the energy theorem in plane-wave scattering this proves to be the case. Its frequency-domain counterpart has been discussed by Tan [1]. Our derivation of the corresponding theorem in the time domain shows that it holds for obstacles that are non-linear and/or time variant in their elastodynamic behavior. The theorem implies that the total amount of energy (or time-averaged power) that is both absorbed and scattered by the obstacle can, in principle, be determined from a measurement at a single position in the far-field region, provided that the incident plane wave is known from a separate measurement. In the computational modeling of elastodynamic scattering problems the theorem can serve as a check on the consistency of the computations.

2. Formulation of the scattering problem

In three-dimensional space $\mathcal{R}^3$ a scattering object and an embedding homogeneous, isotropic, perfectly elastic solid are present. The scattering object occupies the bounded domain $\mathcal{D}$. The boundary surface of $\mathcal{D}$ is denoted by $\partial\mathcal{D}$ and the complement of the union of $\mathcal{D}$ and $\partial\mathcal{D}$ in $\mathcal{R}^3$ by $\mathcal{D}'$. Position in space is characterized by the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal Cartesian reference frame with

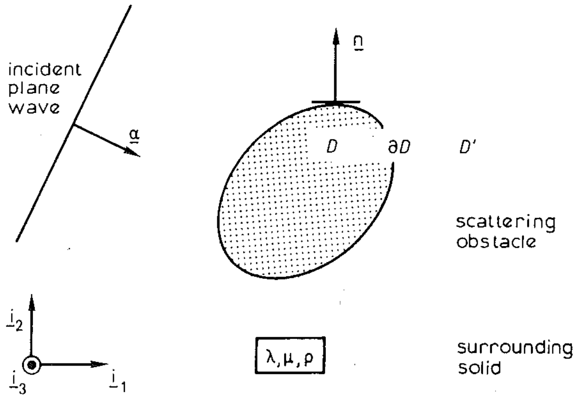

Fig. 1. Scattering configuration with incident plane wave. In the surrounding medium the speed of elastodynamic P-waves is $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$ and the speed of elastodynamic S-waves is $c_S = (\mu/\rho)^{1/2}$.

origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_s\}$ of unit length each. In the order indicated, the base vectors form a right-handed system. The time coordinate is denoted by t. Partial differentiation is denoted by ∂. The subscript notation for vectors and tensors is used and the summation convention applies. Occasionally, vectors are denoted by boldface symbols; for example, $\boldsymbol{x} = x_k \boldsymbol{i}_k$ denotes the position vector. The medium occupying the domain $\mathcal{D}'$ is mechanically characterized by its volume density of mass ρ and its Lamé coefficients λ and μ. The speed of P-waves (compressional waves) in this medium is $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$; the speed of S-waves (shear-waves) in this medium is $c_S = (\mu/\rho)^{1/2}$ (Fig. 1).

The elastodynamic field in the configuration is characterized by the stress $\tau_{ij} = \tau_{ij}(\boldsymbol{x}, t)$ and the particle velocity $v_i = v_i(\boldsymbol{x}, t)$. The stress is a symmetrical tensor, i.e. $\tau_{ij} = \tau_{ji}$, and occasionally it is advantageous to use the relationship $\tau_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ij})$. In $\mathcal{D}'$, where the medium is linear, the total field is written as the sum of the incident field $\{\tau^{i}_{ij}, v^{i}_{i}\}$ and the scattered field $\{\tau^{s}_{ij}, v^{s}_{i}\}$. Note that, in general, the scattered field is not linearly related to the incident field. The incident field is defined in $\mathcal{R}^3$ and satisfies in $\mathcal{D}$ the source-free elastodynamic field equations

$$\partial_i \tau^{i}_{ij} - \rho \partial_t v^{i}_{j} = 0 \quad \text{when } \boldsymbol{x} \in \mathcal{D}, \tag{1}$$

$$\tfrac{1}{2} C_{ijpq}(\partial_p v^{i}_{q} + \partial_q v^{i}_{p}) - \partial_t \tau^{i}_{ij} = 0 \quad \text{when } \boldsymbol{x} \in \mathcal{D}, \tag{2}$$

where

$$C_{ijpq} = \lambda\, \delta_{ij}\, \delta_{pq} + \mu(\delta_{ip}\, \delta_{jq} + \delta_{jp}\, \delta_{iq}). \tag{3}$$

The scattered field is defined in $\mathcal{D}'$ and satisfies in this domain the source-free elastodynamic field equations

$$\partial_i \tau^{s}_{ij} - \rho\, \partial_t v^{s}_{j} = 0 \quad \text{when } \boldsymbol{x} \in \mathcal{D}', \tag{4}$$

$$C_{ijpq}\, \partial_p v^{s}_{q} - \partial_t \tau^{s}_{ij} = 0 \quad \text{when } \boldsymbol{x} \in \mathcal{D}'. \tag{5}$$

At large distances from te scattering object the scattered field admits the representation

$$\{\tau^{s}_{ij}, v^{s}_{j}\} \sim \{TP^{s}_{ij}, VP^{s}_{j}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}|/c_P)/4\pi c_P^2 |\boldsymbol{x}| + \{TS^{s}_{ij}, VS^{s}_{j}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}|/c_S) 4\pi c_S^2 |\boldsymbol{x}| \quad \text{as } |\boldsymbol{x}| \to \infty, \tag{6}$$

where $\boldsymbol{\xi} = \boldsymbol{x}/|\boldsymbol{x}|$ is the unit vector in the direction of observation. The right-hand side of (6) is the expression for the scattered-field values in the far-field region. In this region the field separates into a P-wave part for which the property

$$VP^{s}_{j} = (\xi_p VP^{s}_{p})\xi_j \tag{7}$$

holds, and an S-wave part for which the property

$$\xi_p VS^{s}_{p} = 0 \tag{8}$$

holds. Between the stress and the particle-velocity far-field radiation characteristics the following relations exist:

$$TP^{s}_{ij} = -ZP_{ijq} VP^{s}_{q}, \tag{9}$$

$$TS^{s}_{ij} = -ZS_{ijq} VS^{S}_{q}, \tag{10}$$

where $ZP_{ijq} = C_{ijpq}\xi_p / c_P$ is the plane P-wave impedance and $ZS_{ijq} = C_{ijpq}\xi_p / c_S$ is the plane S-wave impedance of the medium surrounding the scattering object. Note that in the right-hand side of (6) the time argument has been delayed by the travel time for P- and S-waves, respectively, from the origin (which is located in the neighborhood of the obstacle) to the point of observation.

In the analysis we further need the instantaneous power flow P^{i} of the incident wave across $\partial\mathscr{D}$ towards $\mathscr{D}$, i.e.

$$P^{i} = \int_{x \in \partial\mathscr{D}} \nu_i \tau^{i}_{ij} v^{i}_{j} \, dA, \tag{11}$$

where ν_i is the unit vector along the normal to $\partial\mathscr{D}$, pointing away from $\mathscr{D}$, the instantaneous power flow P^{s} that the scattered wave carries away from $\partial\mathscr{D}$, towards $\mathscr{D}'$, i.e.

$$P^{s} = -\int_{x \in \partial\mathscr{D}} \nu_i \tau^{s}_{ij} v^{s}_{j} \, dA, \tag{12}$$

and the instantaneous flow P^{a} of power that is absorbed by the obstacle, i.e.

$$P^{a} = \int_{x \in \partial\mathscr{D}} \nu_i \tau_{ij} v_j \, dA. \tag{13}$$

For the incident field we first take the uniform plane P-wave propagating in the direction of the unit vector α_i:

$$\{\tau^{i}_{ij}, v^{i}_{j}\} = \{TP^{i}_{ij}, VP^{i}_{j}\}(t - \alpha_p x_p / c_P). \tag{14}$$

Between TP^{i}_{ij} and VP^{i}_{j} the following relation exists:

$$TP^{i}_{ij} = -ZP_{ijq} VP^{i}_{q}, \tag{15}$$

where ZP_{ijq} is the same as in (9). Further,

$$VP^{i}_{i} = (\alpha_p VP^{i}_{p})\alpha_i. \tag{16}$$

Equation (16) implies that the P-wave is longitudinal. Secondly, we shall consider the case of an incident S-wave propagating in the direction of the unit vector α_i:

$$\{\tau^{i}_{ij}, v^{i}_{j}\} = \{TS^{i}_{ij}, VS^{i}_{j}\}(t - \alpha_p x_p / c_S). \tag{17}$$

Between TS^{i}_{ij} and VS^{i}_{j} the following relation exists:

$$TS^{\mathrm{i}}_{ij} = -ZS_{ijq} VS^{\mathrm{i}}_{q}, \tag{18}$$

where ZS_{ijq} is the same as in (10). Further,

$$\alpha_p VS^{\mathrm{i}}_p = 0. \tag{19}$$

Equation (19) implies that the S-wave is transverse.

3. Surface-source representation of the scattered field

The basic tool in the derivation of the energy theorem is the time-domain surface source representation of the scattered wave field. For simplicity, we shall only use the surface-source representation of the particle velocity; the relevant expression for the stress is more complicated [2, p. 30]. Let

$$f^{\partial}_j = -\nu_i \tau^{\mathrm{s}}_{ij} \quad \text{when } \boldsymbol{x} \in \partial\mathscr{D} \tag{20}$$

and

$$h^{\partial}_{ij} = -\tfrac{1}{2}(\nu_i v^{\mathrm{s}}_j + \nu_j v^{\mathrm{s}}_i) \quad \text{when } \boldsymbol{x} \in \partial\mathscr{D} \tag{21}$$

denote the scattered-field surface densities of force and strain rate, respectively, and let

$$A_i(\boldsymbol{x}, t) = \int_{t_0}^{\infty} \mathrm{d}t' \int_{\boldsymbol{x}' \in \partial\mathscr{D}} G_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t') f^{\partial}_j(\boldsymbol{x}', t')\, \mathrm{d}A \tag{22}$$

and

$$\Psi_{ijpq}(\boldsymbol{x}, t) = \int_{t_0}^{\infty} \mathrm{d}t' \int_{\boldsymbol{x}' \in \partial\mathscr{D}} G_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t') h^{\partial}_{pq}(\boldsymbol{x}', t')\, \mathrm{d}A \tag{23}$$

denote the corresponding vector and tensor potentials. In (22) and (23),

$$G_{ij} = G_{\mathrm{S}}(\boldsymbol{x}, t)\delta_{ij} + \partial_i \partial_j \int_0^{\infty} [c_{\mathrm{P}}^2 G_{\mathrm{P}}(\boldsymbol{x}, t - t'') - c_{\mathrm{S}}^2 G_{\mathrm{S}}(\boldsymbol{x}, t - t'')] t''\, \mathrm{d}t'', \tag{24}$$

with

$$G_{\mathrm{P,S}} = (4\pi c_{\mathrm{P,S}}^2 |\boldsymbol{x}|)^{-1}\, \delta(t - |\boldsymbol{x}|/c_{\mathrm{P,S}}), \tag{25}$$

denotes the infinite medium Green's function of the three-dimensional particle velocity wave equation

$$(c_{\mathrm{P}}^2 - c_{\mathrm{S}}^2)\, \partial_i\, \partial_k\, G_{kj} + c_{\mathrm{S}}^2\, \partial_k\, \partial_k\, G_{ij} - \partial_t^2\, G_{ij} = -\delta_{ij}\, \delta(\boldsymbol{x}, t), \tag{26}$$

where δ_{ij} denotes the three-dimensional symmetrical unit tensor and $\delta(\boldsymbol{x}, t)$ denotes the four-dimensional unit pulse. Then, the following integral relation for the particle velocity of the scattered field holds [2, p. 29; 3]

$$-\rho^{-1}\, \partial_t\, A_j - \rho^{-1} C_{ikpq}\, \partial_i\, \Psi_{jkpq} = \{1, \tfrac{1}{2}, 0\} v^{\mathrm{s}}_j(\boldsymbol{x}, t), \quad \text{when } \boldsymbol{x} \in \{\mathscr{D}', \partial\mathscr{D}, \mathscr{D}\} \text{ and } t \in \{t_0, \infty\}. \tag{27}$$

In (22)–(27) we have taken into account the condition of causality, i.e. we have assumed that the scattered wave field vanishes everywhere in $\mathscr{D}'$ prior to t_0, where t_0 is the instant at which the incident wave hits the scattering object. A concise derivation of (27) can be obtained with the aid of a Laplace transform with respect to time and a Fourier transform over $\mathscr{D}'$. From the derivation it follows that in the right-hand sides of (22) and (23) the limiting values upon approaching $\partial\mathscr{D}$ via $\mathscr{D}'$ have to be taken.

By letting $|\boldsymbol{x}|\to\infty$ in (22), (23) and (27), we arrive at integral representations for the far-field amplitude radiation characteristic for the particle velocity of the scattered wave. In the expression for $G_{ij}(\boldsymbol{x}-\boldsymbol{x}', t-t')$ that results from (24) we employ the relation

$$|\boldsymbol{x}-\boldsymbol{x}'| = |\boldsymbol{x}| - \xi_k x'_k + \text{vanishing terms} \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{28}$$

and obtain

$$G_{ij}(\boldsymbol{x}-\boldsymbol{x}', t-t') \sim (\delta_{ij}-\xi_i\xi_j)\frac{\delta(t-t'-|\boldsymbol{x}|/c_S+\xi_k x'_k/c_S)}{4\pi c_S^2|\boldsymbol{x}|} + \xi_i\xi_j\frac{\delta(t-t'-|\boldsymbol{x}|/c_P+\xi_k x'_k/c_P)}{4\pi c_P^2|\boldsymbol{x}|} \quad \text{as } |\boldsymbol{x}|\to\infty. \tag{29}$$

Using (29) in (22) and (23) we arrive at

$$A_i(\boldsymbol{x}, t) \sim AP_i(\boldsymbol{\xi}, t-|\boldsymbol{x}|/c_P)/4\pi c_P^2|\boldsymbol{x}| + AS_i(\boldsymbol{\xi}, t-|\boldsymbol{x}|/c_S)/4\pi c_S^2|\boldsymbol{x}| \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{30}$$

where the P-wave contribution is given by

$$AP_i(\boldsymbol{\xi}, t) = \xi_i\xi_j \int_{\boldsymbol{x}'\in\partial\mathscr{D}} f_j^{\partial}(\boldsymbol{x}', t+\xi_k x'_k/c_P)\,\mathrm{d}A, \tag{31}$$

and the S-wave contribution by

$$AS_i(\boldsymbol{\xi}, t) = (\delta_{ij}-\xi_i\xi_j) \int_{\boldsymbol{x}'\in\partial\mathscr{D}} f_j^{\partial}(\boldsymbol{x}', t+\xi_k x'_k/c_S)\,\mathrm{d}A, \tag{32}$$

and

$$\Psi_{ijpq}(\boldsymbol{x}, t) \sim \Psi P_{ijpq}(\boldsymbol{\xi}, t-|\boldsymbol{x}|/c_P)/4\pi c_P^2|\boldsymbol{x}| + \Psi S_{ijpq}(\boldsymbol{\xi}, t-|\boldsymbol{x}|/c_S)/4\pi c_S^2|\boldsymbol{x}| \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{33}$$

where the P-wave contribution is given by

$$\Psi P_{ijpq}(\boldsymbol{\xi}, t) = \xi_i\xi_j \int_{\boldsymbol{x}'\in\partial\mathscr{D}} h_{pq}^{\partial}(\boldsymbol{x}', t+\xi_k x'_k/c_P)\,\mathrm{d}A, \tag{34}$$

and the S-wave contribution by

$$\Psi S_{ijpq}(\boldsymbol{\xi}, t) = (\delta_{ij}-\xi_i\xi_j) \int_{\boldsymbol{x}'\in\partial\mathscr{D}} h_{pq}^{\partial}(\boldsymbol{x}', t+\xi_k x'_k/c_S)\,\mathrm{d}A. \tag{35}$$

The use of (30) and (33) in (27) leads to the asymptotic expression (6) with

$$VP_j^{\mathrm{s}} = -\rho^{-1}\,\partial_t AP_j + \rho^{-1} C_{ikpq}(\xi_i/c_P)\,\partial_t \Psi P_{jkpq}, \tag{36}$$

and

$$VS_j^{\mathrm{s}} = -\rho^{-1}\,\partial_t AS_j + \rho^{-1} C_{ikpq}(\xi_i/c_S)\,\partial_t \Psi S_{jkpq}. \tag{37}$$

It can easily be verified that VP_j^{S} has the same direction as ξ_j, and that VS_j^{s} is perpendicular to ξ_j.

4. The energy theorem

The time-domain energy theorem takes on somewhat different shapes, depending on the time behavior of the elastodynamic wave field. Three cases are considered: (a) transient fields, (b) periodic fields, and (c) perpetuating fields with bounded mean power flows. The three cases will be discussed separately.

1. Transient fields

Transient fields vanish prior to a certain instant and go to zero as $t\to\infty$, and these properties hold at any point in space. In our scattering problem the instant t_0 at wich the incident wave hits the obstacle marks the onset of the scattering phenomenon. By applying Gauss' divergence theorem to the domain $\mathscr{D}$ and to the vector $\frac{1}{2}(\tau^{\mathrm{i}}_{ij}+\tau^{\mathrm{i}}_{ji})v^{\mathrm{i}}_j$, and using (1) and (2), it follows that

$$\int_{t_0}^{\infty} P^{\mathrm{i}}\,\mathrm{d}t=0, \tag{38}$$

where P^{i} is given by (11). This result expresses that the medium with volume density of mass ρ and stiffness C_{ijpq} is lossless. Further, the total energy that is absorbed by the obstacle is

$$W^{\mathrm{a}}=\int_{t_0}^{\infty} P^{\mathrm{a}}\,\mathrm{d}t, \tag{39}$$

where P^{a} is given by (13), while the total energy W^{s} that is carried by the scatterted wave is

$$W^{\mathrm{s}}=\int_{t_0}^{\infty} P^{\mathrm{s}}\,\mathrm{d}t, \tag{40}$$

where P^{s} is given by (12). Let us consider now the expression for the sum of the absorbed energy and the scattered energy. With the aid of $\tau_{ij}=\tau^{\mathrm{i}}_{ij}+\tau^{\mathrm{s}}_{ij}$ and $v_j=v^{\mathrm{i}}_j+v^{\mathrm{s}}_j$ the relevant expression can be written as

$$W^{\mathrm{a}}+W^{\mathrm{s}}=\int_{t_0}^{\infty}\mathrm{d}t\int_{x\in\partial\mathscr{D}} \nu_i(\tau^{\mathrm{i}}_{ij}v^{\mathrm{s}}_j+\tau^{\mathrm{s}}_{ij}v^{\mathrm{i}}_j)\,\mathrm{d}A, \tag{41}$$

where, again, it is understood that τ^{i}_{ij} and τ^{s}_{ij} are symmetrical tensors.

For an *incident P-wave* we substitute in (41) the expressions (14) and use (20) and (21). The result is

$$W^{\mathrm{a}}+W^{\mathrm{s}}=-\int_{t_0}^{\infty}\mathrm{d}t\int_{x\in\partial\mathscr{D}}[TP^{\mathrm{i}}_{ij}(t-\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})h^{\partial}_{ij}(x,t)+VP^{\mathrm{i}}_j(t-\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})f^{\partial}_j(x,t)]\,\mathrm{d}A. \tag{42}$$

We now intoduce the instant tP^{i} at which the incident P-wave reaches the origin of our chosen coordinate system. Then, we have $TP^{\mathrm{i}}_{ij}(t)=0$ and $VP^{\mathrm{i}}(t)=0$ when $-\infty<t<tP^{\mathrm{i}}$, and the right-hand side of (42) can, upon shifting the time integration, be rewritten as

$$W^{\mathrm{a}}+W^{\mathrm{s}}=\int_{tP^{\mathrm{i}}}^{\infty}\mathrm{d}t\int_{x\in\partial\mathscr{D}}[TP^{\mathrm{i}}_{ij}(t)h^{\partial}_{ij}(x,t+\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})+VP^{\mathrm{i}}_j(t)f^{\partial}_j(x,t+\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})]\,\mathrm{d}A. \tag{43}$$

Obviously, the relation between t_0 and tP^{i} is given by

$$t_0=tP^{\mathrm{i}}+\min_{x\in\partial\mathscr{D}}(\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}}). \tag{44}$$

After comparing the integral over $\partial\mathscr{D}$ in (43) with the expression for VP^{s}_j that results from (36) with

$\xi_i = \alpha_i$, and taking into account that (16) holds, we arrive at

$$W^{\mathrm{a}} + W^{\mathrm{s}} = -\rho \int_{tP^{\mathrm{i}}}^{\infty} VP_j^{\mathrm{i}}(t) \left[\int_{tP^{\mathrm{i}}}^{t} VP_j^{\mathrm{s}}(\boldsymbol{\alpha}, t')\,\mathrm{d}t' \right] \mathrm{d}t \quad \text{(incident P-wave)}. \tag{45}$$

Equation (45) constitutes the energy theorem for transient plane P-wave scattering. In the right-hand side the P-wave scattered-field spherical-wave amplitude in the far-field region occurs in the direction of observation $\boldsymbol{\alpha}$, i.e. in the direction of propagation of the incident P-wave.

For an *incident S-wave* we substitute in (41) the expression (17). Next, we introduce the instant tS^{i} at which the incident S-wave reaches the origin and again shift the time integration. After using in the resulting right-hand side the expression for VS_j^{s} that results from (37) with $\xi_i = \alpha_i$ and taking into account that (19) holds, we arrive at

$$W^{\mathrm{a}} + W^{\mathrm{s}} = -\rho \int_{tS^{\mathrm{i}}}^{\infty} VS_j^{\mathrm{i}}(t) \left[\int_{tS^{\mathrm{i}}}^{t} VS_j^{\mathrm{s}}(\boldsymbol{\alpha}, t')\,\mathrm{d}t' \right] \mathrm{d}t \quad \text{(incident S-wave)}. \tag{46}$$

Equation (46) constitutes the energy theorem for an incident S-wave.

2. *Time-periodic fields*

For time-periodic fields, with period T, we introduce the time-averaged values, over a period, of the different acoustic power flows. Let $\langle \cdots \rangle_T$ denote the time average over a period, i.e.

$$\langle \cdots \rangle_T = T^{-1} \int_{t_0}^{t_0+T} \cdots \,\mathrm{d}t. \tag{47}$$

Then the counterparts of (38), (39) and (40) are

$$\langle P^{\mathrm{i}} \rangle_T = 0, \tag{48}$$

$$\langle P^{\mathrm{a}} \rangle_T = T^{-1} \int_{t_0}^{t_0+T} P^{\mathrm{a}}\,\mathrm{d}t, \tag{49}$$

and

$$\langle P^{\mathrm{s}} \rangle_T = T^{-1} \int_{t_0}^{t_0+T} P^{\mathrm{s}}\,\mathrm{d}t, \tag{50}$$

respectively. We now consider the expression for the sum of the time-averaged absorbed power and the time-averaged scattered power. This expression can be written as (cf. (41))

$$\langle P^{\mathrm{a}} \rangle_T + \langle P^{\mathrm{s}} \rangle_T = \left\langle \iint_{x \in \partial\mathcal{D}} \nu_i (\tau_{ij}^{\mathrm{i}} v_j^{\mathrm{s}} + \tau_{ij}^{\mathrm{s}} v_j^{\mathrm{i}})\,\mathrm{d}A \right\rangle_T. \tag{51}$$

For an *incident P-wave* we substitute in (51) the expressions (14) and use (20) and (21). The result is

$$\langle P^{\mathrm{a}} \rangle_T + \langle P^{\mathrm{s}} \rangle_T = -\left\langle \iint_{x \in \partial\mathcal{D}} [TP_{ij}^{\mathrm{i}}(t - \alpha_{\mathrm{P}} x_{\mathrm{P}}/c_{\mathrm{P}}) h_{ij}^{\partial}(x, t) + VP_j^{\mathrm{i}}(t - \alpha_{\mathrm{P}} x_{\mathrm{P}}/c_{\mathrm{P}}) f_j^{\partial}(x, t)]\,\mathrm{d}A \right\rangle_T. \tag{52}$$

After interchanging the time integration with the one over $\partial\mathcal{D}$ and shifting the variable in the resulting time integration, we obtain

$$\langle P^{\mathrm{a}} \rangle_T + \langle P^{\mathrm{s}} \rangle_T = -\int_{x \in \partial\mathcal{D}} \langle TP_{ij}^{\mathrm{i}}(t) h_{ij}^{\partial}(x, t + \alpha_{\mathrm{P}} x_{\mathrm{P}}/c_{\mathrm{P}}) + VP_j^{\mathrm{i}}(t) f_j^{\partial}(x, t + \alpha_{\mathrm{P}} x_{\mathrm{P}}/c_{\mathrm{P}}) \rangle_T\,\mathrm{d}A. \tag{53}$$

After comparing the integral over $\partial\mathscr{D}$ in (53) with the expression for VP_j^s that results from (36) with $\xi_i = \alpha_i$ and taking into account that (16) holds, we arrive at

$$\langle P^a\rangle_T + \langle P^s\rangle_T = -\rho\left\langle VP_j^i(t)\int_{t_0}^{t} VP_j^s(\boldsymbol{\alpha}, t')\,\mathrm{d}t'\right\rangle_T \quad \text{(incident P-wave).} \tag{54}$$

For an *incident S-wave* we arrive along similar lines at

$$\langle P^a\rangle_T + \langle P^s\rangle_T = -\rho\left\langle VS_j^i(t)\int_{t_0}^{t} VS_j^s(\boldsymbol{\alpha}, t')\,\mathrm{d}t'\right\rangle_T \quad \text{(incident S-wave).} \tag{55}$$

Equations (54) and (55) constitute the energy theorems for time-periodic wave scattering. Note that in the time integration of the scattered-field spherical-wave amplitudes the properties

$$\langle VP_j^s\rangle_T = 0 \tag{56}$$

and

$$\langle VS_j^s\rangle_T = 0 \tag{57}$$

hold in view of (36) and (37), respectively.

Obviously, it has been assumed here that the incident waves and the scattered waves are both time periodic with the same period T. Now, with regard to the scattering object this implies that a possible time-varying behavior has to comply with this assumption, i.e. the elastodynamic properties of the scattering object must at most be time periodic with the same period T, too.

3. *Perpetuating fields*

For perpetuating fields we assume that the time-averaged values of the different power flow densities exist. Let $\langle\cdots\rangle_\infty$ denote the relevant time averages. Then

$$\langle\cdots\rangle_\infty = \lim_{T_1\to-\infty, T_2\to\infty} (T_2 - T_1)^{-1}\int_{T_1}^{T_2} \cdots\,\mathrm{d}t. \tag{58}$$

In accordance with this, the fields are assumed to have bounded values as $t\to-\infty$ and as $t\to\infty$. Then, with (1) and (2) it follows that

$$\langle P^i\rangle_\infty = 0. \tag{59}$$

As in the case of transient fields we consider the expression for the sum of the time-averaged absorbed power and the time-averaged scattered power. This can be written as (cf. (41))

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = \left\langle\int_{\boldsymbol{x}\in\partial\mathscr{D}} \nu_i(\tau_{ij}^i v_j^s + \tau_{ij}^s v_j^i)\,\mathrm{d}A\right\rangle_\infty. \tag{60}$$

For an *incident P-wave* we substitute in (60) the expressions (14) and use (20) and (21). The result is

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = -\left\langle\int_{\boldsymbol{x}\in\partial\mathscr{D}} [TP_{ij}^i(t-\alpha_P x_P/c_P)h_{ij}^{\partial}(\boldsymbol{x}, t) + VP_j^i(t-\alpha_P x_P/c_P)f_j^{\partial}(\boldsymbol{x}, t)]\,\mathrm{d}A\right\rangle_\infty. \tag{61}$$

After interchanging the time integration with the one over $\partial\mathscr{D}$ and shifting the variable in the resulting time integration, we obtain

$$\langle P^{\mathrm{a}}\rangle_{\infty}+\langle P^{\mathrm{s}}\rangle_{\infty}=-\int_{x\in\partial\mathscr{D}}\langle TP^{\mathrm{i}}_{ij}(t)h^{\partial}_{ij}(x,t+\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})+VP^{\mathrm{i}}_{j}(t)f^{\partial}_{j}(x,t+\alpha_{\mathrm{P}}x_{\mathrm{P}}/c_{\mathrm{P}})\rangle_{\infty}\,\mathrm{d}A. \tag{62}$$

After comparing the integral over $\partial\mathscr{D}$ in (62) with the expression for VP^{s}_{j} that results from (36) with $\xi_i=\alpha_i$, and taking into account that (16) holds, we arrive at

$$\langle P^{\mathrm{a}}\rangle_{\infty}+\langle P^{\mathrm{s}}\rangle_{\infty}=-\rho\left\langle VP^{\mathrm{i}}_{j}(t)\int_{-\infty}^{t}VP^{\mathrm{s}}_{j}(\boldsymbol{\alpha},t')\,\mathrm{d}t'\right\rangle_{\infty}\quad\text{(incident P-wave)}. \tag{63}$$

For an *incident S-wave* we arrive along similar lines at

$$\langle P^{\mathrm{a}}\rangle_{\infty}+\langle P^{\mathrm{s}}\rangle_{\infty}=-\rho\left\langle VS^{\mathrm{i}}_{j}(t)\int_{-\infty}^{t}VS^{\mathrm{s}}_{j}(\boldsymbol{\alpha},t')\,\mathrm{d}t'\right\rangle_{\infty}\quad\text{(incident S-wave)}. \tag{64}$$

Equations (63) and (64) constitute the energy theorems for the scattering of perpetuating plane waves. Note that in the time integration of the scattered-field spherical-wave amplitudes the properties

$$\langle VP^{\mathrm{s}}_{j}\rangle_{\infty}=0 \quad\text{and}\quad \langle VS^{\mathrm{s}}_{j}\rangle_{\infty}=0$$

hold. In comparison with the case of time-periodic fields no restrictions are, in this case, laid upon the possible time variations in the elastodynamic properties of the scattering object.

5. Conclusions

For the scattering of a plane P- or a plane S-wave by an object of bounded extent embedded in a homogeneous, isotropic, perfectly elastic solid an energy theorem related to the scattering phenomenon has been derived. It relates the energy that is both absorbed and scattered by the object to the spherical-wave amplitudes of the scattered wave field in the far-field region, when observed in the direction of propagation of the incident plane wave. In the relevant time interaction integral, only the interaction between incident P and scattered P, and incident S and scattered S occurs, although for each type of incident wave both types of scattered waves are excited. Depending on the time behaviour of the incident wave (transient, periodic, perpetuating), the energy theorem takes on somewhat different forms. An important consequence of the full time-domain analysis is that the elastodynamic properties of the scattering object hardly need any specification. For example, unlike the case in the frequency-domain analysis, a time-variant and/or non-linear behavior is included.

References

[1] T.H. Tan, "Theorem on the scattering and absorption cross section for scattering of plane, time-harmonic, elastic waves", *J. Acoust. Soc. Amer. 59*, 1265–1267 (1979).

[2] G.C. Herman, "Scattering of transient acoustic waves in fluids and solids", Report 1981-13, Laboratory of Electromagnetic Research, Department of Electrical Engineering, Delft University of Technology, Delft, The Netherlands (1981).

[3] J.D. Achenbach, "*Wave Propagation in Elastic Solids*", North-Holland, Amsterdam (1974) 103.