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Time-domain reciprocity theorems for electromagnetic fields in dispersive media

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Time-domain reciprocity theorems of the time-convolution and the time-correlation type for electromagnetic fields in linear, time-invariant, and locally reacting media are discussed. Inhomogeneous, anisotropic, and arbitrarily dispersive, both active and passive, media are included. The analysis is entirely carried out in space-time, without intermediate recourse to the frequency or the wave vector domain. The application to inverse source and inverse constituency (or inverse profiling) problems is briefly indicated.

1. INTRODUCTION

A field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two admissible physical states that could occur in one and the same domain in space-time. As far as electromagnetic fields are concerned, *Lorentz* [1896] is commonly credited as the first to derive a reciprocity theorem; his theorem applies to "the propagation of light vibrations," i.e., to time-harmonic fields. The interaction quantity that occurs in Lorentz's reciprocity theorem (see also *Van Bladel* [1964, p. 205]) was later denoted by *Rumsey* [1954] (see also *Van Bladel* [1964, p. 234]) as the "reaction" between the sources and the fields in the two states. In their analyses, H. A. Lorentz and Rumsey incorporated general reciprocal anisotropic and lossy media. *Welch* [1960] (see also *Van Bladel* [1964, p. 193]) seems to be the first to have derived a time-domain reciprocity theorem for electromagnetic fields; it is of the correlation type and applies to homogeneous, isotropic, and lossless media. His derivation employs Fourier transforms with respect to time and makes use of the corresponding frequency-domain result. In a subsequent paper, *Welch* [1961] changed the theorem into one of the time-convolution type and included losses of the conduction type. A convolution-type reciprocity theorem applying to general causal dispersive media was presented by *Ru-Shao Cheo* [1965]. His proof was based on space-time arguments only; nonreciprocal media were excluded.

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Geurst [1963] had earlier derived a similar reciprocity relation, using, however, the time Fourier transform in the intermediate steps. A full space-time version of the reciprocity theorem for homogeneous, lossless media can also be found in *Felsen and Marcuvitz* [1973]. *Bojarski* [1983] clearly distinguished between convolution- and correlation-type reciprocity relations and presented the corresponding time-domain reciprocity theorems for homogeneous, isotropic, and lossless media, where the electromagnetic field is easily expressible in terms of its scalar and vector potentials (in the Lorentz gauge) and where the electromagnetic Green's dyad is shift-invariant in space-time. In this connection, he introduced the concept of "effectual" field as the time-reversed counterpart of a "causal" field and emphasized the relationship between time-advanced and time-retarded fields. Time reversal and space reversal were also introduced in the discussion by *Kong* [1972] on the generalization of electromagnetic reciprocity theorems to bianisotropic media, of which he particularly discussed the bianisotropy that is induced in moving media in accordance with the theory of relativity.

The application of reciprocity theorems to radiating apertures was studied by *Van Bladel* [1966]. The present author [*de Hoop*, 1959] investigated their application to the direct scattering of electromagnetic waves (see also *Van Bladel* [1964, p. 254]) and to multiport antennas [*de Hoop*, 1975]. The application to inverse scattering is reviewed by *Fisher and Langenberg* [1984], where an extensive list of references to earlier papers can be found.

The present investigation deals with time-

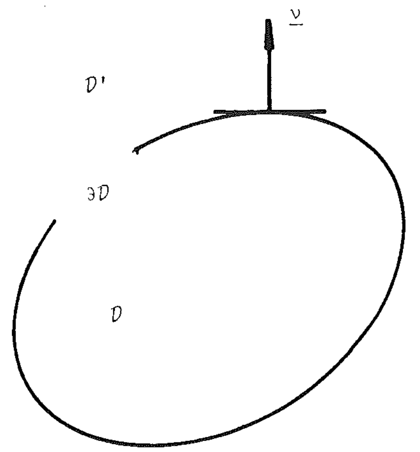

Fig. 1. Time-invariant configuration to which the reciprocity theorems apply.

convolution and time-correlation reciprocity theorems for electromagnetic fields in time-invariant configurations that are linear and locally reacting in their electromagnetic behavior. As regards the space-time geometry in which the two admissible states occur, this implies that this geometry is the Cartesian product $\mathscr{D} \times \mathscr{R}$ of a time-invariant spatial domain $\mathscr{D} \subset \mathscr{R}^3$ and the real time axis $\mathscr{R}$. Further, the constitutive parameters of the media present in the two states are time invariant and independent of the field values. No further restrictions are imposed.

The position of observation in $\mathscr{R}^3$ is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin $\mathscr{O}$ and the three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each. In the indicated order the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors in $\mathscr{R}^3$ is employed and the summation convention applies. The corresponding lower case Latin subscripts are to be assigned the values $\{1, 2, 3\}$. Whenever appropriate, the position vector will be denoted by $\mathbf{x} = x_p \mathbf{i}_p$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂, ∂_p denotes differentiation with respect to x_p, and ∂_t denotes differentiation with respect to t.

The reciprocity theorems are derived for bounded domains $\mathscr{D}$. In the analysis also the boundary $\partial\mathscr{D}$ of $\mathscr{D}$ occurs, as well as the complement $\mathscr{D}'$ of the union of $\mathscr{D}$ and $\partial\mathscr{D}$ in $\mathscr{R}^3$. The unit vector along the normal to $\partial\mathscr{D}$ is denoted by $\mathbf{v}$; it points away from $\mathscr{D}$ (Figure 1).

2. SOME PROPERTIES OF TIME-CONVOLUTION AND TIME-CORRELATION OF SPACE-TIME FUNCTIONS

In this section we present the properties of the time-convolution and the time-correlation of space-time functions as far as they are needed in the derivation of the reciprocity theorems. Let $f_1 = f_1(\mathbf{x}, t)$ and $f_2 = f_2(\mathbf{x}, t)$ be two transient space-time functions. By this we mean that the functions are absolutely integrable on the entire $t \in \mathscr{R}$. Then, time-convolution of f_1 and f_2 is defined as

$$\begin{aligned} C(f_1, f_2; \mathbf{x}, \tau) &= \int_{t \in \mathscr{R}} f_1(\mathbf{x}, t) f_2(\mathbf{x}, \tau - t)\, dt \\ C(f_1, f_2; \mathbf{x}, \tau) &= \int_{t \in \mathscr{R}} f_1(\mathbf{x}, \tau - t) f_2(\mathbf{x}, t)\, dt \\ C(f_1, f_2; \mathbf{x}, \tau) &= C(f_2, f_1; \mathbf{x}, \tau) \end{aligned} \tag{1}$$

Equation (1) shows that time-convolution is symmetrical in f_1 and f_2. Time-correlation of f_1 and f_2 is defined as

$$\begin{aligned} R(f_1, f_2; \mathbf{x}, \tau) &= \int_{t \in \mathscr{R}} f_1(\mathbf{x}, t) f_2(\mathbf{x}, t - \tau)\, dt \\ R(f_1, f_2; \mathbf{x}, \tau) &= \int_{t \in \mathscr{R}} f_1(\mathbf{x}, t + \tau) f_2(\mathbf{x}, t)\, dt \\ R(f_1, f_2; \mathbf{x}, \tau) &= R(f_2, f_1; \mathbf{x}, -\tau) \end{aligned} \tag{2}$$

Equation (2) shows that time-correlation is not symmetrical in f_1 and f_2.

Let, now, $\bar{f}$ denote the time-reversal of f, i.e.,

$$\bar{f}(\mathbf{x}, t) = f(\mathbf{x}, -t) \tag{3}$$

Then, it follows from (1)–(3) that

$$R(f_1, f_2; \mathbf{x}, \tau) = C(f_1, \bar{f}_2; \mathbf{x}, \tau) \tag{4}$$

Using (1), we further obtain the properties

$$C(\bar{f}_1, f_2; \mathbf{x}, \tau) = \bar{C}(f_1, \bar{f}_2; \mathbf{x}, \tau) \tag{5}$$

and

$$C(\bar{f}_1, \bar{f}_2; \mathbf{x}, \tau) = \bar{C}(f_1, f_2; \mathbf{x}, \tau) \tag{6}$$

For the time derivative of time-convolution the rules

$$\begin{aligned} \partial_\tau C(f_1, f_2; \mathbf{x}, \tau) &= C(f_1, \partial_t f_2; \mathbf{x}, \tau) \\ \partial_\tau C(f_1, f_2; \mathbf{x}, \tau) &= C(\partial_t f_1, f_2; \mathbf{x}, \tau) \end{aligned} \tag{7}$$

apply. In view of the property

$$\overline{\partial_t f} = -\partial_t \bar{f} \tag{8}$$

the time derivatives of time-correlation are taken care of by using (4) in conjunction with (7), i.e.,

$$\partial_\tau C(f_1, \bar{f}_2; \mathbf{x}, \tau) = C(\partial_t f_1, \bar{f}_2; \mathbf{x}, \tau)$$
$$\partial_\tau C(f_1, \bar{f}_2; \mathbf{x}, \tau) = -C(f_1, \overline{\partial_t f_2}; \mathbf{x}, \tau) \tag{9}$$

For the incorporation of dispersive media in the reciprocity theorems we also need the time-convolution of three space-time functions. For this, either of the definitions

$$C(f_1, f_2, f_3; \mathbf{x}, \tau) = C(f_1, C(f_2, f_3); \mathbf{x}, \tau)$$
$$C(f_1, f_2, f_3; \mathbf{x}, \tau) = C(C(f_1, f_2), f_3; \mathbf{x}, \tau) \tag{10}$$

holds.

In view of its simpler properties, the time-convolution concept is used throughout the entire subsequent derivations, i.e., both for time-convolution and for time-correlation reciprocity theorems.

3. PROPERTIES OF THE ELECTROMAGNETIC FIELD IN THE CONFIGURATION

In each subdomain of the configuration where the electromagnetic properties vary continuously with position the electromagnetic field vectors are continuously differentiable and satisfy Maxwell's equations

$$\partial_t D_i - \boldsymbol{\varepsilon}_{i,k,q} \partial_k H_q = -J_i \tag{11}$$

$$\boldsymbol{\varepsilon}_{j,k,p} \partial_k E_p + \partial_t B_j = -K_j \tag{12}$$

where

- E_p electric field strength, Vm^{-1};
- H_q magnetic field strength, Am^{-1},
- D_i electric flux density, Cm^{-2};
- B_j magnetic flux density, T;
- J_i volume source density of electric current, Am^{-2};
- K_j volume source density of magnetic current, Vm^{-2};

$\boldsymbol{\varepsilon}_{i,k,q}$ is the Levi-Civita tensor: $\boldsymbol{\varepsilon}_{i,k,q} = +1$ if $\{i, k, q\}$ is an even permutation of $\{1, 2, 3\}$, $\boldsymbol{\varepsilon}_{i,k,q} = -1$ if $\{i, k, q\}$ is an odd permutation of $\{1, 2, 3\}$, and $\boldsymbol{\varepsilon}_{i,k,q} = 0$ in all other cases. Equations (11) and (12) are supplemented by the constitutive relations. For a linear, time-invariant, locally reacting medium these are

$$D_i(\mathbf{x}, t) = \varepsilon_0 \int_{\tau=-\infty}^{\infty} \chi_{i,p}(\mathbf{x}, \tau) E_p(\mathbf{x}, t-\tau)\, d\tau \tag{13}$$

$$B_j(\mathbf{x}, t) = \mu_0 \int_{\tau=-\infty}^{\infty} \kappa_{j,q}(\mathbf{x}, \tau) H_q(\mathbf{x}, t-\tau)\, d\tau \tag{14}$$

where

- ε_0 permittivity in vacuum, Fm^{-1};
- $\chi_{i,p}$ electric relaxation function, s^{-1};
- μ_0 permeability in vacuum, Hm^{-1};
- $\kappa_{j,q}$ magnetic relaxation function, s^{-1}.

In SI, we have $\mu_0 = 4\pi * 10^{-7}\ \mathrm{Hm}^{-1}$ and $\varepsilon_0 = 1/\mu_0 c_0^2$, with $c_0 = 299792458\ \mathrm{ms}^{-1}$. Using the notation of (1), (13) and (14) can be rewritten as

$$D_i(\mathbf{x}, t) = \varepsilon_0\, C(\chi_{i,p}, E_p; \mathbf{x}, t) \tag{15}$$

$$B_j(\mathbf{x}, t) = \mu_0\, C(\kappa_{j,q}, H_q; \mathbf{x}, t) \tag{16}$$

respectively. In (13) and (14), inhomogeneity, anisotropy and dispersion of the medium are included.

If $\{\chi_{i,p}, \kappa_{j,q}\}(\mathbf{x}, \tau) = 0$ when $\tau < 0$, the medium at $\mathbf{x}$ is causal. If

$$\varepsilon_0\, \chi_{i,p}(\mathbf{x}, \tau) = \varepsilon_{i,p}(\mathbf{x})\delta(\tau) \tag{17}$$

$$\mu_0\, \kappa_{j,q}(\mathbf{x}, \tau) = \mu_{j,q}(\mathbf{x})\delta(\tau) \tag{18}$$

where $\delta(\tau)$ is the unit impulse (Dirac distribution), the medium is instantaneously reacting, and $\varepsilon_{i,p}$ and $\mu_{j,q}$ are its permittivity and permeability, respectively. If $\{\chi_{i,p}, \kappa_{j,q}\}(\mathbf{x}, \tau) = 0$ when $\tau > 0$, the medium is anticausal or effectual. From an energy point of view, a medium for which $\{\chi_{i,p}, \kappa_{j,q}\}(\mathbf{x}, \tau) \not\equiv 0$ when $\tau > 0$ is dissipative, a medium for which (17) and (18) hold is lossless, and a medium for which $\{\chi_{i,p}, \kappa_{j,q}\}(\mathbf{x}, \tau) \not\equiv 0$ when $\tau < 0$ is active. A medium that is either dissipative or lossless is also denoted as passive. For our reciprocity theorems no specific type of relaxation function is presupposed.

It is assumed that $\chi_{i,p}$ and $\kappa_{j,q}$ are piecewise continuous functions of position. At an interface between two different media they jump by finite amounts. Across such an interface the tangential components of the electric and the magnetic field strengths are continuous. If an impenetrable object is present, either the tangential components of the electric or the tangential components of the magnetic field strength have zero values at its boundary. Through the relevant boundary conditions, the presence of either interfaces or impenetrable objects is accounted for.

The two states that occur in the reciprocity theorems are denoted by the superscripts a and b, respectively. It is noted that the two states can apply to different source distributions and to different media, but they must be present in one and the same domain in space-time.

4. RECIPROCITY THEOREM OF TIME-CONVOLUTION TYPE

The reciprocity theorem of the time-convolution type follows upon considering the interaction quantity $\varepsilon_{k,i,j}[C(E_i^a, H_j^b; \mathbf{x}, \tau) - C(E_i^b, H_j^a; \mathbf{x}, \tau)]$. Using (11) and (12) for each of the two states, we obtain

$$\varepsilon_{k,i,j}\partial_k C(E_i^a, H_j^b; \mathbf{x}, \tau) = -C(\partial_t B_j^a + K_j^a, H_j^b; \mathbf{x}, \tau) - C(E_i^a, \partial_t D_i^b + J_i^b; \mathbf{x}, \tau) \tag{19}$$

and

$$\varepsilon_{k,i,j}\partial_k C(E_i^b, H_j^a; \mathbf{x}, \tau) = -C(\partial_t B_j^b + K_j^b, H_j^a; \mathbf{x}, \tau) - C(E_i^b, \partial_t D_i^a + J_i^a; \mathbf{x}, \tau) \tag{20}$$

Now, in view of (15) and (16), we have

$$C(\partial_t B_j^b, H_j^a; \mathbf{x}, \tau) - C(\partial_t B_j^a, H_j^b; \mathbf{x}, \tau) = \mu_0 \partial_\tau C(\kappa_{j,q}^b - \kappa_{q,j}^a, H_j^a, H_q^b; \mathbf{x}, \tau) \tag{21}$$

and

$$C(E_i^b, \partial_t D_i^a; \mathbf{x}, \tau) - C(E_i^a, \partial_t D_i^b; \mathbf{x}, \tau) = \varepsilon_0 \partial_\tau C(\chi_{p,i}^a - \chi_{i,p}^b, E_i^a, E_p^b; \mathbf{x}, \tau) \tag{22}$$

where (7) has been used. Subtracting (20) from (19) and employing (21) and (22), we arrive at

$$\begin{aligned}\varepsilon_{k,i,j}\partial_k[C(E_i^a, H_j^b; \mathbf{x}, \tau) - C(E_i^b, H_j^a; \mathbf{x}, \tau)] &= \mu_0 \partial_\tau C(\kappa_{j,q}^b - \kappa_{q,j}^a, H_j^a, H_q^b; \mathbf{x}, \tau) \\ &+ \varepsilon_0 \partial_\tau C(\chi_{p,i}^a - \chi_{i,p}^b, E_i^a, E_p^b; \mathbf{x}, \tau) \\ &- C(K_j^a, H_j^b; \mathbf{x}, \tau) - C(E_i^a, J_i^b; \mathbf{x}, \tau) \\ &+ C(K_j^b, H_j^a; \mathbf{x}, \tau) + C(E_i^b, J_i^a; \mathbf{x}, \tau)\end{aligned} \tag{23}$$

Equation (23) is the local form of the time-convolution reciprocity theorem. The first two terms on the right-hand side are representative of the differences in the properties of the media present in the two states; they vanish at those locations where $\chi_{p,i}^a(\mathbf{x}, \tau) = \chi_{i,p}^b(\mathbf{x}, \tau)$ and $\kappa_{j,q}^b(\mathbf{x}, \tau) = \kappa_{q,j}^a(\mathbf{x}, \tau)$ for all $\tau \in \mathcal{R}$. In case the latter conditions hold, the two media are denoted as each other's adjoints. Note in this respect that the adjoint of a causal (effectual) medium is causal (effectual), too. The last four terms on the right-hand side of (23) are associated with the source distributions; they vanish at those locations where no sources are present. Upon integrating (23) over the subdomains of $\mathcal{D}$ where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\begin{aligned}\int_{\mathbf{x}\in\partial\mathcal{D}} \varepsilon_{k,i,j}\nu_k[C(E_i^a, H_j^b; \mathbf{x}, \tau) - C(E_i^b, H_j^a; \mathbf{x}, \tau)]\,dA &= \int_{\mathbf{x}\in\mathcal{D}} [\mu_0 \partial_\tau C(\kappa_{j,q}^b - \kappa_{q,j}^a, H_j^a, H_q^b; \mathbf{x}, \tau) \\ &+ \varepsilon_0 \partial_\tau C(\chi_{p,i}^a - \chi_{i,p}^b, E_i^a, E_p^b; \mathbf{x}, \tau)]\,dV \\ &+ \int_{\mathbf{x}\in\mathcal{D}} [-C(K_j^a, H_j^b; \mathbf{x}, \tau) - C(E_i^a, J_i^b; \mathbf{x}, \tau) \\ &+ C(K_j^b, H_j^a; \mathbf{x}, \tau) + C(E_i^b, J_i^a; \mathbf{x}, \tau)]\,dV\end{aligned} \tag{24}$$

Equation (24) is the global form, for the domain $\mathcal{D}$, of the time-convolution reciprocity theorem. Note that the contributions from interfaces between different media present in $\mathcal{D}$ have cancelled and that the contributions from the boundaries of impenetrable objects present in $\mathcal{D}$ vanish in view of the boundary conditions stated in section 3.

5. RECIPROCITY THEOREM OF THE TIME-CORRELATION TYPE

The reciprocity theorem of the time-correlation type follows upon considering the interaction quantity $\varepsilon_{k,i,j}[R(E_i^a, H_j^b; \mathbf{x}, \tau) + R(E_i^b, H_j^a; \mathbf{x}, -\tau)]$. On account of (4), this interaction quantity is equivalent to $\varepsilon_{k,i,j}[C(E_i^a, \bar{H}_j^b; \mathbf{x}, \tau) + C(\bar{E}_i^b, H_j^a; \mathbf{x}, \tau)]$. Using (11) and (12) for each of the two states, we obtain

$$\varepsilon_{k,i,j}\partial_k C(E_i^a, \bar{H}_j^b; \mathbf{x}, \tau) = -C(\partial_t B_j^a + K_j^a, \bar{H}_j^b; \mathbf{x}, \tau) - C(E_i^a, \overline{\partial_t D_i^b} + \bar{J}_i^b; \mathbf{x}, \tau) \tag{25}$$

and

$$\varepsilon_{k,i,j}\partial_k C(\bar{E}_i^b, H_j^a; \mathbf{x}, \tau) = -C(\overline{\partial_t B_j^b} + \bar{K}_j^b, H_j^a; \mathbf{x}, \tau) - C(\bar{E}_i^b, \partial_t D_i^a + J_i^a; \mathbf{x}, \tau) \tag{26}$$

Now, in view of (15) and (16), we have

$$-C(\overline{\partial_t B_j^b}, H_j^a; \mathbf{x}, \tau) - C(\partial_t B_j^a, \bar{H}_j^b; \mathbf{x}, \tau) = \mu_0 \partial_\tau C(\bar{\kappa}_{j,q}^b - \kappa_{q,j}^a, H_j^a, \bar{H}_q^b; \mathbf{x}, \tau) \tag{27}$$

and

$$-C(E_i^a, \overline{\partial_t D_i^b}; \mathbf{x}, \tau) - C(\bar{E}_i^b, \partial_t D_i^a; \mathbf{x}, \tau) = \varepsilon_0 \partial_\tau C(\bar{\chi}_{i,p}^b - \chi_{p,i}^a, E_i^a, \bar{E}_p^b; \mathbf{x}, \tau) \tag{28}$$

where (7) has been used. Adding (26) to (25) and employing (27) and (28), we arrive at

$$\begin{aligned}\varepsilon_{k,i,j}\partial_k[C(E_i^a, \bar{H}_j^b; \mathbf{x}, \tau) + C(\bar{E}_i^b; H_j^a; \mathbf{x}, \tau)] &= \mu_0 \partial_\tau C(\bar{\kappa}_{j,q}^b - \kappa_{q,j}^a, H_j^a, \bar{H}_q^b; \mathbf{x}, \tau) \\ &+ \varepsilon_0 \partial_\tau C(\bar{\chi}_{i,p}^b - \chi_{p,i}^a, E_i^a, \bar{E}_p^b; \mathbf{x}, \tau)\end{aligned}$$

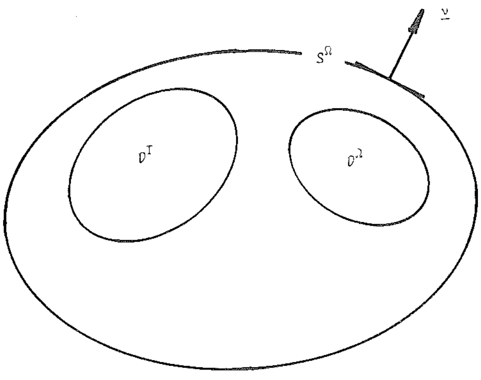

Fig. 2. Configuration illustrative of the inverse source problem: unknown sources radiate in $\mathscr{D}^T$; the field is measured in $\mathscr{D}^\Omega$ and on $\mathscr{S}^\Omega$.

$$- C(K^a_j, \bar{H}^b_j; \mathbf{x}, \tau) - C(E^a_i, \bar{J}^b_i; \mathbf{x}, \tau)$$
$$- C(\bar{K}^b_j, H^a_j; \mathbf{x}, \tau) - C(\bar{E}^b_i, J^a_i; \mathbf{x}, \tau) \qquad (29)$$

Equation (29) is the local form of the time-correlation reciprocity theorem. The first two terms on the right-hand side are representative of the differences in the properties of the media present in the two states; they vanish at these locations where $\bar{\chi}^b_{i,p}(\mathbf{x}, \tau) = \chi^a_{p,i}(\mathbf{x}, \tau)$ and $\bar{\kappa}^b_{j,q}(\mathbf{x}, \tau) = \kappa^a_{q,j}(\mathbf{x}, \tau)$ for all $\tau \in \mathscr{R}$. In case the latter conditions hold, the two media are denoted as each other's time-reverse adjoints. Note in this respect that the time-reverse adjoint of a causal (effectual) medium is an effectual (causal) one. Upon integrating (29) over the subdomains of $\mathscr{D}$ where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\int_{\mathbf{x} \in \partial D} \varepsilon_{k,i,j} \nu_k [C(E^a_i, \bar{H}^b_j; \mathbf{x}, \tau) + C(\bar{E}^b_i, H^a_j; \mathbf{x}, \tau)] \, dA$$
$$= \int_{\mathbf{x} \in D} [\mu_0 \partial_\tau C(\bar{\kappa}^b_{j,q} - \kappa^a_{q,j}, H^a_j, \bar{H}^b_q; \mathbf{x}, \tau)$$
$$+ \varepsilon_0 \partial_\tau C(\bar{\chi}^b_{i,p} - \chi^a_{p,i}, E^a_i, \bar{E}^b_p; \mathbf{x}, \tau)] \, dV$$
$$+ \int_{\mathbf{x} \in D} [-C(K^a_j, \bar{H}^b_j; \mathbf{x}, \tau) - C(E^a_i, \bar{J}^b_i; \mathbf{x}, \tau)$$
$$- C(\bar{K}^b_j, H^a_j; \mathbf{x}, \tau) - C(\bar{E}^b_i, J^a_i; \mathbf{x}, \tau)] \, dV \qquad (30)$$

Equation (30) is the global form, for the domain $\mathscr{D}$, of the time-correlation reciprocity theorem. Note that the contributions from interferences between different media present in $\mathscr{D}$ have cancelled and that the contributions from the boundaries of impenetrable objects present in $\mathscr{D}$ vanish in view of the boundary conditions stated in section 3.

6. APPLICATION TO INVERSE PROBLEMS

In this section we briefly outline the relevance of the reciprocity theorems (24) and (30) to inverse problems. In this respect, we distinguish between inverse source problems and inverse constituency problems. In an inverse source problem the aim is to reconstruct the volume current densities of radiating sources present in some inaccessible domain in space from the measured values of the radiated electromagnetic field in some other domain in space. The constitutive parameters of the medium in which the radiation takes place are assumed to be known. In an inverse constituency problem (also denoted as inverse profiling problem) the aim is to reconstruct the distribution of constitutive parameters in some inaccessible domain in space by irradiating the configuration by known sources located in the embedding and measuring the electromagnetic field response in some other domain in the embedding; the constitutive parameters of the embedding are known. The two types of problems are discussed separately.

Inverse source problem

In the inverse source problem the field in state a is taken to be one that is radiated by the unknown source distributions $\{J^T_i, K^T_j\}$. Let $\mathscr{D}^T \subset \mathscr{R}^3$, be their spatial support. The radiated field $\{E^T_i, H^T_j\}$ is measured in some, accessible, observational domain $\mathscr{D}^\Omega \subset \mathscr{R}^3$. The intersection of $\mathscr{D}^T$ and $\mathscr{D}^\Omega$ is empty (Figure 2). State b is taken to be a computational state, denoted as the "observational" one. The corresponding field $\{E^\Omega_i, H^\Omega_j\}$ that would be radiated by known sources with distributions $\{J^\Omega_i, K^\Omega_j\}$ is computed and its interaction with the measured field in $\mathscr{D}^\Omega$ is evaluated. In general, one could say that the introduction of the observational state is representative of the processing of the measured data. Since only the interaction in $\mathscr{D}^\Omega$ is considered, it makes no sense to take the support of $\{J^\Omega_i, K^\Omega_j\}$ larger than $\mathscr{D}^\Omega$. Finally, the medium in the observational state is taken to be either the adjoint (for the application of (24)) or the time-reverse adjoint (for the application of (30)) of the one in which the unknown sources radiate.

The reciprocity relations (24) and (30) are now applied to the domain interior to the closed surface $\mathscr{S}^\Omega$ that is taken such that $\mathscr{D}^T$ and $\mathscr{D}^\Omega$ are located in its interior. Then, (24) leads to

$$\int_{\mathbf{x} \in D^T} [C(K^T_j, H^\Omega_j; \mathbf{x}, \tau) - C(E^\Omega_i, J^T_i; \mathbf{x}, \tau)] \, dV$$

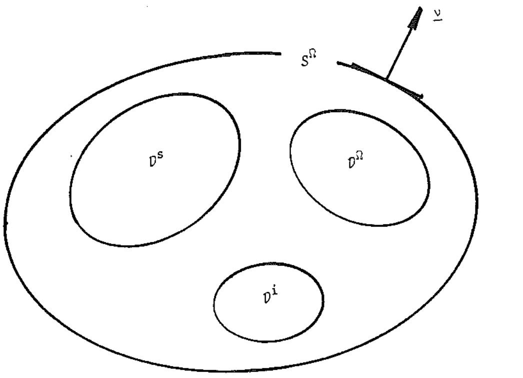

Fig. 3. Configuration illustrative of the inverse constituency problem: known sources in $\mathcal{D}^i$ irradiate the contrasting domain $\mathcal{D}^s$ with unknown properties; the field is measured in $\mathcal{D}^\Omega$ and on $\mathcal{S}^\Omega$.

$$= \int_{\mathbf{x} \in D^\Omega} [C(K_j^\Omega, H_j^T; \mathbf{x}, \tau) - C(E_i^T, J_i^\Omega; \mathbf{x}, \tau)]\, dV - \int_{\mathbf{x} \in S^\Omega} \varepsilon_{k,i,j} \nu_k [C(E_i^T, H_j^\Omega; \mathbf{x}; \tau) - C(E_i^\Omega, H_j^T; \mathbf{x}, \tau)]\, dA \tag{31}$$

and (30) leads to

$$\int_{\mathbf{x} \in D^T} [C(K_j^T, \bar{H}_j^\Omega; \mathbf{x}, \tau) + C(\bar{E}_i^\Omega, J_i^T; \mathbf{x}, \tau)]\, dV = \int_{\mathbf{x} \in \mathcal{D}^\Omega} [-C(\bar{K}_j^\Omega, H_j^T; \mathbf{x}, \tau) - C(E_i^T, \bar{J}_i^\Omega; \mathbf{x}, \tau)]\, dV - \int_{\mathbf{x} \in \mathcal{S}^\Omega} \varepsilon_{k,i,j} \nu_k [C(E_i^T, \bar{H}_j^\Omega; \mathbf{x}, \tau) + C(\bar{E}_i^\Omega, H_j^T; \mathbf{x}, \tau)]\, dA \tag{32}$$

In (31) and (32) the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on $\mathcal{S}^\Omega$. A solution to the inverse source problem is now commonly constructed by taking for $\{J_i^\Omega, K_j^\Omega\}$ a sequence of N linearly independent distributions with spatial support $\mathcal{D}^\Omega$ and fixed, preferably broadband, time behavior (for example, a unit impulse). The corresponding sequence of field distributions $\{E_i^\Omega, H_j^\Omega\}$ is computed. Next, the unknown source distributions $\{J_i^T, K_j^T\}$ are expanded into an appropriate sequence of M linearly independent space-time expansion functions with spatial support $\mathcal{D}^T$; the corresponding expansion coefficients are unknown. Substitution of the results into (31) and (32) and evaluation of the relevant integrals lead to systems of linear algebraic equations with the source expansion coefficients as unknowns. When $M = N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is nonsingular, it turns out to be ill-conditioned in all practical cases. Therefore one usually takes $M > N$, and a best fit of the expanded source distributions is obtained by the application of minimization techniques.

At this point, more must be said about the role of $\mathcal{S}^\Omega$. In practice, one is mostly interested in causal media. Then it is advantageous to choose the fields in (31) causal as well. Given the fact that $\mathcal{S}^\Omega$ surrounds all sources, the integral over $\mathcal{S}^\Omega$ can be replaced by an integral over any sphere $\mathcal{S}_\Delta$ with radius Δ and center at the origin such that $\mathcal{S}_\Delta$ surrounds $\mathcal{S}^\Omega$. (This follows from application of (24) to the domain between $\mathcal{S}_\Delta$ and $\mathcal{S}^\Omega$.) However, for sufficiently large values of Δ the causal field on $\mathcal{S}_\Delta$ is zero, since it propagates with a finite maximum speed away from the sources. Hence under these circumstances, the surface integral in the right-hand side of (31) vanishes. A similar argument does not apply to the surface integral in the right-hand side of (32), since in (32) effectual (or anticausal) fields are involved in all cases. This difference in the roles of the surface integrals in (31) and (32) has been pointed out by *Bojarski* [1983].

Inverse constituency problem

In the inverse constituency problem the field in state a is taken to be the one that irradiates the configuration. Let $\mathcal{D}^i \subset \mathcal{R}^3$ be the spatial support of the irradiating sources with known distributions $\{J_i^i, K_j^i\}$ and let the corresponding field be $\{E_i^i, H_j^i\}$. This field is measured in some accessible, observational domain $\mathcal{D}^\Omega \subset \mathcal{R}^3$. Let, further, $\mathcal{D}^s \subset \mathcal{R}^3$ be the (inaccessible) domain in which the constitutive parameters are unknown. The intersections of $\mathcal{D}^s$ and $\mathcal{D}^i$ and of $\mathcal{D}^s$ and $\mathcal{D}^\Omega$ are empty; $\mathcal{D}^i$ and $\mathcal{D}^\Omega$ may, however, have points in common, or may even completely coincide (Figure 3). State b is taken to be a computational state, denoted as the "observational" one. The corresponding field $\{E_i^\Omega, H_j^\Omega\}$ that would be radiated by known sources with distributions $\{J_i^\Omega, K_j^\Omega\}$ in the known medium with the constitutive parameters $\{\chi_{i,p}^\Omega, \kappa_{j,q}^\Omega\}$ of the adjoint (for the application of (24)) or the time-reverse adjoint (for the application of (30)) of the known embedding is computed and its interaction with the measured field in $\mathcal{D}^\Omega$ is evaluated. Since only the interaction in $\mathscr{D}^\Omega$ is considered, it makes no sense to take the support of $\{J_i^\Omega, K_j^\Omega\}$ larger than $\mathscr{D}^\Omega$. The unknown constitutive parameters of $\mathscr{D}^s$ are denoted by $\{\chi_{i,p}^s, \kappa_{j,q}^s\}$, $\mathscr{D}^s$ being the support of the differences $\{\chi_{i,p}^s - \chi_{p,i}^\Omega, \kappa_{j,q}^s - \kappa_{q,j}^\Omega\}$ and $(\chi_{i,p}^s - \bar{\chi}_{p,i}^\Omega, \kappa_{j,q}^s - \bar{\kappa}_{q,j}^\Omega)$ for the application of (24) and (30), respectively.

The reciprocity relations (24) and (30) are now applied to the domain interior to the surface $\mathscr{S}^\Omega$ that is taken such that $\mathscr{D}^i$, $\mathscr{D}^s$ and $\mathscr{D}^\Omega$ are located in its interior. Then (24) leads to

$$\int_{\mathbf{x}\in\mathscr{D}^s} [C(K_q^s, H_q^\Omega; \mathbf{x}, \tau) - C(E_p^\Omega, J_p^s; \mathbf{x}, \tau)]\, dV$$
$$= \int_{\mathbf{x}\in\mathscr{D}^i} [-C(K_j^i, H_j^\Omega; \mathbf{x}, \tau) + C(E_i^\Omega, J_i^i; \mathbf{x}, \tau)]\, dV$$
$$+ \int_{\mathbf{x}\in\mathscr{D}^\Omega} [C(K_j^\Omega, H_j^i; \mathbf{x}, \tau) - C(E_i^i, J_i^\Omega; \mathbf{x}, \tau)]\, dV$$
$$- \int_{\mathbf{x}\in\mathscr{S}^\Omega} \varepsilon_{k,i,j}\nu_k[C(E_i^i, H_j^\Omega; \mathbf{x}, \tau) - C(E_i^\Omega, H_j^i; \mathbf{x}, \tau)]\, dA \tag{33}$$

in which

$$J_p^s = \varepsilon_0\, \partial_\tau\, C(\chi_{p,i}^s - \chi_{i,p}^\Omega, E_i^i; \mathbf{x}, \tau) \tag{34}$$

is the equivalent contrast volume source density of electric current in $\mathscr{D}^s$, and

$$K_q^s = \mu_0\, \partial_\tau\, C(\kappa_{q,j}^s - \kappa_{j,q}^\Omega, H_j^i; \mathbf{x}, \tau) \tag{35}$$

is the equivalent contrast volume source density of magnetic current in $\mathscr{D}^s$. In the same way, (30) leads to

$$\int_{\mathbf{x}\in\mathscr{D}^s} [C(K_q^s, \bar{H}_q^\Omega; \mathbf{x}, \tau) + C(\bar{E}_p^\Omega, J_p^s; \mathbf{x}, \tau)]\, dV$$
$$= \int_{\mathbf{x}\in\mathscr{D}^i} [-C(K_j^i, \bar{H}_j^\Omega; \mathbf{x}, \tau) - C(\bar{E}_i^\Omega, J_i^i; \mathbf{x}, \tau)]\, dV$$
$$+ \int_{\mathbf{x}\in\mathscr{D}^\Omega} [-C(\bar{K}_j^\Omega, H_j^i; \mathbf{x}, \tau) - C(E_i^i, \bar{J}_i^\Omega; \mathbf{x}, \tau)]\, dV$$
$$- \int_{\mathbf{x}\in\mathscr{S}^\Omega} \varepsilon_{k,i,j}\nu_k[C(E_i^i, \bar{H}_j^\Omega; \mathbf{x}, \tau)$$
$$+ C(\bar{E}_i^\Omega, H_j^i; \mathbf{x}, \tau)]\, dA \tag{36}$$

in which

$$J_p^s = \varepsilon_0\, \partial_\tau\, C(\chi_{p,i}^s - \bar{\chi}_{i,p}^\Omega, E_i^i; \mathbf{x}, \tau) \tag{37}$$

is the equivalent contrast volume source density of electric current in $\mathscr{D}^s$, and

$$K_q^s = \mu_0\, \partial_\tau\, C(\kappa_{q,j}^s - \bar{\kappa}_{j,q}^\Omega, H_j^i; \mathbf{x}, \tau) \tag{38}$$

is the equivalent contrast volume source density of magnetic current in $\mathscr{D}^s$.

In (33) and (36), the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on $\mathscr{S}^\Omega$. The easiest way to address the inverse constituency problem is to consider it as an inverse source problem for the quantities $\{J_p^s, K_q^s\}$. Once values for these have been obtained, the solution of the forward or direct source problem with known values of $\{J_i^i, K_j^i\}$ and $\{J_p^s, K_q^s\}$ yields the values of $\{E_i^i, H_j^i\}$ in $\mathscr{D}^s$ and the temporal deconvolution of either (34) and (35) or (37) and (38) yields, since $\{\chi_{i,p}^\Omega, \kappa_{j,q}^\Omega\}$ are known, the values of $\{\chi_{p,i}^s, \kappa_{q,j}^s\}$. As to the role of the surface integrals over $\mathscr{S}^\Omega$ in the right-hand sides of (33) and (36), the same remarks as for the inverse source problem apply.

To conclude our investigation, we want to emphasize that the uniqueness and the existence of solutions to both the inverse source and the inverse constituency problem are, for the larger part, at the moment open questions.

7. CONCLUSION

Time-domain reciprocity theorems for the electromagnetic field in linear, time-invariant, and locally reacting media have been derived via a full space-time method. Inhomogeneous, anisotropic and arbitrarily dispersive media are included. One of the theorems is of the time-convolution type, the other is of the time-correlation type. The application of the two theorems to inverse source and inverse constituency problems is discussed.

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