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Acoustic Radiation from Impulsive Sources in a Layered Fluid

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The linearized theory of the acoustic radiation generated by impulsive point sources in a layered fluid medium is developed. Space-time expressions for the acoustic pressure and the particle velocity of the acoustic wave field are derived with the aid of the modified Cagniard method. These expressions have the form of a time convolution of the strength of the source as a function of time ('source signature') and a properly defined space-time Green's function that is characteristic for the type of source (volume-injection source, force source) and for the configuration in which the acoustic radiation takes place. The applications to seismics are briefly discussed.

1. Introduction

Acoustic waves belong to the standard diagnostic tools to probe the subsurface structure of the earth in the search for fossile energy resources. In the acquisition of *land seismic* data we distinguish between surface seismics, vertical seismic profiling, and cross-borehole seismics. In surface seismics, the acoustic source is located either on the earth's surface (for a mechanical vibrator) or somewhat below, but close to, it (for an explosion source), while the resulting acoustic wave motion is picked up by a number of acoustic receivers (geophones) placed at the surface of the earth. In this kind of seismics, the offset between source and receivers is predominantly horizontal, and the interpretation of the data is mainly based on the acoustic wave reflection against interfaces between the successive layers out of which the earth is composed. In Vertical Seismic Profiling (VSP), again an acoustic source of the type as used in surface seismics is employed, while the acoustic receivers are now placed in a borehole. In this kind of seismics, the offset between source and receivers is predominantly vertical, and the interpretation of the data is mainly based on the acoustic wave transmission across the successive layers of the earth. In cross-borehole seismics, the acoustic source (usually of an explosion type) is placed in one borehole, while the acoustic receivers are placed in an adjacent one. In this kind of seismics, there is no predominant offset, and the interpretation of the data is based on both the reflection and the transmission properties of the successive layers of the earth.

In the acquisition of *marine seismic data,* both the acoustic source (usually an airgun) and the acoustic receivers (hydrophones) are situated somewhat below, but close to, the surface of the sea; they are towed by a surface vessel. Here, too, the offset between source and receiver is predominantly horizontal.

The problem of seismic prospecting is, in fact, an inverse one: given the measured data, one is to reconstruct the physical properties of the probed structure. In seismic practice, most of this inversion is still carried out by assuming that the earth's structure changes in its mechanical properties (volume density of mass, elastic compliance or stiffness) much more rapidly in the (downward) vertical direction than in the horizontal direction. Accordingly, one takes a horizontally layered model of the earth as point of departure. Using this model, the thickness of the layers and their mechanical parameters are, roughly speaking, inferred by comparing the measured data with the theoretically determined response of a number of model configurations of the indicated type. Hence, there is a need for a versatile and efficient computational method to generate so-called synthetic seismograms for a wide variety of structures of the indicated kind. The modified Cagniard method (CAGNIARD [3], DE HOOP [5,6]) provides such a method (see also ACHENBACH [1], MIKLOWITZ [11], and AKI and RICHARDS [2]).

The present paper discusses this method for the case of acoustic waves generated by an impulsive point source located in a layered fluid medium. The results are also of importance to the case of a layered isotropic solid if the acoustic wave motion in the solid is predominantly of the compressional type (as is the case of fluids), which amounts to neglecting, in the first instance, the influence of shear waves. It is observed, however, that the modified Cagniard method can without difficulty also incorporate the presence of shear waves (cf. DE HOOP and VAN DER HIJDEN [7,8,9]), while with the unavoidable increase in the degree of complexity it can also be applied to arbitrarily anisotropic solid media (VAN DER HIJDEN [13]). Through the method, the total acoustic wave motion is expressed as a time convolution of the strength of the source as a function of time ('source signature') and a properly defined space-time Green's function that is characteristic for the type of source (volume-injection source, force source) and for the configuration in which the acoustic radiation takes place. The space-time Green's function is expressed as the superposition of generalized rays (WIGGINS and HELMBERGER [15]). This representation is exact, and within a finite time interval the number of contributing generalized rays is finite. The expressions for the acoustic pressure and the particle velocity of each generalized ray consist of a single integral over a real, finite range, the integrand of which is an algebraic function of a certain complex ray parameter. The so-called modified Cagniard path maps this complex ray parameter in a one-to-one way to the real time parameter. For large horizontal and vertical offsets, simplified asymptotic representations for the space-time Green's function can be derived (see ROEVER, VINING, and STRICK [12], WIGGINS and HELMBERGER [15], and DE HOOP [16]).

2. Description of the configuration

We investigate theoretically the acoustic wave motion in a layered fluid the mechanical properties of which vary in a single rectilinear direction in space only. This direction is taken to be the vertical one. To specify position in the configuration we employ the coordinates $\{x_1,x_2,x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with the origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. In accordance with the geophysical conventions, $\mathbf{i}_3$ points vertically downwards. The subscript notation for Cartesian vectors and tensors is used. Lower-case Latin subscripts are used for this purpose; they are to be assigned the values 1, 2 and 3. Further, the summation convention applies to repeated subscripts. Whenever appropriate, the position is also specified by the position vector $\mathbf{x}=x_p i_p$, with $\mathbf{x}\in\mathbb{R}^3$. The time coordinate is denoted by t, with $t\in\mathbb{R}$. Partial differentiation with respect to x_p is denoted by ∂_p; the symbol ∂_t is reserved for partial differentiation with respect to t. SI-units are used throughout.

The acoustic properties of the fluid are characterized by its volume density of mass ρ (which is characteristic for the inertia properties of the fluid) and its compressibility κ (which is characteristic for the elastic compliance of the fluid). Both ρ and κ are functions of x_3. They are independent of x_1, x_2, or t, which makes the configuration shift invariant in the horizontal x_1,x_2-plane, and time invariant as well. The real-valued functions $\rho=\rho(x_3)$ and $\kappa=\kappa(x_3)$ are taken to be positive and piecewise constant. The acoustic wave speed associated with ρ and κ is given by $c=(\rho\kappa)^{-1/2}$. Their values in the different subdomains are listed in Table 1.

Table 1. Nomenclature of the configuration

Domain	Vertical coordinate	Volume density of mass	Compress-ibility	Wave speed
D_1	$-\infty<x_3<x_{3;1}$	ρ_1	κ_1	c_1
...	...	...	...	...
D_S	$x_{3;S-1}<x_3<x_{3;S}$	ρ_S	κ_S	c_S
D_{S+1}	$x_{3;S}<x_3<x_{3;S+1}$	ρ_{S+1}	κ_{S+1}	c_{S+1}
...	...	...	...	...
D_{ND}	$x_{3;ND-1}<x_3<\infty$	ρ_{ND}	κ_{ND}	c_{ND}

An impulsive point source, located at $\{0,0,x_{3;S}\}$, generates the acoustic waves. For our further analysis it turns out to be advantageous to introduce an interface at the level $x_3=x_{3;S}$, even if the source is located in the interior of one of the domains. The relevant level has been included already in the nomenclature listed in Table 1. The source starts to act at the instant $t=0$. To determine the wave motion that is causally related to the action of the source, we put the state quantities describing this wave motion equal to zero in the time interval $t<0$ (initial condition). Figure 1 shows schematically the configuration and the location of the source.

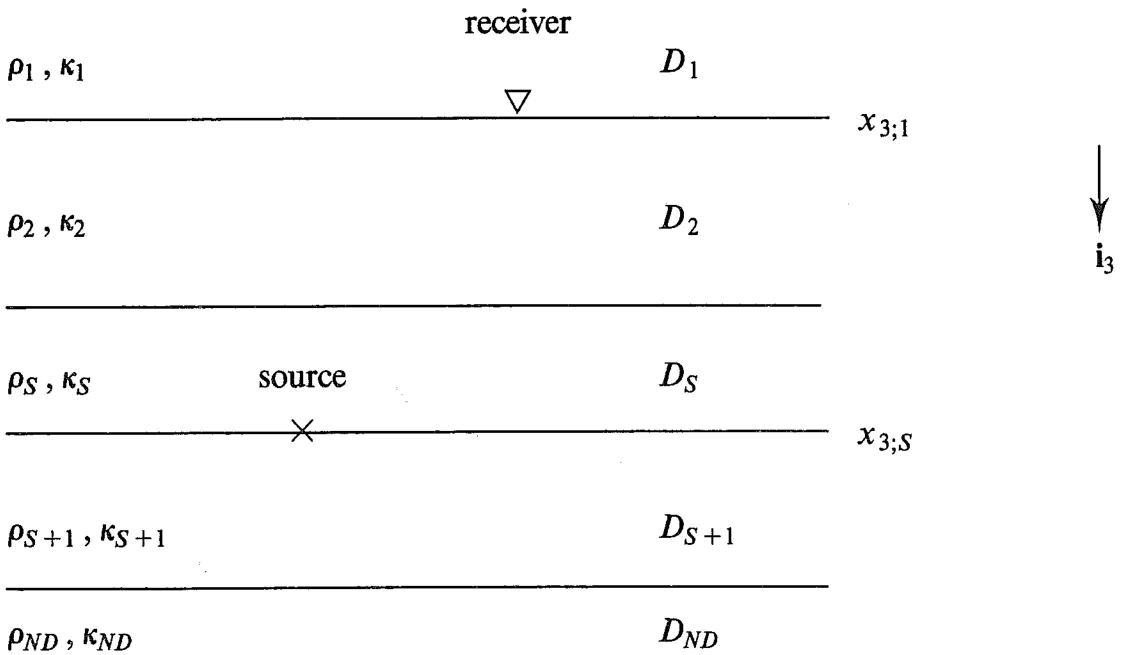

FIGURE 1.
Horizontally layered fluid medium in which acoustic waves are generated by an impulsive point source

3. Basic equations for the acoustic wave motion

The state quantities that characterize the acoustic wave motion are the acoustic pressure p and the particle velocity v_k. In a subdomain of the configuration where ρ and κ vary continuously with position, p and v_k are continuously differentiable functions of $\mathbf{x}$ and t, and they satisfy the acoustic wave equations

$$\kappa\, \partial_t p + \partial_r v_r = q \tag{3.1}$$

$$\partial_k p + \rho\, \partial_t v_k = f_k \tag{3.2}$$

in which q is the volume source density of injection rate and f_k is the volume source density of force. The source densities are assumed to be piecewise continuous functions of $\mathbf{x}$ and t. At the interfaces where ρ and/or κ show a jump discontinuity, the pressure and the normal component of the particle velocity are assumed to be continuous.

For the specific case of a point source located at $\{0,0,x_{3;S}\}$ we have

$$\{q,f_k\} = \{Q(t), F_k(t)\}\delta(x_1,x_2,x_3-x_{3;S}) \tag{3.3}$$

where $\{Q(t),F_k(t)\}$ characterize the strength of the source as a function of time and where δ denotes the three-dimensional unit pulse (Dirac distribution). It is understood that $\{Q(t),F_k(t)\}=0$ when $t<0$. When $Q\neq 0$ and $F_k=0$, the source is of the volume-injection type; such a source is also denoted as an acoustic monopole source. When $Q=0$ and $F_k\neq 0$, the source is of the force type; such a source is also denoted as an acoustic dipole source.

To solve equations (3.1) - (3.3) and their accompanying boundary conditions in the configuration discussed in Section 2, we subject them to a succession of integral transformations that are compatible with the time invariance of the configuration and its shift invariance in the horizontal plane.

4. The transform-domain acoustic wave equations

First, the acoustic state quantities and the volume source densities are subjected to a one-sided Laplace transformation with respect to time over the range $t>0$. To show the notation we give the transformation of the acoustic pressure:

$$\hat{p}(\mathbf{x},s) = \int_{t=0}^{\infty} \exp(-st)p(\mathbf{x},t)dt. \tag{4.1}$$

In (4.1), the Laplace-transform parameter s is taken to be real and sufficiently large positive. Taking $\{Q(t),F_k(t)\}$ to be at most of order $\exp(s_0t)$ as $t\to\infty$, it can be shown that also $\{p(\mathbf{x},t),v_k(\mathbf{x},t)\}$ are at most of this order, and the corresponding right-hand sides of (4.1) exist for $s>s_0$. Considering (4.1) as an integral equation for the space-time quantities, given their Laplace transforms, the solution of this integral equation is unique and vanishes in the interval $t<0$ on account of Lerch's theorem (Widder [14]). Due to this property, the selection of the causal quantities in the $(\mathbf{x},t)$-domain is brought down to selecting their bounded counterparts in the $(\mathbf{x},s)$-domain.

Next, we carry out a Fourier transformation with respect to the horizontal space coordinates over their entire range $(x_1,x_2)\in\mathbb{R}^2$. Again, to show the notation, we give the expression for the acoustic pressure:

$$\tilde{p}(\alpha_1,\alpha_2,x_3,s) = \int_{x_2=-\infty}^{\infty} dx_2 \int_{x_1=-\infty}^{\infty} \exp[is(\alpha_1x_1+\alpha_2x_2)]\,\hat{p}(x_1,x_2,x_3,s)dx_1. \tag{4.2}$$

In (4.2), the Fourier-transform parameters $\{\alpha_1,\alpha_2\}$ are taken to be real, and i is the imaginary unit. Note that in terms of the standard Fourier transformation, $\{s\alpha_1,s\alpha_2\}$ are the Fourier-transform parameters. As a consequence, the inverse transformation is given by

$$\hat{p}(x_1,x_2,x_3,s) = (s/2\pi)^2 \int_{\alpha_1=-\infty}^{\infty} d\alpha_1\cdot \int_{\alpha_2=-\infty}^{\infty} \exp[-is(x_1\alpha_1+x_2\alpha_2)]\,\tilde{p}(\alpha_1,\alpha_2,x_3,s)d\alpha_2. \tag{4.3}$$

Applying the transformations (4.1) and (4.2) to equations (3.1) and (3.2), taking into account that $\partial_1\to -is\alpha_1$, $\partial_2\to -is\alpha_2$, and $\partial_t\to s$, and eliminating $\tilde{v}_1$

and $\tilde{v}_2$ from the resulting equations, we end up with

$$s\gamma Y\tilde{p} + \partial_3\tilde{v}_3 = \tilde{q}' \tag{4.4}$$

$$\partial_3\tilde{p} + s\gamma Y^{-1}\tilde{v}_3 = \tilde{f}' \tag{4.5}$$

in which

$$\gamma = (1/c^2+\alpha_1^2+\alpha_2^2)^{1/2}>0 \tag{4.6}$$

is the vertical slowness,

$$c = (\kappa\rho)^{-1/2}>0 \tag{4.7}$$

is the acoustic wave speed,

$$Y = \gamma/\rho \tag{4.8}$$

is the vertical acoustic wave admittance, and

$$\tilde{q}' = \tilde{q}+\rho^{-1}(i\alpha_1\tilde{f}_1+i\alpha_2\tilde{f}_2) \tag{4.9}$$

$$\tilde{f}' = \tilde{f}_3 \tag{4.10}$$

are the transform-domain 'notional source distributions' of force and volume injection, respectively.

Equations (4.4) and (4.5) are the transform-domain acoustic wave equations. They constitute a system of ordinary differential equations with x_3 as independent variable and $\{\tilde{p},\tilde{v}_3\}$ as dependent variables. At those interfaces between two adjacent domains with different mechanical properties that are source-free, the quantities $\tilde{p}$ and $\tilde{v}_3$ are assumed to be continuous. Since, further, the transform-domain source distributions follow from (3.3) as

$$\{\tilde{q},\tilde{f}_k\} = \{\hat{Q},\hat{F}_k\}\delta(x_3-x_{3;S}) \tag{4.11}$$

the boundary conditions at the interface $x_3=x_{3;S}$ are on account of (4.4) and (4.5)

$$\lim_{x_3\downarrow x_{3;S}} \tilde{v}_3 - \lim_{x_3\uparrow x_{3;S}} \tilde{v}_3 = \tilde{Q}' \tag{4.12}$$

$$\lim_{x_3\downarrow x_{3;S}} \tilde{p} - \lim_{x_3\uparrow x_{3;S}} \tilde{p} = \tilde{F}' \tag{4.13}$$

in which (cf. (4.9) - (4.11))

$$\tilde{Q}' = \hat{Q} + \rho^{-1}(i\alpha_1\hat{F}_1+i\alpha_2\hat{F}_2) \tag{4.14}$$

$$\tilde{F}' = \hat{F}_3 \tag{4.15}$$

are the transform-domain 'notional strengths' of the point source at $\{0,0,x_{3;S}\}$.

Equations (4.4), (4.5), (4.12) and (4.13) will be solved by expanding, in each homogeneous source-free subdomain of the configuration, $\{\tilde{p},\tilde{v}_3\}$ in terms of transform-domain wave constituents that in space-time correspond to down- and upgoing waves.

5. The transform-domain acoustic wave

In the homogeneous source-free subdomain D_N of the configuration, we write

$$\{\tilde{p},\tilde{v}_3\} = \{\tilde{p}_N, \tilde{v}_{3;N}\} = \{\tilde{p}_N^+ + \tilde{p}_N^-, \tilde{v}_{3;N}^+ + \tilde{v}_{3;N}^-\} \text{ with } N\in(1,...,ND\} \tag{5.1}$$

where (cf. (4.4) and (4.5))

$$\tilde{p}_N^+ = W_N^+ \exp[-s\gamma_N(x_3 - x_{3;N-1})] \tag{5.2}$$

$$\tilde{p}_N^- = W_N^- \exp[-s\gamma_N(x_{3;N} - x_3)] \tag{5.3}$$

$$\tilde{v}_{3;N}^\pm = \pm Y_N \tilde{p}_N^\pm \tag{5.4}$$

with (cf. (4.6) - (4.8))

$$\gamma_N = (1/c_N^2 + \alpha_1^2 + \alpha_2^2)^{1/2} > 0 \tag{5.5}$$

$$c_N = (\kappa_N \rho_N)^{-1/2} > 0 \tag{5.6}$$

$$Y_N = \gamma_N / \rho_N. \tag{5.7}$$

The wave $\{\tilde{p}_N^+, \tilde{v}_{3;N}^+\}$ propagates away from the interface $x_3 = x_{3;N-1}$ in the direction of increasing x_3 (and hence downwardly), whereas the wave $\{\tilde{p}_N^-, \tilde{v}_{3;N}^-\}$ propagates away from the interface $x_3 = x_{3;N}$ in the direction of decreasing x_3 (and hence upwardly). Since both $s>0$ and $\gamma_N>0$, the right-hand side of (5.2) stays bounded as $x_3 \to \infty$ and the right-hand side of (5.3) stays bounded as $x_3 \to -\infty$. Hence, the causality condition is satisfied. The quantities $\{W_N^+, W_N^-\}$ are the transform-domain wave amplitudes of the down- and upgoing waves, respectively. In order to satisfy the causality condition in the outer half-spaces, we have $W_1^+ = 0$ and $W_{ND}^- = 0$.

The different wave amplitudes are interrelated by the boundary conditions at the interfaces. At a source-free interface we express this relationship via a scattering description, whereby the amplitudes of the waves propagating away from the interface are expressed in terms of the amplitudes of the waves propagating toward the interface through the scattering matrix. At the interface where the source is located, the scattering description is supplemented by excitation terms that are related to the notional source strengths. Let, accordingly, for any $N\in\{1,...,ND-1\}$,

$$W_N^- = S_N^{-,+} \overline{W}_N^+ + S_N^{-,-} \overline{W}_{N+1}^- + X_N^- \tag{5.8}$$

$$W_{N+1}^+ = S_N^{+,+} \overline{W}_N^+ + S_N^{+,-} \overline{W}_{N+1}^- + X_{N+1}^+ \tag{5.9}$$

where

$$\overline{W}_N^\pm = W_N^\pm \exp(-s\gamma_N d_N) \text{ for any } N\in\{2,...,ND-1\} \tag{5.10}$$

$$\overline{W}_1^+ = 0 \tag{5.11}$$

$$\overline{W}_{ND}^- = 0 \tag{5.12}$$

denote the so-called modified wave amplitudes, and

$$d_N = x_{3;N} - x_{3;N-1} \text{ with } N\in\{2,...,ND-1\} \tag{5.13}$$

is the thickness of the layer occupying the domain D_N. For our single source located at $\{0,0,x_{3;S}\}$, only X_S^- and X_{S+1}^+ differ from zero. Substitution of (5.8) and (5.9) in the boundary conditions (cf. (4.12) and (4.13))

$$\lim_{x_3 \downarrow x_{3;N}} \tilde{v}_{3;N+1} - \lim_{x_3 \uparrow x_{3;N}} \tilde{v}_{3;N} = \tilde{Q}'_N \tag{5.14}$$

$$\lim_{x_3 \downarrow x_{3;N}} \tilde{p}_{N+1} - \lim_{x_3 \uparrow x_{3;N}} \tilde{p}_N = \tilde{F}'_N \tag{5.15}$$

leads, with the aid of (5.2) - (5.4) to

$$S_N^{-,+} = (Y_N - Y_{N+1})/(Y_N + Y_{N+1}) \tag{5.16}$$

$$S_N^{-,-} = 2Y_{N+1}/(Y_{N+1} + Y_N) \tag{5.17}$$

$$S_N^{+,+} = 2Y_N/(Y_N + Y_{N+1}) \tag{5.18}$$

$$S_N^{+,-} = (Y_{N+1} - Y_N)/(Y_{N+1} + Y_N) \tag{5.19}$$

and

$$X_S^- = (\tilde{Q}'_S - Y_{S+1}\tilde{F}'_S)/(Y_S + Y_{S+1}) \tag{5.20}$$

$$X_{S+1}^+ = (\tilde{Q}'_S + Y_S\tilde{F}'_S)/(Y_{S+1} + Y_S). \tag{5.21}$$

The expressions (5.16) and (5.19) are reflection coefficients and the expressions (5.17) and (5.18) are transmission coefficients at the interface $x_3 = x_{3;N}$.

To describe the overall behaviour of the structure, we introduce a matrix formalism, in which the transform-domain wave amplitudes are arranged in the $(2 \times ND - 2)$-by-1 wave matrix $[W]$. Next, we introduce the modified scattering coefficients as

$$\bar{S}_N^{-,+} = S_N^{-,+} \exp(-\gamma_N d_N) \tag{5.22}$$

$$\bar{S}_N^{-,-} = S_N^{-,-} \exp(-\gamma_{N+1} d_{N+1}) \tag{5.23}$$

$$\bar{S}_N^{+,+} = S_N^{+,+} \exp(-\gamma_N d_N) \tag{5.24}$$

$$\bar{S}_N^{+,-} = S_N^{+,-} \exp(-\gamma_{N+1} d_{N+1}) \tag{5.25}$$

and arrange the modified scattering coefficients in the $2(ND-1)$-by-$2(ND-1)$ modified scattering matrix $[\bar{S}]$. Finally, the excitation amplitudes are arranged in the $2(ND-1)$-by- 1 matrix $[X]$, in which in our case only the elements X_S^- and X_{S+1}^+ differ from zero. The relevant arrangements follow from (5.8) and (5.9) as shown in Table 2 (Cisternas, Betancourt and Leiva [4]). The equation in Table 2 is solved by a Neumann iteration. Carrying out Γ steps, with $\Gamma \geqslant 0$, we have

$$[W] = \sum_{\gamma=0}^{\Gamma} [\bar{S}]^\gamma [X] + [\bar{S}]^{\Gamma+1}[W] \tag{5.26}$$

as is easily verified. Now, it is observed that each element of $[\bar{S}]$ contains an exponential factor of the type occurring in the expressions (5.22) - (5.25). In the $(\mathbf{x},t)$-domain these factors correspond to a non-vanishing time delay. Now, in practice, one only observes the acoustic wave motion in some finite time interval, the so-called time window of observation. From a certain value of Γ onward, the last term in (5.26) yields only contributions with a time delay that exceeds the duration of the time window of observation, and the relevant contribution can be ignored.

TABLE 2. Wave-matrix formalism for the layered fluid

$$
\begin{bmatrix} W_1^- \\ W_2^+ \\ \cdots \\ W_S^- \\ W_{S+1}^+ \\ \cdots \\ W_{ND-1}^- \\ W_{ND}^+ \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & \bar{S}_1^{-,-} & & & \\
0 & 0 & \bar{S}_1^{+,-} & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \bar{S}_S^{-,+} & 0 & 0 & \bar{S}_S^{-,-} & \\
 & \bar{S}_S^{+,+} & 0 & 0 & \bar{S}_S^{+,-} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & \bar{S}_{ND-1}^{-,+} & 0 & 0 \\
 & & & \bar{S}_{ND-1}^{+,+} & 0 & 0
\end{bmatrix}
\begin{bmatrix} W_1^- \\ W_2^+ \\ \cdots \\ W_S^- \\ W_{S+1}^+ \\ \cdots \\ W_{ND-1}^- \\ W_{ND}^+ \end{bmatrix}
+
\begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ X_S^- \\ X_{S+1}^+ \\ \cdots \\ \cdots \\ \cdots \end{bmatrix}
$$

Each element of $[\bar{S}]^\gamma[X]$ contains γ interactions at the different interfaces; the value $\gamma=0$ yields the direct excitation of the acoustic wave at the source level. We note that $[\bar{S}]^\gamma$ contains the Laplace-transform parameter s only through the exponential functions occurring in (5.22) - (5.25). Further, $[X]$ contains s only via the source strengths. These properties play a fundamental role in the transformation back to the $(\mathbf{x},t)$-domain via the modified Cagniard method.

After the elements of the wave matrix have been determined, the transform-domain expressions for $\tilde{p}_N$ and $\tilde{v}_{3;N}$ follow from (5.2) - (5.4), while $\tilde{v}_{1;N}$ and $\tilde{v}_{2;N}$ are, on account of (3.2), (4.1) and (4.2), expressed in terms of $\tilde{p}_N$ as

$$\tilde{v}_{1;N} = \rho_N^{-1}\, i\alpha_1\, \tilde{p}_N \tag{5.27}$$

$$\tilde{v}_{2;N} = \rho_N^{-1}\, i\alpha_2\, \tilde{p}_N. \tag{5.28}$$

With this, the determination of the transform-domain acoustic wave quantities has been completed. Through the wave-vector formalism, they are expressed as a superposition of wave constituents that are generated by the source and then undergo successive reflections and transmissions at the interfaces of the layered structure. Each element of the term $[\bar{S}]^\gamma[X]$ in the right-hand side of (5.26) is denoted as a generalized-ray constituent of order γ. Now, the transformation back to the space-time domain with the aid of the modified Cagniard method typically applies to the generalized-ray wave constituents. The relevant steps will be discussed in subsequent sections. As the general form of a generalized-ray constituent we take

$$\tilde{w} = \hat{Q}(s)\Pi(\alpha_1,\alpha_2)\exp[-s\sum_{\lambda\in\Lambda}\gamma_\lambda(\alpha_1,\alpha_2)h_\lambda] \tag{5.29}$$

where $\hat{Q}=\hat{Q}(s)$ is representative for the signature of the source, $\Pi=\Pi(\alpha_1,\alpha_2)$ is the (s-independent) factor that describes the coupling of the generalized-ray constituent to the source as well as the reflections and transmissions it has undergone at the interfaces, h_λ is the total (possible multiple) vertical path length that the generalized ray has traversed in the domain D_λ, γ_λ is the vertical slowness in D_λ, and $\Lambda\in\{1,...,ND\}$.

6. The transformation back to the space-time domain

The first step in the transformation back to the space-time domain consists of applying (4.3) to (5.29). This yields

$$\hat{w}(\mathbf{x},s) = (s/2\pi)^2\hat{Q}(s) \int_{\alpha_1=-\infty}^{\infty} d\alpha_1 \cdot$$

$$\int_{\alpha_2=-\infty}^{\infty} \Pi(\alpha_1,\alpha_2)\exp[-s(i\alpha_1 x_1 + i\alpha_2 x_2 + \sum_{\lambda\in\Lambda} \gamma_\lambda(\alpha_1,\alpha_2)h_\lambda)]d\alpha_2. \tag{6.1}$$

Next, we rewrite (6.1) as

$$\hat{w}(\mathbf{x},s) = s^2\hat{Q}(s)\hat{G}(\mathbf{x},s) \tag{6.2}$$

where

$$\hat{G}(\mathbf{x},s) = (2\pi)^{-2} \int_{\alpha_1=-\infty}^{\infty} d\alpha_1 \cdot$$

$$\int_{\alpha_2=-\infty}^{\infty} \Pi(\alpha_1,\alpha_2)\exp[-s(i\alpha_1 x_1 + i\alpha_2 x_2 + \sum_{\lambda\in\Lambda} \gamma_\lambda(\alpha_1,\alpha_2)h_\lambda)]d\alpha_2 \tag{6.3}$$

is by definition the Green's function of the generalized ray. The modified Cagniard method now aims at rewriting, somehow, (6.3) as

$$\hat{G}(\mathbf{x},s) = \int_{\tau=T}^{\infty} \exp(-s\tau)w^G(\mathbf{x},\tau)d\tau, \tag{6.4}$$

where τ is a real variable of integration, $T>0$, and where $w^G(\mathbf{x},\tau)$ does not depend on s. Suppose that this has been achieved, then the uniqueness theorem of the Laplace transformation with real, positive, transform parameter (Lerch's theorem; see Widder [14]) ensures that

$$G(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T, \\ w^G(\mathbf{x},t) & \text{when } t>T. \end{cases} \tag{6.5}$$

Additional properties of the Laplace transformation then lead to the final space-time result

$$w(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T. \\ \partial_t^2 \int_{\tau=T}^{t} Q(t-\tau)w^G(\mathbf{x},\tau)d\tau & \text{when } t>T. \end{cases} \tag{6.6}$$

Equation (6.6) shows that T is the arrival time of the generalized-ray constituent at the point of observation.

In rewriting (6.3) in the form (6.4) several steps are carried out; they are characteristic for the modified Cagniard method. First, the variables of integration $\{\alpha_1,\alpha_2\}$ in (6.3) are transformed into $\{p,q\}$ via the substitution

$$\alpha_1 = -ip\cos(\phi) - q\sin(\phi) \tag{6.7}$$

$$\alpha_2 = -ip\sin(\phi) + q\cos(\phi) \tag{6.8}$$

where ϕ follows from the polar-coordinate specification of the point of observation in the horizontal plane, i.e. from

$$x_1 = r\cos(\phi), \quad x_2 = r\sin(\phi) \tag{6.9}$$

with $r\geqslant 0$ and $0\leqslant\phi<2\pi$, $p\in I$ and $q\in\mathbb{R}$. Since $d\alpha_1 d\alpha_2 = i^{-1}dpdq$ and $i\alpha_1 x_1 + i\alpha_2 x_2 = pr$, we have

$$\hat{G}(\mathbf{x},s) = (4\pi^2 i)^{-1}\int_{q=-\infty}^{\infty} dq \int_{p=-i\infty}^{i\infty} \overline{\Pi}(p,q)\exp[-s(pr+\sum_{\lambda\in\Lambda}\overline{\gamma}_\lambda(p,q)h_\lambda)]dp \tag{6.10}$$

where $\overline{\Pi}$ results from Π and $\overline{\gamma}_\lambda$ from γ_λ under the substitution (6.7) - (6.8). In the integration with respect to p we next continue the integrand analytically into the complex p-plane, away from the imaginary axis. In this process we encounter the singularities in $\overline{\Pi}$ and $\overline{\gamma}_\lambda$. In view of (5.16) - (5.21) and (5.5), these are the branch points of $\overline{\gamma}_\lambda$ i.e. $p=\pm S_\lambda(q)$, where

$$S_\lambda = (1/c_\lambda^2 + q^2)^{1/2} > 0. \tag{6.11}$$

To make the analytic continuation single-valued, we introduce branch cuts along $\mathrm{Im}(p)=0$, $|\mathrm{Re}(p)|\geqslant S_\lambda$. In the cut p-plane, we then have $\mathrm{Re}(\gamma_\lambda)>0$. Further, it can be shown that the denominators in (5.16) - (5.21) never vanish in the finite part of the p-plane; hence, no poles occur.

Keeping q real, the integration in the complex p-plane is now carried out along a path along which

$$pr + \sum_{\lambda\in\Lambda}\overline{\gamma}_\lambda(p,q)h_\lambda = \tau \tag{6.12}$$

where τ is real and positive; such a path is denoted as a modified Cagniard path. Since $r\geqslant 0$, and $\mathrm{Im}(\overline{\gamma}_\lambda)<0$ and >0 in the upper and lower halves of the complex p-plane, respectively, the modified Cagniard paths are located in the right-half of the p-plane. It is clear that the part of the real p-axis $0\leqslant\mathrm{Re}(p)<\min_{\lambda\in\Lambda}[S_\lambda(q)]$, $\mathrm{Im}(p)=0$ satisfies (6.12). Further, there is a complex path that satisfies (6.12); it has the asymptotic representation

$$p \sim \tau/(r\mp i\sum_{\lambda\in\Lambda}h_\lambda) \quad \text{as } \tau\to\infty \tag{6.13}$$

in the upper/lower half of the p-plane. This part is denoted as the body-wave path; its representation in the first quadrant of the p-plane will be denoted as

$p=p^B(\tau,q)$. Since the left-hand side of (6.12) satisfies Schwarz's reflection principle, the representation in the fourth quadrant is then $p=p^{B*}(\tau,q)$, where $*$ denotes complex conjugation. The point of intersection of $p=p^B$ with the real p-axis follows from the consideration that at that point τ attains its minimum value. Let $p=p_0(q)$ denote the relevant (real) value of p, then we have

$$r - p \sum_{\lambda\in\Lambda}(h_\lambda/\overline{\gamma}_\lambda) = 0 \quad \text{at}\, p = p_0(q). \tag{6.14}$$

Since $p=p_0(q)$ is necessarily located between $p=0$ and each of the branch points $p=S_\lambda(q)$ (all $\overline{\gamma}_\lambda$ with $\lambda\in\Lambda$ must be real at p_0), we can write

$$p_0(q) = S_\lambda(q)\sin[\theta_\lambda(q)] \text{ for all } \lambda\in\Lambda, \tag{6.15}$$

with $0\leqslant\theta_\lambda(q)\leqslant\pi/2$. Since, then,

$$\overline{\gamma}_\lambda = S_\lambda(q)\cos[\theta_\lambda(q)] \text{ at}\, p = p_0 \tag{6.16}$$

equation (6.14) can be rewritten as

$$r - \sum_{\lambda\in\Lambda} h_\lambda \tan[\theta_\lambda(q)] = 0 \quad \text{at}\, p = p_0(q). \tag{6.17}$$

Let $\tau=T^B(q)$ at $p=p_0(q)$, then T^B follows from (6.12) as

$$T^B(q) = \sum_{\lambda\in\Lambda}\{S_\lambda(q)h_\lambda/\cos[\theta_\lambda(q)]\}. \tag{6.18}$$

In the first instance we replace the integration along $p\in I$ in (6.10) by an integration along $p=p^B(\tau,q)$ and $p=p^{B*}(\tau,q)$. In view of Cauchy's theorem, Jordan's lemma, and the properties of $p=p^B(\tau,q)$, this is admissible provided that $\overline{\Pi}=\overline{\Pi}(p,q)$ has either no other singularities than $\{p=S_\lambda(q), \lambda\in\Lambda\}$ or additional branch points (due to reflection against a layer in which the generalized-ray constituent under consideration does not propagate) that are outside the range $0\leqslant p\leqslant\min_{\lambda\in\Lambda}[S_\lambda(q)]$. The generalized-ray constituent under consideration then only contains a body-wave part. If $\overline{\Pi}=\overline{\Pi}(p,q)$ does have an additional branch point in the indicated range, the body-wave part of the modified Cagniard path must, depending on the location of the point of observation with respect to the source, be supplemented by a loop integral around the branch cut belonging to this branch point and joining the points $p=p_0(q)-i0$ and $p=p_0(q)+i0$, where $p=p^{B*}(\tau,q)$ and $p=p^B(\tau,q)$, respectively, were tempted to cross the real p-axis. This part of the path of integration is denoted as the head-wave part, and its contribution to the generalized-ray constituent is called the head-wave contribution. The body-wave and the head-wave contributions to the generalized-ray constituent will be investigated in Sections 7 and 8, respectively.

7. Body-wave contribution to a generalized-ray constituent

The body-wave contribution to a generalized-ray Green's function follows from (6.10) and (6.12) as

$$\hat{G}^B(\mathbf{x},s) = (2\pi^2)^{-1} \int_{q=-\infty}^{\infty} dq \int_{\tau=T^B(q)}^{\infty} \exp(-s\tau)\mathrm{Im}[\overline{\Pi}(p^B,q)(\partial p^B/\partial\tau)]d\tau \tag{7.1}$$

where the parts along $p=p^B(\tau,q)$ and $p^{B*}(\tau,q)$ have been taken together, Schwarz's reflection principle (that also applies to $\overline{\Pi}$) has been used, and where the Jacobian of the transformation from p^B to τ follows from (6.12) as

$$\partial p^B/\partial\tau = [r - p^B \sum_{\lambda\in\Lambda}(h_\lambda/\overline{\gamma}_\lambda)]^{-1}. \tag{7.2}$$

Next, we interchange in the right-hand side of (7.1) the order of the integrations. This yields

$$\hat{G}^B(\mathbf{x},s) = \int_{\tau=T^B(0)}^{\infty} \exp(-s\tau)d\tau(2\pi^2)^{-1} \int_{q=-Q^B(\tau)}^{Q^B(\tau)} \mathrm{Im}[\overline{\Pi}(p^B,q)(\partial p^B/\partial\tau)]dq \tag{7.3}$$

in which $q=Q^B(\tau)$ is the inverse of the mapping $\tau=T^B(q)$ for $q\geqslant 0$. (Note that, in view of (6.12), $T^B(q)$ is an even function of q, while differentiation of the left-hand side with respect to q shows that $\partial_q T^B(q)>0$ if $q>0$). Now, (7.3) is of the form of (6.4) and hence

$$w^{G;B}(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T^B(0), \\ (2\pi^2)^{-1} \displaystyle\int_{q=-Q^B(\tau)}^{Q^B(\tau)} \mathrm{Im}[\overline{\Pi}(p^B,q)(\partial p^B/\partial\tau)]dq & \text{when } t>T^B(0). \end{cases} \tag{7.4}$$

With this, the body-wave contribution to the generalized-ray constituent follows as

$$w^B(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T^B(0), \\ \partial_t^2 \displaystyle\int_{\tau=T^B(0)}^{t} Q(t-\tau)w^{G;B}(\mathbf{x},\tau)d\tau & \text{when } t>T^B(0). \end{cases} \tag{7.5}$$

8. Head-wave contribution to a generalized-ray constituent

Let us consider the head-wave contribution that is due to the occurrence of $\overline{\gamma}_\mu=\overline{\gamma}_\mu(p,q)$ in $\overline{\Pi}=\overline{\Pi}(p,q)$, where $\mu\in\{1,\ldots,ND\}$, but $\mu\notin\Lambda$. Then, the integral along the body-wave contour must be supplemented by a loop integral around the branch cut associated with $\overline{\gamma}_\mu$ and joining the points $p_0(q)-i0$ and $p_0(q)+i0$ where $p=p^{B*}(\tau,q)$ and $p=p^B(\tau,q)$, respectively, were tempted to cross the real axis. Along this loop, too, the parametrization (6.12) has to be carried out. The relevant values of p in the upper and lower halves of the p-

plane will be denoted by $p=p^H(\tau,q)$ and $p=p^{H*}(\tau,q)$, respectively. Let $\tau=T^H(q)$ denote the value of τ at $p=S_\mu(q)$, then (6.12) leads to

$$T^H(q) = S_\mu(q)r + \sum_{\lambda\in\Lambda} [S_\lambda^2(q) - S_\mu^2(q)]^{1/2}h_\lambda. \tag{8.1}$$

Now, for a head-wave contribution to occur, we must have $S_\mu(q)<p_0(q)$, or, using (6.15),

$$S_\mu(q)<S_\lambda(q)\sin[\theta_\lambda(q)] \text{ with } \lambda\in\Lambda. \tag{8.2}$$

For those values of q where (8.2) is satisfied, we introduce the 'critical angles' $\{\theta_\lambda^\mu(q);\lambda\in\Lambda\}$, with $0\leqslant\theta_\lambda^\mu(q)\leqslant\pi/2$, through

$$\sin[\theta_\lambda^\mu(q)] = S_\mu(q)/S_\lambda(q). \tag{8.3}$$

Then (8.2) implies

$$\sin[\theta_\lambda(q)]>\sin[\theta_\lambda^\mu(q)] \tag{8.4}$$

which in its turn implies

$$\tan[\theta_\lambda(q)]>\tan[\theta_\lambda^\mu(q)]. \tag{8.5}$$

Using (8.5) in combination with (6.17), the condition $S_\mu(q)<p_0(q)$ is then equivalent to

$$r> \sum_{\lambda\in\Lambda} h_\lambda \tan[\theta_\lambda^\mu(q)], \tag{8.6}$$

or

$$r> \sum_{\lambda\in\Lambda} h_\lambda \frac{S_\mu(q)/S_\lambda(q)}{[1-S_\mu^2(q)/S_\lambda^2(q)]^{1/2}}. \tag{8.7}$$

When $q=0$, (8.7) reduces to

$$r> \sum_{\lambda\in\Lambda} h_\lambda \frac{c_\lambda/c_\mu}{(1-c_\lambda^2/c_\mu^2)^{1/2}} \tag{8.8}$$

which is the condition for 'total reflection' against an interface of the layer D_μ in accordance with Snell's law of refraction at the other interfaces. Further, since

$$S_{\lambda,\mu}(q) \sim |q| + \mathcal{O}(q^{-1}) \text{ as } |q|\to\infty \tag{8.9}$$

equation (6.15) leads to

$$\theta_\lambda(q) \sim \Theta + \mathcal{O}(q^{-1}) \text{ as } |q|\to\infty \tag{8.10}$$

and hence

$$p_0(q) \sim |q| \sin(\Theta) + \mathcal{O}(q^{-1}) \text{ as } |q|\to\infty. \tag{8.11}$$

Since $\sin(\Theta)<1$, we always have $p_0(q)<S_\mu(q)$ from a certain value of $|q|$ onward, and hence (8.2) is, beyond this value, no longer satisfied. Let (8.2) be satisfied in the range $-Q_\mu<q<Q_\mu$, then the head-wave contribution to the

generalized-ray Green's function can be written as

$$\hat{G}^{H}(\mathbf{x},s) = (2\pi^2)^{-1} \int_{q=-Q_\mu}^{Q_\mu} dq \int_{\tau=T^H(q)}^{T^B(q)} \exp(-s\tau)\mathrm{Im}[\overline{\Pi}(p^H,q)(\partial p^H/\partial\tau)]d\tau \qquad (8.12)$$

where the parts along $p=p^H(\tau,q)$ and $p=p^{H*}(\tau,q)$ have been taken together, Schwarz's reflection principle has been used, and where the Jacobian of the transformation from p^H to τ follows from (6.12) as

$$\partial p^H/\partial\tau = [r - p^H \sum_{\lambda\in\Lambda}(h_\lambda/\overline{\gamma}_\lambda)]^{-1}. \qquad (8.13)$$

Next, in the right-hand side of (8.12) we interchange the order of the integrations. This yields

$$\hat{G}^{H}(\mathbf{x},s) = \int_{\tau=T^H(0)}^{T^B(0)} \exp(-s\tau)d\tau\, (2\pi^2)^{-1} \int_{q=-Q^H(\tau)}^{Q^H(\tau)} \mathrm{Im}[\overline{\Pi}(p^H,q)(\partial p^H/\partial\tau)]dq \qquad (8.14)$$
$$+ \int_{\tau=T^B(0)}^{T_\mu} \exp(-s\tau)d\tau\, (2\pi^2)^{-1}\left[\int_{q=-Q^H(\tau)}^{-Q^B(\tau)} + \int_{q=Q^B(\tau)}^{Q^H(\tau)}\right]\mathrm{Im}[\overline{\Pi}(p^H,q)(\partial p^H/\partial\tau)]dq$$

in which $q=Q^H(\tau)$ is the inverse of the mapping $\tau=T^H(q)$ for $q\geqslant 0$. (Note that, in view of (6.12), $T^H(q)$ is an even function of q, while differentiation of the left-hand side with respect to q shows that $\partial_q T^H(q)>0$ if $q>0$.) In the τ,q-plane, the curves $\tau=T^B(q)$ and $\tau=T^H(q)$ have the point $q=Q_\mu$, $\tau=T_\mu$ in common. This follows from the definition of Q_μ and T_μ. At this point they have, however, also a common tangent, as follows by differentiation of (6.12) with respect to q and taking into account (6.14). Now, (8.14) is of the form of (6.4) and hence

$$w^{G;H}(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T^H(0), \\ (2\pi^2)^{-1} \displaystyle\int_{q=-Q^H(\tau)}^{Q^H(\tau)} \mathrm{Im}[\overline{\Pi}(p^H,q)(\partial p^H/\partial\tau)]dq & \text{when } T^H(0)<T^B(0), \\ (2\pi^2)^{-1}\left[\displaystyle\int_{q=-Q^H(\tau)}^{-Q^B(\tau)} + \int_{q=Q^B(\tau)}^{Q^H(\tau)}\right]\mathrm{Im}[\overline{\Pi}(p^H,q)(\partial p^H/\partial\tau)]dq & \\ & \text{when } T^B(0)<\tau<T_\mu. \end{cases} \qquad (8.15)$$

With this, the head-wave contribution to the generalized-ray constituent follows as

$$w^H(\mathbf{x},t) = \begin{cases} 0 & \text{when } t<T^H(0), \\ \partial_t^2 \displaystyle\int_{\tau=T^H(0)}^{t} Q(t-\tau)w^{G;H}(\mathbf{x},\tau)d\tau & \text{when } t>T^H(0). \end{cases} \qquad (8.16)$$

Note that, for (8.16) to occur, (8.8) must be satisfied, i.e. the horizontal offset between source and receiver must be large enough.

9. Numerical implementation

Except for the simplest case where in the summation in the exponential function in (5.29) only a single term is present, the modified Cagniard path must be determined with the aid of numerical methods. First, for each q and given r and $\{h_\lambda;\lambda\in\Lambda\}$, (6.14) is solved for p_0. Using (6.12), the mappings $\tau=T^B(q)$ and $q=Q^B(\tau)$ follow. Next, for each q and given r and $\{h_\lambda;\lambda\in\Lambda\}$, (6.12) is solved for p^B in the range $\tau>T^B(q)$, and this value is used in the body-wave contributions. If head-wave contributions are present, (8.1) is used to obtain the mappings $\tau=T^H(q)$ and $q=Q^H(\tau)$. Subsequently, (6.12) is used to construct the value of p^H that is to be used in the head-wave contribution. For the evaluation of the generalized-ray Green's functions, the integrations occurring in (7.4) and (8.15) have to be carried out numerically. This has to be done carefully because of the inverse square root singularity, due to $\partial p/\partial\tau$, at the end points of the interval of integration. The application of a local stretching procedure circumvents this difficulty. The final evaluation of the convolution integrals (7.5) and (8.16) presents usually no difficulty. Numerical results along these lines can be found in De Hoop and Van der Hijden [7,8,9], Van der Hijden [13], and Drijkoningen and Fokkema [10].

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APPENDIX

The modified cagniard method for a generalized ray that propagates in a single medium only

For a generalized-ray constituent that propagates in a single medium only, the summation in the exponential function in (5.29) contains a single term only. The modified Cagniard path then follows from an expression of the form

$$pr + [S(q)^2 - p^2]^{1/2}h = \tau \tag{A.1}$$

in which

$$S(q) = (1/c^2+q^2)^{1/2}>0. \tag{A.2}$$

The value of $p_0=p_0(q)$ then follows from (cf. (6.14))

$$r - p[S(q)^2 - p^2]^{-1/2}h = 0. \tag{A.3}$$

Equation (A.3) leads to

$$p_0 = rS(q)/(r^2+h^2)^{1/2}. \tag{A.4}$$

Substitution of (A.4) in (A.1) yields

$$T^B = T^B(q) = S(q)(r^2+h^2)^{1/2}. \tag{A.5}$$

Using (A.2), it follows from (A.5) that

$$Q^B = Q^B(\tau) = [\tau^2/(r^2+h^2) - 1/c^2]^{1/2}. \tag{A.6}$$

Solving p from (A.1), we obtain

$$p^B = \frac{\tau r+ih[\tau^2-T^B(q)^2]^{1/2}}{r^2+h^2} \tag{A.7}$$

from which the Jacobian of the transformation from p^B to τ is found as

$$\partial p^B/\partial\tau = \frac{r+ih\tau[\tau^2-T^B(q)^2]^{-1/2}}{r^2+h^2}. \tag{A.8}$$

Let, further, the presence of γ_μ in one of the reflection coefficients be responsible for the occurrence of a head wave. Then (cf. (8.1)),

$$T^H = T^H(q) = S_\mu(q)r + (1/c^2-1/c_\mu^2)^{1/2}h \tag{A.9}$$

in which

$$S_\mu = (1/c_\mu^2+q^2)^{1/2}. \tag{A.10}$$

Using (A.9) and (A.10), it follows that

$$Q^H = Q^H(\tau) = \{[\tau/r-(1/c^2-1/c_\mu^2)^{1/2}h/r]^2-1/c_\mu^2\}^{1/2}. \tag{A.11}$$

Solving p from (A.1), we now obtain

$$p^H = \frac{\tau r - h[T^B(q)^2-\tau^2]^{1/2}}{r^2+h^2} \tag{A.12}$$

from which the Jacobian of the transformation from p^H to τ is found as

$$\partial p^H/\partial\tau = \frac{r+h\tau[T^B(q)^2-\tau^2]^{-1/2}}{r^2+h^2}. \tag{A.13}$$

The end point of the q-interval in which a head-wave contribution is present, follows as

$$Q_\mu = [(1/c^2-1/c_\mu^2)r^2/h^2-1/c_\mu^2]^{1/2} \tag{A.14}$$

which leads to

$$T_\mu = (r^2+h^2)(1/c^2-1/c_\mu^2)^{1/2}/h. \tag{A.15}$$