Time-domain reciprocity theorems for acoustic wave fields in fluids with relaxation

Adrianus T. de Hoop
Delft University of Technology, Faculty of Electrical Engineering, Laboratory of Electromagnetic Research, P. O. Box 5031, 2600 GA Delft, The Netherlands

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Time-domain reciprocity theorems of the time-convolution and the time-correlation type for acoustic wave fields in linear, time-invariant, and locally reacting fluids are discussed. Inhomogeneity, inertial anisotropy, and arbitrary relaxation effects in inertia and compressibility properties, both of the active and the passive type, are included. The theorems also apply to the "equivalent fluid model" of a solid in which only compressional waves are considered and shear is neglected. The analysis is entirely carried out in space-time, without intermediate recourse to the frequency or the wave vector domains. The application to inverse source and inverse constituency (or inverse profiling/scattering, or imaging) problems is briefly indicated.

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INTRODUCTION

A wave field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two admissible physical states that could occur in one and the same domain in space-time. As far as acoustic wave fields are concerned, Lord Rayleigh[1] is commonly credited as the first to derive a reciprocity theorem; it applies to harmonic sound vibrations in a homogeneous, ideal fluid. (He denotes it as Helmholtz's theorem, but gives no reference to Helmholtz.) A clear distinction between convolution- and correlation-type reciprocity relations is made by Bojarski,[2] who has presented the time-domain reciprocity theorems for homogeneous, isotropic, and lossless media, both for acoustic wave fields and for electromagnetic fields. In this connection, he introduced the concept of an "effectual" (or "effectal") wave field as the time-reversed counterpart of a "causal" wave field, and emphasized the relationship between time-advanced and time-retarded wave fields. The application of reciprocity theorems to inverse scattering is reviewed by Fisher and Langenberg,[3] where an extensive list of references to earlier papers also can be found.

The present investigation deals with time-convolution and time-correlation reciprocity theorems for acoustic wave fields in time-invariant configurations that are linear and locally reacting in their acoustic behavior. As regards the space-time geometry in which the two admissible states occur, this implies that this geometry is the Cartesian product $D \times R$ of a time-invariant spatial domain $D \subset R^3$ and the real time axis R. Further, the constitutive parameters of the fluids present in the two states are time invariant and independent of the wave field values. No further restrictions are imposed.

The position of observation in R^3 is specified by the coordinates $\{x_1,x_2,x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors in R^3 is employed and the summation convention applies. The corresponding lowercase Latin subscripts are to be assigned the values $\{1,2,3\}$. Whenever appropriate, the positon vector will be denoted by $\mathbf{x} = x_p i_p$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂, ∂_p denotes differentiation with respect to x_p, and ∂_t denotes differentiation with respect to t.

The reciprocity theorems will be derived for bounded domains D. In the analysis the boundary ∂D of D also occurs, as well as the complement D' of the union of D and ∂D in R^3. The unit vector along the normal to ∂D is denoted by $\boldsymbol{\nu}$; it is oriented away from D (Fig. 1).

I. SOME PROPERTIES OF THE TIME CONVOLUTION AND THE TIME CORRELATION OF SPACE-TIME FUNCTIONS

In this section, we present the properties of the time convolution and the time correlation of space-time functions as far as they are needed in the derivation of the reciprocity theorems. Let $f_1 = f_1(\mathbf{x},t)$ and $f_2 = f_2(\mathbf{x},t)$ be two transient space-time functions. By this we mean that the functions are absolutely integrable on the entire $t \in R$. Then, the time convolution of f_1 and f_2 is defined as

$$C(f_1,f_2;\mathbf{x},\tau) = \int_{t\in R} f_1(\mathbf{x},t) f_2(\mathbf{x},\tau - t)\,dt$$

$$= \int_{t\in R} f_1(\mathbf{x},\tau - t) f_2(\mathbf{x},t)\,dt$$

$$= C(f_2,f_1;\mathbf{x},\tau). \qquad (1)$$

Equation (1) shows that the time convolution is symmetri-

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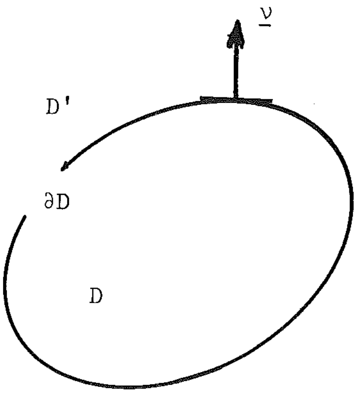

FIG. 1. Time-invariant configuration to which the reciprocity theorems apply.

cal in f_1 and f_2. The time correlation of f_1 and f_2 is defined as

$$R(f_1,f_2;\mathbf{x},\tau) = \int_{t\in R} f_1(\mathbf{x},t) f_2(\mathbf{x},t-\tau)\,dt$$
$$= \int_{t\in R} f_1(\mathbf{x},t+\tau) f_2(\mathbf{x},t)\,dt$$
$$= R(f_2,f_1;\mathbf{x},-\tau). \quad (2)$$

Equation (2) shows that the time correlation is not symmetrical in f_1 and f_2.

Now, let $\bar{f}$ denote the time reverse of f, i.e.,

$$\bar{f}(\mathbf{x},t) = f(\mathbf{x},-t). \quad (3)$$

Then, it follows from Eqs. (1)–(3) that

$$R(f_1,f_2;\mathbf{x},\tau) = C(f_1,\bar{f}_2;\mathbf{x},\tau). \quad (4)$$

Using Eq. (1), we further obtain the properties

$$C(\bar{f}_1,f_2;\mathbf{x},\tau) = \bar{C}(f_1,\bar{f}_2;\mathbf{x},\tau) \quad (5)$$

and

$$C(\bar{f}_1,\bar{f}_2;\mathbf{x},\tau) = \bar{C}(f_1,f_2;\mathbf{x},\tau). \quad (6)$$

For the time derivative of the time convolution, the rules

$$\partial_\tau C(f_1,f_2;\mathbf{x},\tau) = C(f_1,\partial_t f_2;\mathbf{x},\tau) = C(\partial_t f_1,f_2;\mathbf{x},\tau) \quad (7)$$

apply. In view of the property

$$\overline{\partial_t f} = -\partial_t \bar{f}, \quad (8)$$

the time derivatives of the time correlation are taken care of by using Eq. (4) in conjunction with Eq. (7), i.e.,

$$\partial_\tau C(f_1,\bar{f}_2;\mathbf{x},\tau) = C(\partial_t f_1,\bar{f}_2;\mathbf{x},\tau) = -C(f_1,\overline{\partial_t f}_2;\mathbf{x},\tau). \quad (9)$$

For the incorporation of relaxation effects in the reciprocity theorems, we also need the time convolution of three space-time functions. For this, either of the definitions

$$C(f_1,f_2,f_3;\mathbf{x},\tau) = C(f_1,C(f_2,f_3);\mathbf{x},\tau)$$
$$= C(C(f_1,f_2),f_3;\mathbf{x},\tau) \quad (10)$$

holds.

In view of its simpler properties, the time-convolution concept is used throughout the entire subsequent derivations, i.e., both for the time-convolution and for the time-correlation reciprocity theorems.

II. PROPERTIES OF THE ACOUSTIC WAVE FIELD IN THE CONFIGURATION

In each subdomain of the configuration where the acoustic properties vary continuously with position, the acoustic wave field quantities are continuously differentiable and satisfy the equations

$$-\dot{\theta} + \partial_r v_r = q, \quad (11)$$
$$\partial_k p + \dot{\Phi}_k = f_k, \quad (12)$$

where

p = acoustic pressure (Pa),
v_r = particle velocity ($\mathrm{m\,s^{-1}}$),
$\dot{\theta}$ = dilatation rate ($\mathrm{s^{-1}}$),
$\dot{\Phi}_k$ = mass flow density rate ($\mathrm{kg\,m^{-2}\,s^{-2}}$),
q = volume source density of injection rate ($\mathrm{s^{-1}}$),
f_k = volume source density of force ($\mathrm{N\,m^{-3}}$).

Equations (11) and (12) are supplemented by the constitutive relations. For a general linear, time-invariant, locally reacting fluid these are, in their linearized version,

$$\dot{\theta}(\mathbf{x},t) = -\partial_t \int_{\tau=-\infty}^{\infty} \phi(\mathbf{x},\tau) p(\mathbf{x},t-\tau)\,d\tau, \quad (13)$$

$$\dot{\Phi}_k(\mathbf{x},t) = \partial_t \int_{\tau=-\infty}^{\infty} \gamma_{k,r}(\mathbf{x},\tau) v_r(\mathbf{x},t-\tau)\,d\tau, \quad (14)$$

where

ϕ = fluidity relaxation function ($\mathrm{Pa^{-1}\,s^{-1}}$),
$\gamma_{k,r}$ = inertia relaxation function ($\mathrm{kg\,m^{-3}\,s^{-1}}$).

Using the notation of Eq. (1), Eqs. (13) and (14) can be rewritten as

$$\dot{\theta}(\mathbf{x},t) = -\partial_t C(\phi,p;\mathbf{x},t), \quad (15)$$
$$\dot{\Phi}_k(\mathbf{x},t) = \partial_t C(\gamma_{k,r},v_r;\mathbf{x},t), \quad (16)$$

respectively. In Eqs. (13) and (14), inhomogeneity, anisotropy, and relaxation of the fluid are included. The inertial anisotropy in Eq. (14) is introduced to make the theorems also applicable to acoustic waves in solids where only compression in the stress behavior is considered and shear is neglected; inertial anisotropy manifests itself, for example, in the macroscopic seismic behavior of rock with microlayering (cf. Schoenberg[4]) and the macroscopic Biot theory of the propagation of acoustic waves in solids with fluid-filled pores (cf. Biot[5]). In this respect, it is noted that Lord Rayleigh[6] has considered this type of anisotropy in connection with the mechanical ether theory for the propagation of light in crystals.

If $\{\phi,\gamma_{k,r}\}(\mathbf{x},\tau) = 0$ when $\tau<0$, the fluid at $\mathbf{x}$ is, to use the terminology of linear, time-invariant systems, causal. If

$$\phi(\mathbf{x},\tau) = \kappa(\mathbf{x})\delta(\tau), \quad (17)$$
$$\gamma_{k,r}(\mathbf{x},\tau) = \rho_{k,r}(\mathbf{x})\delta(\tau), \quad (18)$$

where $\delta(\tau)$ is the unit impulse (Dirac distribution), the fluid is instantaneously reacting, and κ and $\rho_{k,r}$ are its compressibility and its (tensorial) volume density of mass, respectively. If $\{\phi,\gamma_{k,r}\}(\mathbf{x},\tau) = 0$ when $\tau>0$, the fluid is anticausal or effectual. From an energy point of view, a fluid for which $\{\phi,\gamma_{k,r}\}(\mathbf{x},\tau) \neq 0$ when $\tau>0$ is dissipative, a fluid for which Eqs. (17) and (18) hold is lossless, and a fluid for which $\{\phi,\gamma_{k,r}\}(\mathbf{x},\tau) \neq 0$ when $\tau<0$ is active. A fluid that is either dissipative or lossless is also denoted as passive. For our reciprocity theorems, no specific type of relaxation function is presupposed. Conservation of energy of course requires an acoustically active fluid to be stimulated, through the constitutive parameters, by some other physical phenomenon (for example, by the passage of light through it).

It is assumed that ϕ and $\gamma_{k,r}$ are piecewise continuous functions of position. At an interface between two different fluids, they jump by finite amounts. Across such an interface, the acoustic pressure and the normal component of the particle velocity are continuous. If an acoustically impenetrable object is present, either the pressure (at a void) or the normal component of the particle velocity (at an immovable rigid object) has zero value at its boundary. Through the relevant boundary conditions, the presence of either interfaces or impenetrable objects is accounted for.

The two states that occur in the reciprocity theorems will be denoted by the superscripts "a" and "b," respectively. It is noted that the two states can apply to different source distributions and to fluids with different properties, but they must be present in one and the same domain in space-time.

III. THE RECIPROCITY THEOREM OF THE TIME-CONVOLUTION TYPE

The reciprocity theorem of the time-convolution type follows upon considering the interaction quantity $C(p^a,v_k^b;\mathbf{x},\tau) - C(p^b,v_k^a;\mathbf{x},\tau)$. Using Eqs. (11) and (12) for each of the two states, we obtain

$$\partial_k C(p^a,v_k^b;\mathbf{x},\tau) = C(f_k^a - \dot{\Phi}_k^a,v_k^b;\mathbf{x},\tau) + C(p^a,q^b + \dot{\theta}^b;\mathbf{x},\tau) \tag{19}$$

and

$$\partial_k C(p^b,v_k^a;\mathbf{x},\tau) = C(f_k^b - \dot{\Phi}_k^b,v_k^a;\mathbf{x},\tau) + C(p^b,q^a + \dot{\theta}^a;\mathbf{x},\tau). \tag{20}$$

Now, in view of Eqs. (15) and (16), we have

$$C(\dot{\Phi}_k^b,v_k^a;\mathbf{x},\tau) - C(\dot{\Phi}_k^a,v_k^b;\mathbf{x},\tau) = \partial_\tau C(\gamma_{k,r}^b - \gamma_{r,k}^a,v_k^a,v_r^b;\mathbf{x},\tau) \tag{21}$$

and

$$C(p^b,\dot{\theta}^a;\mathbf{x},\tau) - C(p^a,\dot{\theta}^b;\mathbf{x},\tau) = \partial_\tau C(\phi^b - \phi^a,p^a,p^b;\mathbf{x},\tau), \tag{22}$$

where Eq. (7) has been used. Subtracting Eq. (20) from Eq. (19) and employing Eqs. (21) and (22), we arrive at

$$\begin{aligned}\partial_k [C(p^a,v_k^b;\mathbf{x},\tau) - C(p^b,v_k^a;\mathbf{x},\tau)] &= \partial_\tau C(\gamma_{k,r}^b - \gamma_{r,k}^a,v_k^a,v_r^b;\mathbf{x},\tau)\\ &\quad - \partial_\tau C(\phi^b - \phi^a,p^a,p^b;\mathbf{x},\tau)\\ &\quad + C(f_k^a,v_k^b;\mathbf{x},\tau) + C(p^a,q^b;\mathbf{x},\tau)\\ &\quad - C(f_k^b,v_k^a;\mathbf{x},\tau) - C(p^b,q^a;\mathbf{x},\tau).\end{aligned} \tag{23}$$

Equation (23) is the local form of the time-convolution reciprocity theorem. The first two terms at the right-hand side are representative for the differences in the properties of the fluids present in the two states; they vanish at those locations where $\gamma_{k,r}^b(\mathbf{x},\tau) = \gamma_{r,k}^a(\mathbf{x},\tau)$ and $\phi^b(\mathbf{x},\tau) = \phi^a(\mathbf{x},\tau)$ for all $\tau \in R$. In case the latter conditions hold, the two media are denoted as each other's adjoints. Note in this respect that the adjoint of a causal (effectual) fluid is causal (effectual), also. The last four terms at the right-hand side of Eq. (23) are associated with the source distributions; they vanish at those locations where no sources are present. Upon integrating Eq. (23) over the subdomains of D where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\begin{aligned}\int_{\mathbf{x}\in\partial D} \nu_k [C(p^a,v_k^b;\mathbf{x},\tau) - C(p^b,v_k^a;\mathbf{x},\tau)]\,dA &= \int_{\mathbf{x}\in D} [\partial_\tau C(\gamma_{k,r}^b - \gamma_{r,k}^a,v_k^a,v_r^b;\mathbf{x},\tau)\\ &\quad - \partial_\tau C(\phi^b - \phi^a,p^a,p^b;\mathbf{x},\tau)]\,dV\\ &\quad + \int_{\mathbf{x}\in D} [C(f_k^a,v_k^b;\mathbf{x},\tau) + C(p^a,q^b;\mathbf{x},\tau)\\ &\quad - C(f_k^b,v_k^a;\mathbf{x},\tau) - C(p^b,q^a;\mathbf{x},\tau)]\,dV.\end{aligned} \tag{24}$$

Equation (24) is the global form, for the domain D, of the time-convolution reciprocity theorem. Note that the contributions from interfaces between different fluids present in D have canceled and that the contributions from the boundaries of acoustically impenetrable objects present in D vanish in view of the boundary conditions stated in Sec. II.

IV. THE RECIPROCITY THEOREM OF THE TIME-CORRELATION TYPE

The reciprocity theorem of the time-correlation type follows upon considering the interaction quantity $R(p^a,v_k^b;\mathbf{x},\tau) + R(p^b,v_k^a;\mathbf{x},-\tau)$. On account of Eqs. (4) and (5), this interaction quantity is equivalent to $C(p^a,\bar{v}_k^b;\mathbf{x},\tau) + C(\bar{p}^b,v_k^a;\mathbf{x},\tau)$. Using Eqs. (11) and (12) for each of the two states, we obtain

$$\partial_k C(p^a,\bar{v}_k^b;\mathbf{x},\tau) = C(f_k^a - \dot{\Phi}_k^a,\bar{v}_k^b;\mathbf{x},\tau) + C(p^a,\bar{q}^b + \overline{\dot{\theta}}^b;\mathbf{x},\tau) \tag{25}$$

and

$$\partial_k C(\bar{p}^b,v_k^a;\mathbf{x},\tau) = C(\bar{f}_k^b - \overline{\dot{\Phi}}_k^b,v_k^a;\mathbf{x},\tau) + C(\bar{p}^b,q^a + \dot{\theta}^a;\mathbf{x},\tau). \tag{26}$$

Now, in view of Eqs. (15) and (16), we have

$$-C(\overline{\dot{\Phi}}_k^b,v_k^a;\mathbf{x},\tau) - C(\dot{\Phi}_k^a,\bar{v}_k^b;\mathbf{x},\tau) = \partial_\tau C(\bar{\gamma}_{k,r}^b - \gamma_{r,k}^a,v_k^a,\bar{v}_r^b;\mathbf{x},\tau) \tag{27}$$

and

$$C(p^a,\overline{\dot{\theta}}^b;\mathbf{x},\tau) + C(\bar{p}^b,\dot{\theta}^a;\mathbf{x},\tau) = \partial_\tau C(\bar{\phi}^b - \phi^a,p^a,\bar{p}^b;\mathbf{x},\tau), \tag{28}$$

where Eq. (9) has been used. Adding Eq. (26) to Eq. (25) and employing Eqs. (27) and (28), we arrive at

$$\begin{aligned}\partial_k [C(p^a,\bar{v}_k^b;\mathbf{x},\tau) + C(\bar{p}^b,v_k^a;\mathbf{x},\tau)] &= \partial_\tau C(\bar{\gamma}_{k,r}^b - \gamma_{r,k}^a,v_k^a,\bar{v}_r^b;\mathbf{x},\tau)\\ &\quad + \partial_\tau C(\bar{\phi}^b - \phi^a,p^a,\bar{p}^b;\mathbf{x},\tau)\\ &\quad + C(f_k^a,\bar{v}_k^b;\mathbf{x},\tau) + C(p^a,\bar{q}^b;\mathbf{x},\tau)\\ &\quad + C(\bar{f}_k^b,v_k^a;\mathbf{x},\tau) + C(\bar{p}^b,q^a;\mathbf{x},\tau).\end{aligned} \tag{29}$$

Equation (29) is the local form of the time-correlation reciprocity theorem. The first two terms at the right-hand side are representative of the differences in the properties of the fluids present in the two states; they vanish at those locations where $\bar{\gamma}^b_{k,r}(\mathbf{x},\tau) = \gamma^a_{r,k}(\mathbf{x},\tau)$ and $\bar{\phi}^b(\mathbf{x},\tau) = \phi^a(\mathbf{x},\tau)$ for all $\tau \in R$. In case the latter conditions hold, the two fluids are denoted as each other's time-reverse adjoints. Note in this respect that the time-reverse adjoint of a causal (effectual) fluid is an effectual (causal) one. Upon integrating Eq. (29) over the subdomains of D where both sides are continuously differentiable, applying Gauss' diverence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\int_{\mathbf{x}\in\partial D} \nu_k [C(p^a,\bar{v}^b_k;\mathbf{x},\tau) + C(\bar{p}^b,v^a_k;\mathbf{x},\tau)]dA$$
$$= \int_{\mathbf{x}\in D} [\partial_\tau C(\bar{\gamma}^b_{k,r} - \gamma^a_{r,k},v^a_k,\bar{v}^b_r;\mathbf{x},\tau) + \partial_\tau C(\bar{\phi}^b - \phi^a,p^a,\bar{p}^b;\mathbf{x},\tau)]dV$$
$$+ \int_{\mathbf{x}\in D} [C(f^a_k,\bar{v}^b_k;\mathbf{x},\tau) + C(p^a,\bar{q}^b;\mathbf{x},\tau) + C(\bar{f}^b_k,v^a_k;\mathbf{x},\tau) + C(\bar{p}^b,q^a;\mathbf{x},\tau)]dV. \quad (30)$$

Equation (30) is the global form, for the domain D, of the time-correlation reciprocity theorem. Note that the contributions from interfaces between different fluids present in D have canceled and that the contributions from the boundaries of acoustically impenetrable objects present in D vanish in view of the boundary conditions stated in Sec. II.

V. APPLICATION TO INVERSE PROBLEMS

In this section, we briefly outline the relevance of the reciprocity theorems given in Eqs. (24) and (30) to inverse problems. In this respect, we distinguish between inverse source problems and inverse constituency problems. In an inverse source problem, the aim is to reconstruct the volume source densities of the injection rate and force of acoustic sources present in some inaccessible domain in space from the measured values of the emitted acoustic wave field in some other domain in space. The constitutive parameters of the fluids in which the acoustic radiation takes place are assumed to be known. In an inverse constituency problem (also denoted as inverse profiling or imaging problem), the aim is to reconstruct the distribution of constitutive parameters in some inaccessible domain in space by irradiating the configuration by known acoustic sources located in the embedding and measuring the acoustic wave field response in some other domain in the embedding; the constitutive parameters of the embedding are known. The two types of problems will be discussed separately.

A. Inverse source problem

In the inverse source problem, the acoustic wave field in state "a" is taken to be one that is radiated by the unknown source distributions $\{q^T,f^T_k\}$. Let $D^T \subset \mathbb{R}^3$ be their spatial support. The radiated wave field $\{p^T,v^T_k\}$ is measured in some accessible observational domain $D^\Omega \subset R^3$. The intersection of D^T and D^Ω is empty (Fig. 2). State "b" is taken to be a computational state, denoted as the "observational" one. The corresponding wave field $\{p^\Omega,v^\Omega_k\}$ that would be radiated by known sources with distributions $\{q^\Omega,f^\Omega_k\}$ is computed and its interaction with the measured acoustic wave field in D^Ω is evaluated. In general, one could say that the introduction of the observational state is representative of the processing of the measured data. Since only the interaction in D^Ω is considered, it makes no sense to take the support of $\{q^\Omega,f^\Omega_k\}$ larger than D^Ω. Finally, the fluid in the observational state is taken to be either the adjoint [for the application of Eq. (24)] or the time-reverse adjoint [for the application of Eq. (30)] of the one in which the unknown sources radiate.

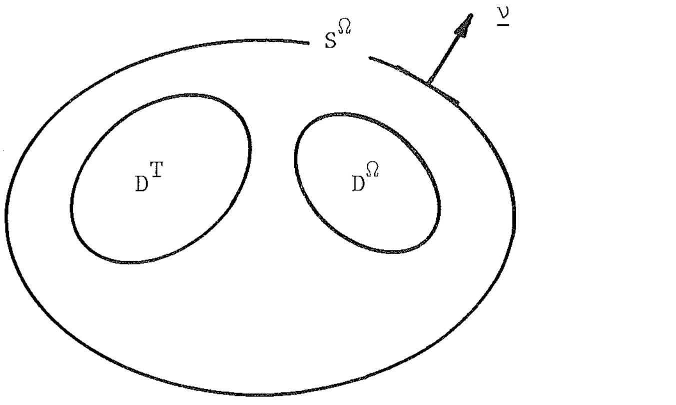

FIG. 2. Configuration illustrative for the inverse source problem: Unknown acoustic sources radiate in D^T; the acoustic wave field is measured in D^Ω and on S^Ω.

The reciprocity relations given in Eqs. (24) and (30) are now applied to the domain interior to the closed surface S^Ω that is taken such that D^T and D^Ω are located in its interior. Then, Eq. (24) leads to

$$\int_{\mathbf{x}\in D^T} [-C(f^T_k,v^\Omega_k;\mathbf{x},\tau) + C(p^\Omega,q^T;\mathbf{x},\tau)]dV$$
$$= \int_{\mathbf{x}\in D^\Omega} [-C(f^\Omega_k,v^T_k;\mathbf{x},\tau) + C(p^T,q^\Omega;\mathbf{x},\tau)]dV - \int_{\mathbf{x}\in S^\Omega} \nu_k [C(p^T,v^\Omega_k;\mathbf{x},\tau) - C(p^\Omega,v^T_k;\mathbf{x},\tau)]dA, \quad (31)$$

and Eq. (30) leads to

$$\int_{\mathbf{x}\in D^T} [-C(f^T_k,\bar{v}^\Omega_k;\mathbf{x},\tau) - C(\bar{p}^\Omega,q^T;\mathbf{x},\tau)]dV$$
$$= \int_{\mathbf{x}\in D^\Omega} [C(\bar{f}^\Omega_k,v^T_k;\mathbf{x},\tau) + C(p^T,\bar{q}^\Omega;\mathbf{x},\tau)]dV - \int_{\mathbf{x}\in S^\Omega} \nu_k [C(p^T,\bar{v}^\Omega_k;\mathbf{x},\tau) + C(\bar{p}^\Omega,v^T_k;\mathbf{x},\tau)]dA. \quad (32)$$

In Eqs. (31) and (32), the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on S^Ω. A solution to the inverse source problem is now commonly constructed by taking for $\{q^\Omega,f^\Omega_k\}$ a sequence of N linearly independent distributions with spatial support D^Ω and fixed, preferably broadband, time behavior (for example, an impulse). The corresponding sequence of acoustic wave field distributions $\{p^{\Omega},v_k^{\Omega}\}$ is computed. Next, the unknown source distributions $\{q^{T},f_k^{T}\}$ are expanded into an appropriate sequence of M linearly independent space-time expansion functions with spatial support D^{T}; the corresponding expansion coefficients are unknown. Substitution of the results in Eqs. (31) and (32) and evaluation of the relevant integrals lead to systems of linear algebraic equations with the source expansion coefficients as unknowns. When $M=N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is nonsingular, it turns out to be ill-conditioned in most practical cases. Therefore, one usually takes $M>N$, and a best fit of the expanded source distributions is obtained by the application of minimization techniques.

At this point, some more must be said about the role of S^{Ω}. In practice, one is mostly interested in causal media. Then, it is advantageous to choose the wave fields in Eq. (31) to be causal as well. Given the fact that S^{Ω} surrounds all sources, the integral over S^{Ω} can be replaced by an integral over any sphere S_{Δ} with radius Δ and center at the origin such that S_{Δ} surrounds S^{Ω}. [This follows from the application of Eq. (24) to the domain in between S_{Δ} and S^{Ω}.] However, for sufficiently large values of Δ, the causal wave field on S_{Δ} is zero, since it propagates with a finite maximum speed away from the sources. Hence, under these circumstances, the surface integral in the right-hand side of Eq. (31) vanishes. A similar argument does not apply to the surface integral in the right-hand side of Eq. (32), since in Eq. (32) effectual (or anticausal) wave fields are involved in all cases. This difference in the roles of the surface integrals in Eqs. (31) and (32) has been pointed out by Bojarski.[2]

B. Inverse constituency problem

In the inverse constituency problem, the acoustic wave field in state "a" is taken to be the one that irradiates the configuration. Let $D^{i}\subset R^{3}$ be the spatial support of the irradiating sources with known distributions $\{q^{i},f_k^{i}\}$ and let the corresponding acoustic wave field be $\{p^{i},v_k^{i}\}$. This wave field is measured in some accessible, observational domain $D^{\Omega}\subset R^{3}$. Further, let $D^{s}\subset R^{3}$ be the (inaccessible) domain in which the constitutive parameters are unknown. The intersections of D^{s} and D^{i} and of D^{s} and D^{Ω} are empty; D^{i} and D^{Ω} may, however, have points in common, or may even completely coincide (Fig. 3). State "b" is taken to be a computational state, denoted as the "observational" one. The corresponding wave field $\{p^{\Omega},v_k^{\Omega}\}$ that would be emitted by known acoustic sources with distributions $\{q^{\Omega},f_k^{\Omega}\}$ in the known fluid with the constitutive parameters $\{\phi^{\Omega},\gamma_{k,r}^{\Omega}\}$ of the adjoint [for the application of Eq. (24)] or the time-reverse adjoint [for the application of Eq. (30)] of the known embedding is computed and its interaction with the measured acoustic wave field in D^{Ω} is evaluated. Since only the interaction in D^{Ω} is considered, it makes no sense to take the support of $\{q^{\Omega},f_k^{\Omega}\}$ larger than D^{Ω}. The unknown constitutive parameters of D^{s} are denoted by $\{\phi^{s},\gamma_{k,r}^{s}\}$, D^{s} being the support of the differences $\{\phi^{s}-\phi^{\Omega},\ \gamma_{k,r}^{s}-\gamma_{r,k}^{\Omega}\}$ and $(\phi^{s}$

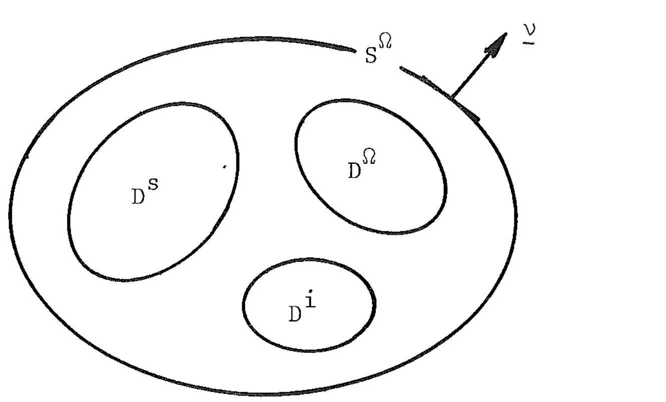

FIG. 3. Configuration illustrative for the inverse constituency problem: Known acoustic sources in D^{i} irradiate the contrasting domain D^{s} with unknown properties; the acoustic wave field is measured in D^{Ω} and on S^{Ω}.

$-\bar{\phi}^{\Omega},\ \gamma_{k,r}^{s}-\bar{\gamma}_{r,k}^{\Omega})$ for the application of Eqs. (24) and (30), respectively.

The reciprocity relations given in Eqs. (24) and (30) are now applied to the domain interior to the surface S^{Ω} that is taken such that D^{i}, D^{s}, and D^{Ω} are located in its interior. Then Eq. (24) leads to

$$\begin{aligned}\int_{\mathbf{x}\in D^{s}}&[-C(f_r^{s},v_r^{\Omega};\mathbf{x},\tau)+C(p^{\Omega},q^{s};\mathbf{x},\tau)]dV\\&=\int_{\mathbf{x}\in D^{i}}[C(f_k^{i},v_k^{\Omega};\mathbf{x},\tau)-C(p^{\Omega},q^{i};\mathbf{x},\tau)]dV\\&\quad+\int_{\mathbf{x}\in D^{\Omega}}[-C(f_k^{\Omega},v_k^{i};\mathbf{x},\tau)+C(p^{i},q^{\Omega};\mathbf{x},\tau)]dV\\&\quad-\int_{\mathbf{x}\in S^{\Omega}}\nu_k[C(p^{i},v_k^{\Omega};\mathbf{x},\tau)-C(p^{\Omega},v_k^{i};\mathbf{x},\tau)]dA,\end{aligned}\qquad(33)$$

in which

$$q^{s}=\partial_{\tau}C(\phi^{\Omega}-\phi^{s},p^{i};\mathbf{x},\tau)\qquad(34)$$

is the equivalent contrast volume source density of injection rate in D^{s}, and

$$f_r^{s}=\partial_{\tau}C(\gamma_{k,r}^{\Omega}-\gamma_{r,k}^{s},v_k^{i};\mathbf{x},\tau)\qquad(35)$$

is the equivalent contrast volume source density of force in D^{s}. In the same way, Eq. (30) leads to

$$\begin{aligned}\int_{\mathbf{x}\in D^{s}}&[-C(f_r^{s},\bar{v}_r^{\Omega};\mathbf{x},\tau)-C(\bar{p}^{\Omega},q^{s};\mathbf{x},\tau)]dV\\&=\int_{\mathbf{x}\in D^{i}}[C(f_k^{i},\bar{v}_k^{\Omega};\mathbf{x},\tau)+C(\bar{p}^{\Omega},q^{i};\mathbf{x},\tau)]dV\\&\quad+\int_{\mathbf{x}\in D^{\Omega}}[C(\bar{f}_k^{\Omega},v_k^{i};\mathbf{x},\tau)+C(p^{i},\bar{q}^{\Omega};\mathbf{x},\tau)]dV\\&\quad-\int_{\mathbf{x}\in S^{\Omega}}\nu_k[C(p^{i},\bar{v}_k^{\Omega};\mathbf{x},\tau)+C(\bar{p}^{\Omega},v_k^{i};\mathbf{x},\tau)]dA,\end{aligned}\qquad(36)$$

in which

$$q^{s}=\partial_{\tau}C(\bar{\phi}^{\Omega}-\phi^{s},p^{i};\mathbf{x},\tau)\qquad(37)$$

is the equivalent contrast volume source density of injection rate in D^{s}, and

$$f_r^{s}=\partial_{\tau}C(\bar{\gamma}_{k,r}^{\Omega}-\gamma_{r,k}^{s},v_k^{i};\mathbf{x},\tau)\qquad(38)$$

is the equivalent contrast volume source density of force in D^s.

In Eqs. (33) and (36), the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on S^Ω. The easiest way to address the inverse constituency problem is to consider it as an inverse source problem for the quantities $\{q^s, f_r^s\}$. Once values for these have been obtained, the solution of the forward or direct source problem with known values of $\{q^i, f_k^i\}$ and $\{q^s, f_r^s\}$ yields the values of $\{p^i, v_k^i\}$ in D^s and the temporal deconvolution of either Eqs. (34) and (35) or Eqs. (37) and (38) yields, since $\{\phi^\Omega, \gamma_{k,r}^\Omega\}$ are known, the values of $\{\phi^s, \gamma_{k,r}^s\}$. As to the role of the surface integrals over S^Ω in the right-hand sides of Eqs. (33) and (36), the same remarks as for the inverse source problem apply.

To conclude our investigation, we want to emphasize that the uniqueness and the existence of solutions to both the inverse source and the inverse constituency problem are, for the larger part, at the moment open questions.

VI. CONCLUSION

Time-domain reciprocity theorems for the acoustic wave field in linear, time-invariant, and locally reacting fluids have been derived via a full space-time method. Inhomogeneity, anisotropy, and relaxation of the fluid are included. The anisotropy is of the inertial type; it is introduced to make the theorems applicable also to acoustic waves in anisotropic solids where only compression in the stress behavior is considered and shear is neglected. One of the theorems is of the time-convolution type, the other of the time-correlation type. The application of the two theorems to inverse source and inverse constituency problems is discussed.

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