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Time domain Born approximation to the far-field scattering of plane electromagnetic waves by a penetrable object

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The low-contrast or (first-order) Born approximation is applied to the time domain scattering of a plane electromagnetic wave by a penetrable object of bounded extent, embedded in an isotropic and homogeneous medium. Closed-form analytic expressions are obtained for the spherical wave far-field scattering amplitude related to homogeneous objects of the following shapes: an ellipsoid, an elliptic cone of finite height, an elliptic cylinder of finite height, and a tetrahedron. Relaxation effects and anisotropy of the scattering object are included. The results are, among others, useful as test cases for time domain inverse scattering algorithms.

1. INTRODUCTION

In inverse scattering theory, the ultimate aim is to find the geometrical and/or physical parameters of unknown configurations from measured data of the scattered wave resulting from a known incident wave used to probe the unknown objects [*Das and Boerner*, 1978; *Boerner et al.*, 1981; *Boerner and Mo*, 1981; *Bojarski*, 1967, 1968, 1972]. For this purpose, computer algorithms are developed and to test these algorithms, often, as a first step at least, a linearized scheme is used, based on the low-contrast or Born approximation. As wave phenomena do take place in space-time, an investigation in space-time seems the most appropriate one. Hence, analytical time domain scattering results are needed for simple, though not trivial, cases that can be used as test cases for computational time domain inversion algorithms.

In the present paper, closed-form analytic expressions are derived for the spherical wave, far-field, scattering amplitude in case a plane electromagnetic wave is incident upon a penetrable object of bounded extent situated in a homogeneous and isotropic embedding. The analysis is entirely carried out in the space-time domain. To incorporate, in a simple notational manner, anisotropy of the scattering object, all formulas are presented in the subscript notation for vectors and tensors. Relaxation effects in the scattering object are also included in our time domain analysis. Homogeneous objects of the following shapes are considered: an ellipsoid, an elliptic cone of finite height, an elliptic cylinder of finite height, and a tetrahedron. A structure consisting of a composition of objects considered here can, in the Born approximation, be dealt with by superposition.

In a recent paper by *de Hoop* [1985], the time domain scattering of a scalar plane wave by an obstacle is, in the Born approximation, dealt with. The present paper deals with the general vectorial problem of electromagnetic scattering.

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2. FORMULATION OF THE SCATTERING PROBLEM

In three-dimensional space $\mathbf{R}^3$ a penetrable, scattering object occupying the bounded domain $\mathscr{D}^s$ is present. The domain outside the object is denoted by $\mathscr{D}$ (Figure 1). To locate position in space, we employ the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal Cartesian reference frame with origin O and three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each. These base vectors form, in the indicated order, a right-handed system. The time coordinate is denoted by t. Partial differentiation is denoted by ∂. The subscript notation for vectors and tensors is used and the summation convention applies. Occasionally, vectors are denoted by boldface symbols; for example, $\mathbf{x} = x_k \mathbf{i}_k$ denotes the position vector. In $\mathscr{D}$, an isotropic, homogeneous, linear, time-invariant and lossless medium is present. Its constitutive parameters are the permittivity ε and the permeability μ; ε and μ are real, positive constants. In any source-free subdomain of $\mathscr{D}$, the electro-

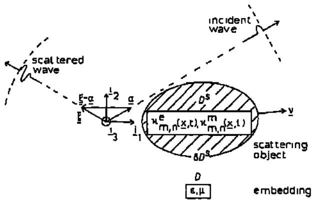

Fig 1 Scattering configuration.

magnetic field quantities satisfy the electromagnetic field equations

$$-\varepsilon_{m,p,q}\partial_p H_q + \varepsilon\partial_t E_m = 0 \tag{1}$$

$$\varepsilon_{m,p,q}\partial_p E_q + \mu\partial_t H_m = 0 \tag{2}$$

In these equations, E_m denotes the electric field strength, H_m the magnetic field strength, and $\varepsilon_{m,p,q}$ the completely antisymmetrical unit tensor of rank 3 (Levi-Civita tensor), defined as $\varepsilon_{m,p,q} = \{+1, -1\}$ when $\{m, p, q\}$ is an {even, odd} permutation of $\{1, 2, 3\}$, while $\varepsilon_{m,p,q} = 0$ when not all subscripts are different.

The scattering object shows a contrast with respect to the surrounding medium. This contrast is accounted for by the electric contrast susceptibility relaxation function $\kappa^e_{m,n} = \kappa^e_{m,n}(\mathbf{x}, t)$ and the magnetic contrast susceptibility relaxation function $\kappa^m_{m,n} = \kappa^m_{m,n}(\mathbf{x}, t)$. Then, the electromagnetic field quantities satisfy in $\mathscr{D}^s$ the equations

$$-\varepsilon_{m,p,q}\partial_p H_q + \varepsilon\partial_t E_m + \varepsilon\partial_t \int_{\tau=0}^{\infty} \kappa^e_{m,n}(\mathbf{x}, \tau)E_n(\mathbf{x}, t-\tau)\, d\tau = 0 \tag{3}$$

$$\varepsilon_{m,p,q}\partial_p E_q + \mu\partial_t H_m + \mu\partial_t \int_{\tau=0}^{\infty} \kappa^m_{m,n}(\mathbf{x}, \tau)H_n(\mathbf{x}, t-\tau)\, d\tau = 0 \tag{4}$$

For a scatterer that is instantaneously reacting, and has a permittivity $\varepsilon^s_{m,n}(\mathbf{x})$ and a permeability $\mu^s_{m,n}(\mathbf{x})$, we have $\kappa^e_{m,n}(\mathbf{x}, t) = [\varepsilon^s_{m,n}(\mathbf{x})/\varepsilon - 1]\delta(t)$ and $\kappa^m_{m,n}(\mathbf{x}, t) = [\mu^s_{m,n}(\mathbf{x})/\mu - 1]\delta(t)$. In (3)–(4), causality has been taken into account

The scatterer is irradiated by a uniform plane electromagnetic wave $\{E^i_m, H^i_n\}$ propagating in the direction of the unit vector $\boldsymbol{\alpha}$. E^i_m and H^i_n are related by

$$E^i_m = (\mu/\varepsilon)^{1/2}\varepsilon_{m,p,q}H^i_p\alpha_q \tag{5}$$

and

$$H^i_n = (\varepsilon/\mu)^{1/2}\varepsilon_{n,p,q}\alpha_p E^i_q \tag{6}$$

The scattered field $\{E^s_m, H^s_n\}$ is defined as

$$E^s_m = E_m - E^i_m \qquad H^s_n = H_n - H^i_n \tag{7}$$

With (1)–(4), it follows that E^s_m and H^s_n satisfy in $\mathbb{R}^3$ the equations

$$-\varepsilon_{m,p,q}\partial_p H^s_q + \varepsilon\partial_t E^s_m = \{-J^s_m, 0\} \qquad \mathbf{x} \in \{\mathscr{D}^s, \mathscr{D}\} \tag{8}$$

$$\varepsilon_{m,p,q}\partial_p E^s_q + \mu\partial_t H^s_m = \{-K^s_m, 0\} \qquad \mathbf{x} \in \{\mathscr{D}^s, \mathscr{D}\} \tag{9}$$

where

$$J^s_m = \varepsilon\partial_t \int_{\tau=0}^{\infty} \kappa^e_{m,n}(\mathbf{x}, \tau)E_n(\mathbf{x}, t-\tau)\, d\tau \tag{10}$$

is the contrast volume density of electric current of the scattering object, and

$$K^s_m = \mu\partial_t \int_{\tau=0}^{\infty} \kappa^m_{m,n}(\mathbf{x}, \tau)H_n(\mathbf{x}, t-\tau)\, d\tau \tag{11}$$

is the contrast volume density of magnetic current of the scattering object

From (8)–(9), it follows that $\{E^s_m, H^s_n\}$ can be expressed in terms of the retarded potentials

$$\{A^s_p, F^s_q\}(\mathbf{x}, t) = \int_{\mathbf{x}' \in \mathscr{D}^s} [\{J^s_p, K^s_q\}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)/4\pi|\mathbf{x} - \mathbf{x}'|]\, dV \tag{12}$$

in which $c = (\varepsilon\mu)^{-1/2}$ is the wave speed in the surrounding medium $\mathscr{D}$. Introducing for the time-differentiated and time-integrated forms of the retarded potentials the notation

$$\{DA_p, DF_q\}(\mathbf{x}, t) = \{\partial_t A_p, \partial_t F_q\}(\mathbf{x}, t) \tag{13}$$

and

$$\{IA_p, IF_q\}(\mathbf{x}, t) = \int_{\tau=-\infty}^{t} \{A_p(\mathbf{x}, \tau), F_q(\mathbf{x}, \tau)\}\, d\tau \tag{14}$$

the expressions for $\{E^s_m, H^s_n\}$ are

$$E^s_m(\mathbf{x}, t) = -\mu DA^s_m + \varepsilon^{-1}\partial_m\partial_p IA^s_p - \varepsilon_{m,p,q}\partial_p F^s_q \tag{15}$$

$$H^s_n(\mathbf{x}, t) = -\varepsilon DF^s_n + \mu^{-1}\partial_n\partial_q IF^s_q + \varepsilon_{n,q,p}\partial_q A^s_p \tag{16}$$

In (12), (15) and (16) the condition of causality has been taken into account. This means, of course, that quantities related to the scattered field vanish prior to t_0, where t_0 is the instant at which the incident wave hits the object. A concise derivation of (15)–(16) can be obtained with the aid of a Laplace transformation with respect to time and a Fourier transformation over $\mathbf{R}^3$. The expressions in (15)–(16) also follow directly from corresponding expressions of *Felsen and Marcuvitz* [1973].

In the representations (15)–(16), the right-hand sides yield, at any instant of discontinuity in time and at any surface of discontinuity in space of J_p^s and/or K_q^s, half the sum of the limiting values at either side of the relevant discontinuity.

By letting $|\mathbf{x}| \to \infty$ in (15)–(16), we arrive at integral representations for the far-field amplitude radiation characteristics of the scattered wave. Upon employing the relation

$$|\mathbf{x}-\mathbf{x}'| = |\mathbf{x}| - \xi_s x_s' + \text{vanishing terms} \qquad |\mathbf{x}| \to \infty \quad (17)$$

in which $\boldsymbol{\xi} = \mathbf{x}/|\mathbf{x}|$ is the unit vector in the direction of observation, the vector potentials are obtained as

$$\{A_p^s, F_q^s\}(\mathbf{x}, t) \sim \{A_p^{s,\infty}, F_q^{s,\infty}\}(\boldsymbol{\xi}, t - |\mathbf{x}|/c)/4\pi|\mathbf{x}| \qquad |\mathbf{x}| \to \infty \quad (18)$$

where

$$\{A_p^{s,\infty}, F_q^{s,\infty}\}(\boldsymbol{\xi}, t) = \int_{\mathbf{x}' \in \mathscr{D}^s} \{J_p^s, K_q^s\}(\mathbf{x}', t + \xi_s x_s'/c)\, dV \quad (19)$$

while for the spatial derivatives we have

$$\partial_m\{A_p^s, F_q^s\}(\mathbf{x}, t) \sim -(\xi_m/c)\{DA_p^{s,\infty}, DF_q^{s,\infty}\} \cdot (\boldsymbol{\xi}, t - |\mathbf{x}|/c)/4\pi|\mathbf{x}| \qquad |\mathbf{x}| \to \infty \quad (20)$$

Writing the far-field approximation to the scattered field as

$$\{E_m^s, H_n^s\}(\mathbf{x}, t) \sim \{E_m^{s,\infty}, H_n^{s,\infty}\}(\boldsymbol{\xi}, t - |\mathbf{x}|/c)/4\pi|\mathbf{x}| \qquad |x| \to \infty \quad (21)$$

we have

$$E_m^{s,\infty} = -\mu(\delta_{m,p} - \xi_m \xi_p)DA_p^{s,\infty} + \varepsilon_{m,p,q}(\xi_p/c)DF_q^{s,\infty} \quad (22)$$

$$H_n^{s,\infty} = -\varepsilon(\delta_{n,q} - \xi_n \xi_q)DF_q^{s,\infty} - \varepsilon_{n,q,p}(\xi_q/c)DA_p^{s,\infty} \quad (23)$$

It is easily verified that the plane wave relation, as expressed in (5)–(6) for the incident plane wave, also holds for the spherical wave amplitudes in the far-field approximation of the scattered field (22)–(23). Now, the scattering problem would have been solved, if E_m and H_n, and hence J_m^s and K_n^s, were known in $\mathscr{D}^s$. For low-contrast scattering objects, the total field values are, in the (first-order) Born approximation, replaced by the corresponding values of the incident wave. In the subsequent sections, this approximation will be carried out for a number of homogeneous objects having different geometrical shapes.

3. BORN APPROXIMATION

In the low-contrast, or (first-order) Born approximation the unknown values of the total field in $\mathscr{D}^s$ are replaced by the known values of the incident field. After this has been done, the volume densities of the contrast current J_p^s and K_q^s are explicitly known in $\mathscr{D}^s$, namely, (cf. (10)–(11))

$$J_p^s = \varepsilon\partial_t \int_{\tau=0}^{\infty} \kappa_{p,m}^e(\mathbf{x}, \tau)E_m^i(\mathbf{x}, t-\tau)\, d\tau \qquad \mathbf{x} \in \mathscr{D}^s \quad (24)$$

and

$$K_q^s = \mu\partial_t \int_{\tau=0}^{\infty} \kappa_{q,n}^m(\mathbf{x}, \tau)H_n^i(\mathbf{x}, t-\tau)\, d\tau \qquad \mathbf{x} \in \mathscr{D}^s \quad (25)$$

In the following, this approximation will be applied to homogeneous scatterers of different shapes. The incident field will be taken to be the uniform plane wave

$$\{E_m^i, H_n^i\}(\mathbf{x}, t) = \{e_m, h_n\}a(t - \alpha_s x_s/c) \quad (26)$$

where $a(t)$ is the, somehow normalized, wave shape of the incident wave, and $\{e_m, h_n\}$ are constant amplitude vectors. In particular, we shall determine the far-field radiation characteristics of the scattered field. With (24)–(26) we obtain, after introducing the vector

$$\mathbf{u} = (\boldsymbol{\xi} - \boldsymbol{\alpha})/c \quad (27)$$

through which the amplitude radiation characteristics depend on the direction of observation $\boldsymbol{\xi}$, the direction of propagation $\boldsymbol{\alpha}$ of the incident wave, and the wave speed c in the surrounding medium,

$$\mu DA_p^{s,\infty} = e_m V \int_{\tau=0}^{\infty} \kappa_{p,m}^e(\tau)\Upsilon(\mathbf{u}, t-\tau)\, d\tau \quad (28)$$

and

$$\varepsilon DF_q^{s,\infty} = h_n V \int_{\tau=0}^{\infty} \kappa_{q,n}^m(\tau)\Upsilon(\mathbf{u}, t-\tau)\, d\tau \quad (29)$$

in which

$$\Upsilon(\mathbf{u}, t) = c^{-2}V^{-1} \int_{\mathbf{x} \in \mathscr{D}^s} D^2a(t + u_s x_s)\, dV \quad (30)$$

is a shape factor and V is the volume of the scatterer. The shape factor Υ depends on the geometrical shape of the scatterer, on the normalized wave shape of the incident wave and on $\mathbf{u}$. For any geometry, we have

$$\Upsilon(0, t) = c^{-2}D^2a(t) \quad (31)$$

This implies that for observation in the far-field region and "behind" the scatterer, the amplitude of the scattered field is independent of the shape of the scatterer and fully determined by the latter's contrast and volume.

Since $a(t + u_s x_s)$ satisfies the homogeneous equation $|\mathbf{u}|^{-2}\partial_r\partial_r a - \partial_t^2 a = 0$ when $\mathbf{u} \neq \mathbf{0}$, we can, in (30), replace the operator $c^{-2}D^2$ by $c^{-2}|\mathbf{u}|^{-2}\partial_r\partial_r$ and apply Gauss' divergence theorem. This results into the expression

$$\Upsilon(\mathbf{u}, t) = c^{-2}|\mathbf{u}|^{-2}V^{-1}\int_{\mathbf{x}\in\partial\mathcal{D}^s}(\nu_r u_r)Da(t + u_s x_s)\,dA \qquad (32)$$

$$\mathbf{u} \neq \mathbf{0}$$

where $\partial\mathcal{D}^s$ denotes the boundary of the scatterer and $\mathbf{\nu}$ is the unit vector along the outward normal to $\partial\mathcal{D}^s$. Equation (32) shows that in the far-field region, the plane-wave scattering by a homogeneous object can, in the Born approximation, be viewed upon as a surface effect, when the observation is not "behind" the scatterer (cf. (31)).

4. SHAPE FACTOR FOR DIFFERENT HOMOGENEOUS OBJECTS

In this section the shape factors for the ellipsoid, the elliptic cone of finite height, the elliptic cylinder of finite height and the tetrahedron will be presented. Details of the calculation of these shape factors are given in the appendices.

4.1. *Ellipsoid*

For the ellipsoid defined by

$$\mathcal{D}^s = \{\mathbf{x};\, 0 \le x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 < 1\} \qquad (33)$$

the shape factor Υ is found to be

$$\Upsilon = (3c^{-2}/2\Gamma^2)\{a(t + \Gamma) + a(t - \Gamma) - \Gamma^{-1}[Ia(t + \Gamma) - Ia(t - \Gamma)]\} \qquad (34)$$

where

$$\Gamma = [(u_1 a_1)^2 + (u_2 a_2)^2 + (u_3 a_3)^2]^{1/2} \ge 0 \qquad (35)$$

By taking the limit $\Gamma \to 0$ in (34), which corresponds to $\mathbf{u} \to \mathbf{0}$, it can be verified that $\Upsilon \to c^{-2}D^2a(t)$, which is in accordance with (31).

4.2. *Elliptic cone of finite height*

For the elliptic cone of finite height defined by

$$\mathcal{D}^s = \{\mathbf{x};\, 0 < x_1^2/a_1^2 + x_2^2/a_2^2 < x_3^2/h^2,\, 0 < x_3 < h\} \qquad (36)$$

the shape factor Υ is found to be

$$\Upsilon = (6/\pi c^2)\int_{-1}^{1}(\Gamma\tau + u_3 h)^{-2} \cdot \{(\Gamma\tau + u_3 h)Da(t + \Gamma\tau + u_3 h) - 2a(t + \Gamma\tau + u_3 h) + 2(\Gamma\tau + u_3 h)^{-1}[Ia(t + \Gamma\tau + u_3 h) - Ia(t)]\}(1 - \tau^2)^{1/2}\,d\tau \qquad (37)$$

where

$$\Gamma = [(u_1 a_1)^2 + (u_2 a_2)^2]^{1/2} \ge 0 \qquad (38)$$

For the limiting cases $u_3 \to 0$, $\Gamma \neq 0$ and $u_3 \neq 0$, $\Gamma \to 0$, the shape factor follows immediately from (37). In the case $\mathbf{u} \to \mathbf{0}$ (i.e., $\Gamma \to 0$ and $u_3 \to 0$), the shape factor follows from (37) as $\Upsilon = c^{-2}D^2a(t)$, which is in accordance with (31).

4.3. *Elliptic cylinder of finite height*

For the elliptic cylinder of finite height defined by

$$\mathcal{D}^s = \{\mathbf{x};\, 0 \le x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\, -h/2 < x_3 < h/2\} \qquad (39)$$

the shape factor Υ is found to be

$$\Upsilon = (2/\pi c^2 u_3 h)\int_{-1}^{1}[Da(t + \Gamma\tau + u_3 h/2) - Da(t + \Gamma\tau - u_3 h/2)](1 - \tau^2)^{1/2}\,d\tau \qquad (40)$$

where

$$\Gamma = [(u_1 a_1)^2 + (u_2 a_2)^2]^{1/2} \ge 0 \qquad (41)$$

For the limiting case $u_3 \to 0$, $\Gamma \neq 0$, the shape factor reduces to

$$\Upsilon = (2/\pi c^2)\int_{-1}^{1}D^2a(t + \Gamma\tau)(1 - \tau^2)^{1/2}\,d\tau \qquad (42)$$

For the limiting case $\Gamma \to 0$, $u_3 \neq 0$, we have

$$\Upsilon = (1/c^2 u_3 h)[Da(t + u_3 h/2) - Da(t - u_3 h/2)] \qquad (43)$$

The case $\mathbf{u} \to \mathbf{0}$ (i.e., $\Gamma \to 0$ and $u_3 \to 0$) leads, either through (42) or (43), to $\Upsilon = c^{-2}D^2a(t)$, which is in accordance with (31).

4.4. *Tetrahedron*

For the tetrahedron defined by

$$\mathcal{D}^s = \left\{\mathbf{x};\, \mathbf{x} = \sum_{N=1}^{4}\lambda_N \mathbf{x}^N,\, 0 < \lambda_N < 1,\, \sum_{N=1}^{4}\lambda_N = 1\right\} \qquad (44)$$

in which $\mathbf{x}^N$ denotes the position vector of the Nth vertex ($N = 1, 2, 3, 4$), the shape factor Υ is found to be

$$\Upsilon = 6c^{-2}[(\Gamma^{PQ}\Gamma^{PR}\Gamma^{PS})^{-1}Ia(t + u_i x_i^P) + (\Gamma^{QR}\Gamma^{QS}\Gamma^{QP})^{-1}Ia(t + u_i x_i^Q) + (\Gamma^{RS}\Gamma^{RP}\Gamma^{RQ})^{-1}Ia(t + u_i x_i^R) + (\Gamma^{SP}\Gamma^{SQ}\Gamma^{SR})^{-1}Ia(t + u_i x_i^S)] \quad (45)$$

in which

$$\Gamma^{PQ} = u_i x_i^P - u_i x_i^Q = -\Gamma^{QP} \quad (46)$$

and $\{P, Q, R, S\}$ is a permutation of $\{1, 2, 3, 4\}$.

Note that the summation convention does not apply to the upper case superscripts and subscripts, only to the subscripts denoted by lower case letters. As special cases we have to consider: (1) $\xi - \alpha$ is perpendicular to a single edge, (2) $\xi - \alpha$ is perpendicular to two nonintersecting edges, (3) $\xi - \alpha$ is perpendicular to three edges (i.e., perpendicular to a face).

4.4.1. *$\xi - \alpha$ perpendicular to a single edge.* Let $\xi - \alpha$ be perpendicular to edge $\mathbf{x}^Q - \mathbf{x}^P$; then $\Gamma^{PQ} = \Gamma^{QP} = 0$. The shape factor is now

$$\Upsilon = 6c^{-2}\{(\Gamma^{PR}\Gamma^{PS})^{-1}[a(t + u_i x_i^P) - ((\Gamma^{PR})^{-1} + (\Gamma^{PS})^{-1})Ia(t + u_i x_i^P)] + (\Gamma^{RP})^{-2}(\Gamma^{RS})^{-1}Ia(t + u_i x_i^R) + (\Gamma^{SP})^{-2}(\Gamma^{SR})^{-1}Ia(t + u_i x_i^S)\} \quad (47)$$

4.4.2. *$\xi - \alpha$ perpendicular to two nonintersecting edges.* Let $\xi - \alpha$ be perpendicular to $\mathbf{x}^Q - \mathbf{x}^P$ and $\mathbf{x}^S - \mathbf{x}^R$; then, $\Gamma^{PQ} = \Gamma^{QP} = 0$ and $\Gamma^{RS} = \Gamma^{SR} = 0$. The shape factor is now

$$\Upsilon = 6c^{-2}\{(\Gamma^{PR})^{-2}[a(t + u_i x_i^P) + a(t + u_i x_i^R)] - 2(\Gamma^{PR})^{-3}[Ia(t + u_i x_i^P) - Ia(t + u_i x_i^R)]\} \quad (48)$$

4.4.3. *$\xi - \alpha$ perpendicular to three edges.* Let $\xi - \alpha$ be perpendicular to $\mathbf{x}^Q - \mathbf{x}^P$, $\mathbf{x}^R - \mathbf{x}^P$ and $\mathbf{x}^R - \mathbf{x}^Q$; then, $\Gamma^{PQ} = \Gamma^{QP} = 0$, $\Gamma^{PR} = \Gamma^{RP} = 0$ and $\Gamma^{QR} = \Gamma^{RQ} = 0$. The shape factor is now

$$\Upsilon = 6c^{-2}\{(\Gamma^{PS})^{-3}[\tfrac{1}{2}(\Gamma^{PS})^2 Da(t + u_i x_i^P) - \Gamma^{PS}a(t + u_i x_i^P) + Ia(t + u_i x_i^P)] + (\Gamma^{SP})^{-3}Ia(t + u_i x_i^S)\} \quad (49)$$

As before, $\{P, Q, R, S\}$ is a permutation of $\{1, 2, 3, 4\}$. Finally, if $\mathbf{u} \to \mathbf{0}$, the shape factor follows from (46)–(49) as $\Upsilon = c^{-2}D^2a(t)$, which is in accordance with (31).

5. CONCLUSION

With the aid of analytical techniques, expressions have been obtained for the far-field electric- and magnetic-field radiation characteristics in a homogeneous surrounding medium, when a plane electromagnetic wave is incident upon a number of contrasting penetrable objects of canonical geometry and the Born approximation is applicable. The synthetic data thus obtained for simple, though not trivial, geometries can, apart from their interest in direct scattering theory, be used as test cases for computational time domain inversion algorithms.

APPENDIX A. CALCULATION OF THE SHAPE FACTOR OF THE ELLIPSOID

The ellipsoid is defined by (33). Its volume is given by

$$V = 4\pi a_1 a_2 a_3/3 \quad (A1)$$

In order to perform the integration in the right-hand side of (30), we introduce as variables of integration

$$y_1 = x_1/a_1 \qquad y_2 = x_2/a_2 \qquad y_3 = x_3/a_3 \quad (A2)$$

In $\mathbf{y}$ space, the domain of integration is the unit sphere $0 \le y_1^2 + y_2^2 + y_3^2 < 1$. By carrying out in $\mathbf{y}$ space the integration with the use of the spherical coordinates $\{\rho, \theta, \phi\}$ with $0 \le \rho < 1$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, around the vector $u_1 a_1 \mathbf{i}_1 + u_2 a_2 \mathbf{i}_2 + u_3 a_3 \mathbf{i}_3$ as polar axis, we obtain

$$u_s x_s = (u_1 a_1)y_1 + (u_2 a_2)y_2 + (u_3 a_3)y_3 = \Gamma\rho\cos(\theta) \quad (A3)$$

where Γ is given by (35), while

$$dV = a_1 a_2 a_3 \rho^2 \sin(\theta)\, d\rho\, d\theta\, d\phi \quad (A4)$$

With (A1)–(A4) the shape factor, defined in (30), is rewritten as

$$\Upsilon = (3/4\pi c^2) \int_0^1 \int_0^\pi \int_0^{2\pi} D^2 a(t + \Gamma\rho\cos(\theta))\rho^2\, d\rho \sin(\theta)\, d\theta\, d\phi \quad (A5)$$

The integration in (A5) is performed by replacing the differentiation of $a(t + \Gamma\rho\cos(\theta))$ with respect to t by differentiation with respect to ρ or to θ, and (34) results.

APPENDIX B CALCULATION OF THE SHAPE FACTOR OF THE ELLIPTIC CONE OF FINITE HEIGHT AND OF THE ELLIPTIC CYLINDER OF FINITE HEIGHT

In the process of the calculation of the shape factor of the elliptic cone of finite height defined by (36) and of the elliptic cylinder of finite height defined by (39), the same steps have to be taken. Consequently, this appendix does apply to both objects Only the volumes of these objects differ; the volume of the elliptic cone is given by

$$V = \pi a_1 a_2 h/3 \tag{B1}$$

and the volume of the elliptic cylinder by

$$V = \pi a_1 a_2 h \tag{B2}$$

In order to perform the integration in the right-hand side of (30), we introduce as variables of integration

$$y_1 = x_1/a_1 \qquad y_2 = x_2/a_2 \tag{B3}$$

In the y plane the domain of integration is the circle $0 \leq y_1^2 + y_2^2 < (x_3/h)^3$ in the case of the elliptic cone and the circle $0 \leq y_1^2 + y_2^2 < 1$ in the case of the elliptic cylinder. By introducing in this y plane as variables of integration the polar coordinates $\{\rho, \theta\}$ around $u_1 a_1 \mathbf{i}_1 + u_2 a_2 \mathbf{i}_2$ as polar axis with $0 \leq \rho < x_3/h$ in the case of the elliptic cone and $0 \leq \rho < 1$ in the case of the elliptic cylinder and $0 \leq \theta < 2\pi$, we obtain

$$u_s x_s = (u_1 a_1) y_1 + (u_2 a_2) y_2 + u_3 x_3 = \rho \Gamma \cos(\theta) + u_3 x_3 \tag{B4}$$

where Γ is given by either (38) or (41), while

$$dx_1\, dx_2\, dx_3 = a_1 a_2 \rho\, d\rho\, d\theta\, dx_3 \tag{B5}$$

With (B1) and (B3)–(B5) the shape factor, defined in (30), is for the case of the elliptic cone rewritten as

$$\Upsilon = (3/c^2 \pi h) \cdot \int_0^h \int_0^{2\pi} \int_0^{x_3/h} D^2 a(t + \Gamma \rho \cos(\theta) + u_3 x_3) \rho\, dx_3\, d\theta\, d\rho \tag{B6}$$

This expression is further modified by reducing the integral with respect to θ to one over the interval (0, π) and introducing $\tau = \cos(\theta)$ as variable of integration. This results into the expression

$$\Upsilon = (6/c^2 \pi h) \int_0^h \int_{-1}^1 \int_0^{x_3/h} (1 - \tau^2)^{-1/2} \cdot D^2 a(t + \Gamma \rho \tau + u_3 x_3) \rho\, dx_3\, d\tau\, d\rho \tag{B7}$$

The integration in (B7) is performed by replacing the differentiation of $a(t + \Gamma \rho \tau + u_3 x_3)$ with respect to t by a differentiation with respect to ρ or to x_3. The final result is obtained by performing consecutively the integration with respect to ρ, an integration by parts with respect to τ and, finally, the integration with respect to x_3.

For the derivation of the shape factor of the elliptic cylinder the same steps as outlined above for the elliptic cone have to be executed

APPENDIX C: CALCULATION OF THE SHAPE FACTOR OF THE TETRAHEDRON

The tetrahedron is defined by (44). Its volume is given by

$$V = |\varepsilon_{m,p,q}(x_m^S - x_m^P)(x_p^Q - x_p^P)(x_q^R - x_q^P)|/6 \tag{C1}$$

in which $\{P, Q, R, S\}$ is a permutation of $\{1, 2, 3, 4\}$.

In (44), λ_1, λ_2, λ_3 and λ_4 denote the barycentric coordinates of a point in the interior, or on the boundary, of the tetrahedron. In order to perform the integration in the right-hand side of (30), we replace λ_1 by $1 - \lambda_2 - \lambda_3 - \lambda_4$ and carry out the integration over the ranges $0 < \lambda_2 < 1$, $0 < \lambda_3 < 1 - \lambda_2$, $0 < \lambda_4 < 1 - \lambda_2 - \lambda_3$. Further, we take into account the value of the Jacobian $\partial(x_1, x_2, x_3)/\partial(\lambda_2, \lambda_3, \lambda_4) = 6V$. With these substitutions the shape factor, defined in (30), is rewritten as

$$\Upsilon = 6c^{-2} \int_0^1 \int_0^{1-\lambda_2} \int_0^{1-\lambda_2-\lambda_3} - \lambda_3 D^2 a(t + u_i x_i^1 + \sum_{M=2}^{4} \lambda_M u_i(x_i^M - x_i^1))\, d\lambda_2\, d\lambda_3\, d\lambda_4 \tag{C2}$$

The integration in (C2) is performed by replacing the differentiation of $a(t + u_i x_i^1 + \sum_{M=2}^4 \lambda_M u_i(x_i^M - x_i^1))$ with respect to t by a differentiation with respect to λ_2, λ_3 and λ_4, respectively. The special cases are most easily dealt with by redoing the integrations that need modification.

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