Reciprocity theorems for acoustic wave fields in fluid/solid configurations

Adrianus T. de Hoop[a)]
Schlumberger–Doll Research, Old Quarry Road, P.O. Box 307, Ridgefield, Connecticut 06877-4108

(Received 4 August 1989; accepted for publication 24 November 1989)

The time Laplace-transform domain convolution-type reciprocity relation for acoustic waves in a fluid/solid configuration is derived. Arbitrary inhomogeneity, anisotropy, and loss mechanisms are taken into account. Reciprocity between transmitting and receiving transducers located in either the fluid or the solid parts of the configuration is established. It is shown how the reciprocity relation leads to the source-type wave field integral representations for direct source problems and to the integral-equation formulation of inverse source, and direct and inverse scattering problems through the associated contrast source representations. Since neither a fluid inclusion in a solid nor a solid inclusion in a fluid leads to a regular perturbation problem in the integral-equation formulation, the embedding must be adapted to the location of the contrast sources as far as the type of medium (fluid or solid) is concerned. Applications to acoustic emission, and to acoustic imaging and profile inversion are briefly indicated.

PACS numbers: 43.20.Tb, 43.30.Jx, 43.30.Ky

INTRODUCTION

Reciprocity relations for wave fields can be regarded as lying at the root of the construction of algorithms for solving direct as well as inverse source and scattering problems. For acoustic wave fields present in either fluid or solid configurations some general aspects in this respect have recently been discussed in publications by the present author.[1,2] In these papers some historical background on acoustic reciprocity theorems can be found. An additional early reference to applying reciprocity to the inverse source problem of reconstructing earthquake mechanisms from observed seismograms is a paper by Burridge and Knopoff.[3] In the present paper the reciprocity relations pertaining to acoustic wave fields present in configurations that contain both fluids and solids (such as the ones that occur in underwater sound, ocean acoustics, and borehole acoustics in geophysical exploration) are addressed. These relations belong to a more difficult category, especially when inverse source and inverse scattering problems are analyzed with the aid of them, since neither in the case of a solid inclusion in a fluid nor in the case of a fluid inclusion in a solid can the constitutive parameters of the inclusion be regarded as regular perturbations of the ones of the embedding. As a result, the contrast source representations that usually serve as a point of departure for constructing algorithms to solve these kinds of problems are not immediately obvious. The derivation of these representations is the central part of the present paper. Applications of the resulting relations to problems of acoustic emission, acoustic scattering by objects, and acoustic imaging and object reconstruction are briefly indicated. Also, the reciprocity properties of transmitting and receiving transducers in fluid/solid configurations are discussed.

The present investigation deals with the acoustic reciprocity theorem of the time-convolution type in a time-invariant fluid/solid configuration that is linear, locally reacting, and passive in its acoustic behavior. Since in the reciprocity theorems of the time-convolution type causality is preserved (see de Hoop[1,2]), the reciprocity theorem for the fluid/solid configuration will be presented in the time Laplace-transform domain. In this domain, the condition of causality of the wave field quantities in space-time is replaced by the condition of boundedness of their Laplace-transform counterparts in all space, which condition must hold for all values of the time Laplace-transform variable s in the right half $\text{Re}(s) > 0$ of the complex s plane. From the s-domain results the time-domain counterparts easily follow upon using some standard rules of the one-sided Laplace transformation, while the results for sinusoidally in time varying acoustic field quantities are obtained by replacing s by $j\omega$, where j is the imaginary unit and ω the angular frequency, on the condition that imaginary values of s are approached via the right half of the complex s plane. Arbitrary inhomogeneity, anisotropy, and loss mechanisms are taken into account.

I. THE ACOUSTIC WAVE FIELD IN THE FLUID/SOLID CONFIGURATION

The configuration in which the acoustic wave field is present consists of either a fluid inclusion of bounded extent in a solid embedding or a solid inclusion of bounded extent in a fluid embedding. In either case the embedding is, outside some sphere of finite radius, assumed to be homogeneous, isotropic, and lossless in its acoustic properties. The domain occupied by the fluid is denoted by D^f and the domain occupied by the solid by D^s; both D^f and D^s may consist of a finite number of unconnected subdomains.

The position of observation in the configuration is specified by the coordinates $\{x_1,x_2,x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ of unit

[a)] On leave from Delft University of Technology, Faculty of Electrical Engineering, Laboratory of Electromagnetic Research, P.O. Box 5031, 2600 GA Delft, The Netherlands.

length each. In the indicated order, the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors is used and the summation convention applies. The corresponding lowercase Latin subscripts are to be assigned the values {1,2,3}. Whenever appropriate, the position vector will be denoted by $\mathbf{x} = x_m \mathbf{i}_m$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_m denotes differentiation with respect to x_m; ∂_t is a reserved symbol for differentiation with respect to t.

For any causal space-time function $u = u(\mathbf{x},t)$ the one-sided Laplace transform is introduced as

$$\hat{u}(\mathbf{x},s) = \int_{t=0}^{\infty} \exp(-st) u(\mathbf{x},t) dt , \tag{1}$$

where the instant $t = 0$ marks the onset of the events. Obviously, for bounded $|u(\mathbf{x},t)|$, $\hat{u}(\mathbf{x},s)$ is an analytic function of the complex transform parameter s in the right half $\mathrm{Re}(s) > 0$ of the complex s plane. For ease of notation the caret over a symbol denoting its s-domain counterpart will be omitted in the remainder of the paper. The acoustic wave motion is started from a configuration at rest; then, under the one-sided Laplace transformation the operator ∂_t is replaced by an algebraic factor s.

In each subdomain of D^f where the fluid's acoustic properties vary continuously with position, the acoustic wave field quantities are continuously differentiable and satisfy the s-domain equations

$$-\partial_k \sigma + s\rho^f_{k,r} w_r = f^f_k , \tag{2}$$

$$\partial_r w_r - s\kappa\sigma = q , \tag{3}$$

in which σ is the scalar acoustic traction (opposite of the acoustic pressure) in the fluid, w_r is the fluid particle velocity, $\rho^f_{k,r}$ is the fluid volume density of mass, κ is the fluid compressibility, f^f_k is the fluid volume source density of force, and q is the fluid volume source density of injection rate. Anisotropy in the inertia properties of the fluid is taken into account. (This kind of anisotropy shows up in the effective-medium theory of finely layered fluids.[4]) For lossy fluids (i.e., fluids with relaxation), $\rho^f_{k,r}$ and κ are s-dependent, subject to the condition of causality which entails analyticity in $\mathrm{Re}(s) > 0$. Across interfaces between different kinds of fluids in D^f, σ and $\nu_r w_r$ are continuous, where ν_r denotes the unit vector along the normal to such an interface.

In each subdomain of D^s where the solid's acoustic properties vary continuously with position, the acoustic wave field quantities are continuously differentiable and satisfy the s-domain equations

$$-\Delta_{k,m,p,q}\, \partial_m \tau_{p,q} + s\rho^s_{k,r} v_r = f^s_k , \tag{4}$$

$$\Delta_{i,j,m,r}\, \partial_m v_r - sS_{i,j,p,q} \tau_{p,q} = h_{i,j} , \tag{5}$$

in which $\tau_{p,q}$ is the solid stress, v_r is the solid particle velocity, $\rho^s_{k,r}$ is the solid volume density of mass, $S_{i,j,p,q}$ is the solid compliance, f^s_k is the solid volume source density of force, and $h_{i,j}$ is the solid volume source density of rate of deformation. Full anisotropy in the inertia and (visco)elastic properties of the solid is taken into account. For lossy solids (i.e., solids with relaxation), $\rho^s_{k,r}$ and $S_{i,j,p,q}$ are s-dependent, subject to the condition of causality which entails analyticity in $\mathrm{Re}(s) > 0$. Across interfaces between different kinds of solids in D^s (which are assumed to be in rigid contact), v_r and $\Delta_{k,m,p,q}\, \nu_m \tau_{p,q}$ are continuous, where ν_m is the unit vector to the normal along such an interface. The symmetrical unit tensor of rank four

$$\Delta_{k,m,p,q} = \tfrac{1}{2} (\delta_{k,p}\delta_{m,q} + \delta_{k,q}\delta_{m,p}) , \tag{6}$$

where $\delta_{k,p}$ is the symmetrical unit tensor of rank two (Kronecker tensor), is characteristic for elastodynamics and automatically selects from any tensor of rank two with which it is contracted the symmetrical part.

Across an interface between a fluid part and a solid part, the continuity conditions $\nu_m w_m = \nu_m v_m$ and $\sigma = \nu_k \Delta_{k,m,p,q} \nu_m \tau_{p,q}$ apply, together with the explicit boundary condition $(\delta_{k,r} - \nu_k \nu_r)\Delta_{r,m,p,q} \nu_m \tau_{p,q} = 0$, in which the tensor $\delta_{k,r} - \nu_k \nu_r$ selects from any tensor with which it is contracted the tangential part to the surface to which ν_r is the local unit vector along the normal.

Both in the fluid and the solid subdomains, acoustically impenetrable objects may be present. Their presence is accounted for by explicit boundary conditions that hold upon approaching the boundary surface of the object from the outside. For an object located in the fluid these are either $\sigma \to 0$ (void) or $\nu_r w_r \to 0$ (immovable rigid object). For an object located in the solid these are either $\Delta_{k,m,p,q} \nu_m \tau_{p,q} \to 0$ (void) or $v_r \to 0$ (immovable rigid object).

The provisions that have been made for the media at "infinity" ensure that outside some sphere of finite radius and center at the origin of the chosen reference frame the farfield representations for the quantities associated with outgoing acoustic waves in a fluid or a solid apply (see de Hoop[5,6]). In these representations, too, causality plays an essential role.

II. THE LOCAL AND GLOBAL RECIPROCITY RELATIONS

A general wave field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two different physical states that could occur in one and the same domain in space-time. For time-invariant configurations the application of the one-sided Laplace transformation of Eq. (1) to the convolution-type reciprocity theorem leads to an equivalent s-domain result. For this to be applicable to the fluid/solid configuration under investigation, the fluids in the two states should be present in one and the same time-invariant domain D^f and the solids in the two states should be present in one and the same time-invariant domain D^s. The two states will be distinguished by the superscripts A and B, respectively. First, the local reciprocity theorems will be derived; they apply to the subdomains D^f and D^s separately. From them, the global reciprocity theorem for the entire fluid/solid configuration will be obtained.

The local reciprocity relation in the fluid follows upon considering the interaction quantity $\partial_k(\sigma^A w^B_k - \sigma^B w^A_k)$ and evaluating this quantity with the use of Eqs. (2) and (3) for the states A and B, respectively. The result is

$$\begin{aligned}\partial_k (\sigma^A w^B_k - \sigma^B w^A_k) &= s(\rho^{f;A}_{k,r} - \rho^{f;B}_{r,k}) w^A_r w^B_k - s(\kappa^A - \kappa^B)\sigma^A \sigma^B \\ &\quad - f^{f;A}_k w^B_k + f^{f;B}_r w^A_r + q^B \sigma^A - q^A \sigma^B .\end{aligned} \tag{7}$$

The local reciprocity relation in the solid follows upon considering the interaction quantity $\Delta_{k,m,p,q}\partial_m(\tau^A_{p,q}v^B_k - \tau^B_{p,q}v^A_k)$ and evaluating this quantity with the use of Eqs. (4) and (5) for the states A and B, respectively. The result is

$$\Delta_{k,m,p,q}\partial_m(\tau^A_{p,q}v^B_k - \tau^B_{p,q}v^A_k) = s(\rho^{s;A}_{k,r} - \rho^{s;B}_{r,k})v^A_r v^B_k - s(S^A_{i,j,p,q} - S^B_{p,q,i,j})\tau^A_{p,q}\tau^B_{i,j} - f^{s;A}_k v^B_k + f^{s;B}_r v^A_r + h^B_{p,q}\tau^A_{p,q} - h^A_{i,j}\tau^B_{i,j}. \quad (8)$$

Equation (7) holds at any point of D^f in the neighborhood of which the properties of the fluids in the states A and B vary continuously with position. Equation (8) holds at any point of D^s in the neighborhood of which the properties of the solids in the states A and B vary continuously with position. As far as the right-hand sides of Eqs. (7) and (8) are concerned, the terms fall into two categories. In the first set of terms, the medium properties of the fluids and solids in the states A and B occur. In the fluid, these terms vanish if $\rho^{f;A}_{k,r} = \rho^{f;B}_{r,k}$ and $\kappa^A = \kappa^B$. If these properties hold, the fluid in state B is denoted as the adjoint of the fluid present in state A. In case these properties hold for one and the same fluid, this fluid is denoted as self-adjoint, or reciprocal. In the solid the relevant terms vanish if $\rho^{s;A}_{k,r} = \rho^{s;B}_{r,k}$ and $S^A_{i,j,p,q} = S^B_{p,q,i,j}$. If these properties hold, the solid in state B is denoted as the adjoint of the solid present in state A. In case these properties hold for one and the same solid, this solid is denoted as self-adjoint or reciprocal. The second set of terms at the right-hand sides is associated with the source distributions in the states A and B. Obviously, these terms vanish in a source-free subdomain.

The global reciprocity relation that holds for some domain D in the fluid/solid configuration is obtained by integrating Eq. (7) over the domain $D\cap D^f$ that D and D^f have in common, integrating Eq. (8) over the domain $D\cap D^s$ that D and D^s have in common, adding the two results, and applying Gauss' divergence theorem to the resulting left-hand side. With this, the following relation is obtained:

$$\int_{\partial D\cap D^f}\nu_k(\sigma^A w^B_k - \sigma^B w^A_k)dA + \int_{\partial D\cap D^s}\Delta_{k,m,p,q}\nu_m(\tau^A_{p,q}v^B_k - \tau^B_{p,q}v^A_k)dA$$
$$= \int_{D\cap D^f}[s(\rho^{f;A}_{k,r} - \rho^{f;B}_{r,k})w^A_r w^B_k - s(\kappa^A - \kappa^B)\sigma^A\sigma^B - f^{f;A}_k w^B_k + f^{f;B}_r w^A_r + q^B\sigma^A - q^A\sigma^B]dV$$
$$+ \int_{D\cap D^s}[s(\rho^{s;A}_{k,r} - \rho^{s;B}_{r,k})v^A_r v^B_k - s(S^A_{i,j,p,q} - S^B_{p,q,i,j})\tau^A_{p,q}\tau^B_{i,j} - f^{s;A}_k v^B_k + f^{s;B}_r v^A_r + h^B_{p,q}\tau^A_{p,q} - h^A_{i,j}\tau^B_{i,j}]dV, \quad (9)$$

where the contributions from interfaces have canceled in view of the pertaining boundary conditions of the continuity type, while the contributions from the boundary surfaces of impenetrable objects have vanished in view of the pertaining boundary conditions of the explicit type. Equation (9) is the global reciprocity relation that will be used in the considerations that follow. In it, ∂D is the boundary surface of the domain D and ν_m is the unit vector along its normal, pointing away from D, $\partial D\cap D^f$ is the part of ∂D that is located in the fluid, and $\partial D\cap D^s$ is the part of ∂D that is located in the solid.

In some of the applications, D will be the entire three-dimensional space. To address this situation, Eq. (9) is first applied to the domain interior to the sphere S_Δ of radius Δ and with center at the origin of the chosen reference frame, after which the limit $\Delta\to\infty$ is taken. From some Δ onward, S_Δ will be entirely situated in either a homogeneous, isotropic, ideal fluid (in the case of a fluid embedding), or a homogeneous, isotropic, perfectly elastic solid (in the case of a solid embedding). In both cases, the corresponding farfield representations for the acoustic wave field quantities can, for sufficiently large values of Δ, be used on S_Δ, from which the contribution from S_Δ can be shown to vanish in the limit $\Delta\to\infty$ (cf. de Hoop[5,6]).

III. RECIPROCITY PROPERTIES OF TRANSMITTING AND RECEIVING TRANSDUCERS

A first set of corollaries of the global reciprocity relation are the reciprocity properties of transmitting and receiving transducers. First, the case is considered where the action of the transducers is represented by equivalent volume source densities, distributed over the domains occupied by the transducers. Let transducer 1 occupy the bounded domain T_1 and transducer 2 occupy the bounded domain T_2. Let state A be identified with the state where transducer 1 is transmitting and transducer 2 is receiving, and state B with the state where transducer 2 is transmitting and transducer 1 is receiving. Then, the application of Eq. (9) to the entire fluid/solid configuration leads to

$$\int_{T_1\cap D^f}(f^{f;A}_k w^B_k + q^A\sigma^B)dV + \int_{T_1\cap D^s}(f^{s;A}_k v^B_k + h^A_{i,j}\tau^B_{i,j})dV = \int_{T_2\cap D^f}(f^{f;B}_r w^A_r + q^B\sigma^A)dV + \int_{T_2\cap D^s}(f^{s;B}_r v^A_r + h^B_{p,q}\tau^A_{p,q})dV. \quad (10)$$

Equation (10) covers all four possible cases as far as the location of the two transducers (either of them in the fluid or the solid) is concerned. In deriving Eq. (10) it has been assumed that the fluid and the solid present in the configuration are self-adjoint. Note that for the applicability of the description of the action of the transducers by equivalent volume source distributions, either of them must be completely immersed in the fluid part or completely immersed in the solid part of the configuration and hence one term at the left-hand side and one term at the right-hand side are always missing.

Second, the case is considered where the action of the transducers is represented by equivalent surface source densities, distributed over the boundary surfaces ∂T_1 and ∂T_2 of the respective domains occupied by the transducers. Let, further, the states A and B be defined as before. Then, application of Eq. (9) to the entire fluid/solid configuration as far as this is located outside the domains occupied by the transducers leads to

$$\begin{aligned}
&\int_{\partial T_1 \cap D^f} \nu_k (\sigma^A w_k^B - \sigma^B w_k^A) dA \\
&\quad + \int_{\partial T_1 \cap D^s} \Delta_{k,m,p,q} \nu_m (\tau^A_{p,q} v^B_k - \tau^B_{p,q} v^A_k) dA \\
&= \int_{\partial T_2 \cap D^f} \nu_k (\sigma^B w_k^A - \sigma^A w_k^B) dA \\
&\quad + \int_{\partial T_2 \cap D^s} \Delta_{k,m,p,q} \nu_m (\tau^B_{p,q} v^A_k - \tau^A_{p,q} v^B_k) dA . \qquad (11)
\end{aligned}$$

Equation (11) covers all possible cases as far as the location of the two transducers is concerned; either of them may even partly be located in the fluid and partly in the solid. To comply with the conditions of the uniqueness of the acoustic radiation problem, the description of the action of the transducers by equivalent surface sources is restricted to prescribing on that part of their boundary surfaces that is located in the fluid the value of either the scalar traction σ or the normal component of the particle velocity $\nu_k w_k$, and on that part of their boundary surfaces that is located in the solid the value of either the traction $\Delta_{k,m,p,q} \nu_m \tau_{p,q}$ or the particle velocity v_k. How these boundary values are related to the actual response of the transducers is discussed in a paper by Kino.[7]

Equations (10) and (11) are also useful as tests on the consistency and (sometimes) accuracy of computer codes that serve to calculate numerically the values of the acoustic wave field quantities in fluid/solid configurations such as arise in ocean acoustics and borehole acoustics. In such an application, the fluid and solid constituents need not be self-adjoint in their acoustic properties, provided that, for the check to be carried out, the medium properties in the computational state B are taken mathematically to be the adjoints of the medium properties applying to the actual physical state A.

IV. DIRECT AND INVERSE SOURCE PROBLEMS

The reciprocity relation of Eq. (9) can be used to address acoustic direct and inverse source problems. In the first category, sources with known volume or surface distributions emit acoustic radiation into a known embedding; in the second category the embedding is still known, but the source distributions are not. Important applications of these phenomena in fluid/solid configurations are found in exploration geophysics (borehole acoustics and borehole seismics), underwater sound and ocean acoustics, where examples of the inverse source problem are acoustic emission and microseismic activity. In such applications the sources, known or unknown, emit acoustic radiation into a known embedding. Assume that the causal acoustic wave field (Green's state) at position $\mathbf{x}$ radiated by a point source located at position $\mathbf{x}'$ in the embedding can be determined and let these acoustic Green's states be collectively denoted by the superscript G. Assume, further, that the source occupies the bounded domain D^T and denote the acoustic quantities of the wave field radiated by it by the superscript T. Then, the observed acoustic quantities associated with the emission can be expressed as an interaction between the source distribution and an appropriate Green's state. Depending on where the observation is carried out (in the fluid or in the solid), and which quantity is observed (the scalar traction and the particle velocity in the fluid, or the stress and the particle velocity in the solid), different Green's states have to be used. These states will be specified below.

If the observed quantity is the scalar traction in the fluid, the Green's state should be the one corresponding to a nonvanishing point source of the type

$$q^G = Q\delta(\mathbf{x} - \mathbf{x}'), \quad \text{with } \mathbf{x}' \in D^f . \qquad (12)$$

If the observed quantity is the particle velocity in the fluid, the Green's state should be the one corresponding to a nonvanishing point source of the type

$$f_r^{f;G} = F_r^f \delta(\mathbf{x} - \mathbf{x}') , \quad \text{with } \mathbf{x}' \in D^f . \qquad (13)$$

If the observed quantity is the particle velocity in the solid, the Green's state should be the one corresponding to a nonvanishing point source of the type

$$f_r^{s;G} = F_r^s \delta(\mathbf{x} - \mathbf{x}') , \quad \text{with } \mathbf{x} \in D^s . \qquad (14)$$

If the observed quantity is the stress in the solid, the Green's state should be the one corresponding to a nonvanishing point source of the type

$$h^G_{p,q} = H_{p,q} \delta(\mathbf{x} - \mathbf{x}') , \quad \text{with } \mathbf{x}' \in D^s . \qquad (15)$$

In all these cases the point-source solutions to the acoustic wave equations should be calculated in a fluid/solid configuration with acoustic properties that are adjoint to the one of the actual embedding. In Eqs. (12)–(15), the factors multiplying the three-dimensional Dirac delta functions are constants; they will be used to indicate which of the four acoustic Green's states applies in a particular case.

One way to address the direct and inverse acoustic source problems is to model the emitting sources as equivalent volume source distributions occupying the domain D^T. In the direct source problem the location of this domain is known. In the inverse source problem the location of this domain is usually estimated from travel time measurements. Then, upon identifying state A in Eq. (9) with the acoustic emission state T and state B with one of the Green's states G, application of Eq. (9) to the entire fluid/solid configuration yields [see also Eq. (10)]

$$\begin{aligned}
&\int_{D^T \cap D^f} [f_k^{f;T}(\mathbf{x}) w_k^G(\mathbf{x},\mathbf{x}') + q^T(\mathbf{x}) \sigma^G(\mathbf{x},\mathbf{x}')] dV(\mathbf{x}) \\
&\quad + \int_{D^T \cap D^s} [f_k^{s;T}(\mathbf{x}) v_k^G(\mathbf{x},\mathbf{x}') + h^T_{i,j}(\mathbf{x}) \tau^G_{i,j}(\mathbf{x},\mathbf{x}')] dV(\mathbf{x}) \\
&= \{ Q\sigma^T(\mathbf{x}'), F_r^f w_r^T(\mathbf{x}') \} \chi_{D^f}(\mathbf{x}') \\
&\quad + \{ F_r^s v_r^T(\mathbf{x}'), H_{p,q} \tau^T_{p,q}(\mathbf{x}') \} \chi_{D^s}(\mathbf{x}') . \qquad (16)
\end{aligned}$$

In these expressions, $\chi_D(\mathbf{x}')$ denotes the characteristic function of the set D, i.e.,

$$\chi_D(\mathbf{x}') = \{1,\tfrac{1}{2},0\}, \quad \text{for } \mathbf{x}' \in \{D,\partial D,D'\}, \tag{17}$$

where ∂D is the boundary surface of the set D and D' is the complement of $D \cup \partial D$ in three-dimensional space R^3. Hence, the first two results in the right-hand side of Eq. (16) apply to observation in the fluid, and the second two results to observation in the solid. Which Green's function is to be used in the left-hand side to yield a particular observation result in the right-hand side, follows implicitly from the pertaining nonvanishing constant that multiplies that result.

In a number of cases it is, however, either known or likely that the acoustic emission is due to a source whose action can better be modeled by the presence of equivalent surface sources rather than by equivalent volume sources. For example, this is the case when the acoustic emission is due to the formation of (micro)cracks. In such cases, a relation different from Eq. (16) is needed to model the direct as well as the inverse source problem. Let D^T be the (estimated) domain occupied by the emitting source and let ν_m denote the unit vector along the normal to the boundary surface ∂D^T of D^T (for example, the surface of the crack), oriented away from D^T. Then, application of Eq. (9) to the entire fluid/solid configuration as far as it is located outside ∂D^T yields

$$\begin{aligned}
&\int_{\partial D^T \cap D^f} \nu_k(\mathbf{x}) [\sigma^T(\mathbf{x}) w_k^G(\mathbf{x},\mathbf{x}') - w_k^T(\mathbf{x}) \sigma^G(\mathbf{x},\mathbf{x}')] dA(\mathbf{x}) \\
&\quad + \int_{\partial D^T \cap D^s} \Delta_{k,m,p,q} \nu_m(\mathbf{x}) [\tau_{p,q}^T(\mathbf{x}) v_k^G(\mathbf{x},\mathbf{x}') \\
&\quad - v_k^T(\mathbf{x}) \tau_{p,q}^G(\mathbf{x},\mathbf{x}')] dA(\mathbf{x}) \\
&\quad = \{Q\sigma^T(\mathbf{x}'), F_r^f w_r^T(\mathbf{x}')\} \chi_{D^f \cap D^{T'}}(\mathbf{x}') \\
&\quad + \{F_r^s v_r^T(\mathbf{x}'), H_{p,q} \tau_{p,q}^T(\mathbf{x}')\} \chi_{D^s \cap D^{T'}}(\mathbf{x}') .
\end{aligned} \tag{18}$$

The first two results in the right-hand side of Eq. (18) apply to observation in the fluid as far as the latter is located outside ∂D^T and the second two results to observation in the solid as far as the latter is located outside ∂D^T. Which Green's function is to be used in the left-hand side to yield a particular observation result in the right-hand side, follows implicitly from the pertaining nonvanishing constant that multiplies that result.

To use Eqs. (16) and (18) in a practical inverse source situation, the emitted acoustic wave field quantities are recorded at a number of positions and for a number of values of the time Laplace transform parameter s. Let M be the total number of measured data. Then, first, the location of D^T or ∂D^T is estimated from the observed travel times of the acoustic waves. Next, the nature of the source is reconstructed by parametrizing the relevant source distributions (of the volume or the surface type); this is usually done by expanding them in terms of an appropriate sequence of spatial expansion functions that form a discrete base in the function space to which the volume source densities belong. The coefficients in this expansion will in general depend on the time Laplace transform parameter s. The expansion coefficients satisfy the system of equations that results from evaluating the left-hand sides of Eqs. (16) or (18) for a number of values of s and requiring equality with the measured data. Assume that P is the total number of parameters. Then, the system of linear algebraic equations to be satisfied is of the size $P \times M$. Finally, the system is solved either exactly (for $P = M$) or by minimizing (for $P < M$) a given error (for example, the accumulated square error) in the satisfaction of the equality signs in the equations.

Note that it is essential that the action of the sources as far as they are located in the fluid part of the embedding is modeled by fluid volume or surface sources, while the action of the sources as far as they are located in the solid part of the embedding is modeled by solid volume or surface sources.

V. DIRECT AND INVERSE SCATTERING PROBLEMS

In this section the problem of the acoustic scattering by a penetrable object is addressed. Assume that the object occupies the bounded domain Ω. The object may partly consist of a fluid, partly of a solid; these two parts are, in general, disjoint. Let Ω^f be the domain occupied by the fluid part of the object and Ω^s the domain occupied by its solid part. The presence of the object will be accounted for by expressing its contrast with a known embedding. The latter consists, as before, of a fluid part D^f and a solid part D^s. For the contrast description to be valid, it is essential that Ω^f is a subdomain (not necessarily a proper one) of D^f and Ω^s is a subdomain (not necessarily a proper one) of D^s. Note in this respect that in each case the type of embedding must be adjusted to the type of scattering object as far as the nature of its spatial support is concerned. Once the embedding has been selected, it is assumed that the four Green's states corresponding to a fluid/solid configuration with acoustic properties adjoint to the ones of the actual embedding (see Sec. IV) can be calculated.

The total acoustic wave field in the configuration $\{\sigma, w_r\}$ in D^f, $\{\tau_{p,q}, v_r\}$ in D^s is written as

$$\{\sigma, w_r\} = \{\sigma^{\text{in}} + \sigma^{\text{sc}}, w_r^{\text{in}} + w_r^{\text{sc}}\} \quad \text{in } D^f, \tag{19}$$

$$\{\tau_{p,q}, v_r\} = \{\tau_{p,q}^{\text{in}} + \tau_{p,q}^{\text{sc}}, v_r^{\text{in}} + v_r^{\text{sc}}\} \quad \text{in } D^s, \tag{20}$$

where $\{\sigma^{\text{in}}, w_r^{\text{in}}\}$ in D^f,$\{\tau_{p,q}^{\text{in}}, v_r^{\text{in}}\}$ in D^s is the incident acoustic wave field (i.e., the acoustic wave field that would be present in the configuration if the scattering object showed no contrast with the embedding) and $\{\sigma^{\text{sc}}, w_r^{\text{sc}}\}$ in D^f,$\{\tau_{p,q}^{\text{sc}}, v_r^{\text{sc}}\}$ in D^s is the scattered acoustic wave field. The constitutive parameters of the object are denoted by $\{\rho_{k,r}^{f;\Omega}, \kappa^{\Omega}\}$ in Ω^f and $\{\rho_{k,r}^{s;\Omega}, S_{i,j,p,q}^{\Omega}\}$ in Ω^s.

Now, the incident wave field is, by its nature, generated by sources that are located outside Ω. These sources also generate the total wave field, and hence the scattered wave field is source-free outside Ω. By using in the interior of Ω the systems of acoustic wave equations (2)–(5) pertaining to the total and the incident wave fields, and subtracting the corresponding equations in these systems, one arrives at the contrast source equations of the scattered wave field. In the fluid part of the embedding these are

$$-\partial_k \sigma^{\text{sc}} + s\rho_{k,r}^f w_r^{\text{sc}} = f_k^{f;\text{sc}}, \quad \text{for } \mathbf{x} \in D^f, \tag{21}$$

$$\partial_r w_r^{\text{sc}} - s\kappa\sigma^{\text{sc}} = q^{\text{sc}}, \quad \text{for } \mathbf{x} \in D^f, \tag{22}$$

in which

$$f_k^{f;\mathrm{sc}} = -s(\rho_{k,r}^{f;\Omega} - \rho_{k,r}^{f})w_r, \quad \text{for } \mathbf{x}\in\Omega^f, \tag{23}$$

$$q^{\mathrm{sc}} = s(\kappa^{\Omega} - \kappa)\sigma, \quad \text{for } \mathbf{x}\in\Omega^f, \tag{24}$$

are the fluid part contrast volume source densities of the scattering object, while in the solid part of the embedding they are

$$-\Delta_{k,m,p,q}\partial_m \tau_{p,q}^{\mathrm{sc}} + s\rho_{k,r}^{s} v_r^{\mathrm{sc}} = f_k^{s;\mathrm{sc}}, \quad \text{for } \mathbf{x}\in D^s, \tag{25}$$

$$\Delta_{i,j,m,r}\partial_m v_r^{\mathrm{sc}} - sS_{i,j,p,q}\tau_{p,q}^{\mathrm{sc}} = h_{i,j}^{\mathrm{sc}}, \quad \text{for } \mathbf{x}\in D^s, \tag{26}$$

in which

$$f_k^{s;\mathrm{sc}} = -s(\rho_{k,r}^{s;\Omega} - \rho_{k,r}^{s})v_r, \quad \text{for } \mathbf{x}\in\Omega^s, \tag{27}$$

$$h_{i,j}^{\mathrm{sc}} = s(S_{i,j,p,q}^{\Omega} - S_{i,j,p,q})\tau_{p,q}, \quad \text{for } \mathbf{x}\in\Omega^s, \tag{28}$$

are the solid part contrast volume source densities of the scattering object. The contrast volume source densities vanish outside the scattering object. Then, upon identifying in Eq. (9) state A with the scattered state to be denoted by the superscript sc and state B with one of the Green's states defined through Eqs. (12)–(15), application of Eq. (9) to the entire fluid/solid configuration yields [see also Eq. (16)]

$$\begin{aligned}
&\int_{\Omega^f} [f_k^{f;\mathrm{sc}}(\mathbf{x})w_k^G(\mathbf{x},\mathbf{x}') + q^{\mathrm{sc}}(\mathbf{x})\sigma^G(\mathbf{x},\mathbf{x}')]dV(\mathbf{x}) \\
&\quad + \int_{\Omega^s} [f_k^{s;\mathrm{sc}}(\mathbf{x})v_k^G(\mathbf{x},\mathbf{x}') + h_{i,j}^{\mathrm{sc}}(\mathbf{x})\tau_{i,j}^G(\mathbf{x},\mathbf{x}')]dV(\mathbf{x}) \\
&= \{Q\sigma^{\mathrm{sc}}(\mathbf{x}'), F_r^f w_r^{\mathrm{sc}}(\mathbf{x}')\}\chi_{D^f}(\mathbf{x}') \\
&\quad + \{F_r^s v_r^{\mathrm{sc}}(\mathbf{x}'), H_{p,q}\tau_{p,q}^{\mathrm{sc}}(\mathbf{x}')\}\chi_{D^s}(\mathbf{x}') .
\end{aligned} \tag{29}$$

Equation (29), together with Eqs. (19), (20), (23), (24), (27), and (28), forms the basis for the further treatment of both the direct and the inverse scattering problems.

A. The direct scattering problem

In the direct (or forward) scattering problem the location of the scattering object and its acoustic properties are known. Now, upon taking successively $\mathbf{x}'\in\Omega^f$ and $\mathbf{x}'\in\Omega^s$ in Eq. (29), and combining the results with Eqs. (19), (20), (23), (24), (27), and (28), a system of integral equations is obtained in which $\{\sigma,w_r\}$ for $\mathbf{x}'\in\Omega^f$ and $\{\tau_{p,q},v_r\}$ for $\mathbf{x}\in\Omega^s$ are the unknowns. The integral equations are of the second kind and have usually to be solved with the aid of numerical discretization methods. After the unknown acoustic wave field quantities in the interior of the scattering object have been evaluated, the contrast volume source densities are known and the contrast source representations of the scattered wave field can be reused to calculate this wave field everywhere in the configuration. Since the incident wave field was already known, this brings the solution of the direct scattering problem to an end.

B. The inverse scattering problem

In the inverse scattering problem both the location of the scattering object and its acoustic properties are unknown. The object is irradiated by some known incident acoustic wave field and the response of the object, i.e., its scattered acoustic wave field, is measured at a number of locations and for a number of values of the time Laplace-transform parameter s. The next step is to consider the inverse scattering problem as an inverse source problem for the determination of the contrast volume source densities (see Sec. IV). Once these have been determined, the contrast source representations for the scattered acoustic wave field are used to evaluate the total acoustic wave field in the interior of the scattering object. Subsequent use of Eqs. (23) and (24) and Eqs. (27) and (28) then leads to the distribution of the constitutive parameters in the fluid and the solid parts of the object, respectively.

In an actual application of this method the first thing to do will be to determine the fluid/solid interfaces that separate the inclusions from the embedding. Only after that has been accomplished, one can introduce contrast volume source densities to model the presence of the scattering object.

VI. CONCLUSION

The time Laplace-transform domain convolution-type reciprocity relation for acoustic waves in a fluid/solid configuration with arbitrary inhomogeneity, anisotropy, and loss mechanisms has been derived. First, reciprocity properties of transmitting and receiving transducers located in either the fluid or the solid parts of the configuration have been established. Second, it is shown how the recipocity relation leads to the integral-equation formulation of inverse source, and direct and inverse scattering problems through the introduction of the appropriate contrast source representations. Since neither the presence of a fluid inclusion in a solid nor the presence of a solid inclusion in a fluid leads to a regular perturbation problem in the integral-equation formulation, the embedding must be adapted to the location of the contrast sources as far as the type of medium (fluid or solid) is concerned. Applications to acoustic emission and to acoustic imaging and object reconstruction are briefly indicated.

[1]A. T. de Hoop, "Time-domain reciprocity theorems for acoustic wave fields in fluids with relaxation," J. Acoust. Soc. Am. **84**, 1877–1882 (1988).
[2]A. T. de Hoop, "Time-domain reciprocity theorems for elastodynamic wave fields in solids with relaxation and their application to inverse problems," Wave Motion **10**, 479–489 (1988).
[3]R. Burridge and L. Knopoff, "Body force equivalents for seismic dislocations," Bull. Seismol. Soc. Am. **54**, 1875–1888 (1964).
[4]M. Schoenberg and P. N. Sen, "Properties of a periodically stratified acoustic half-space and its relation to a Biot fluid," J. Acoust. Soc. Am. **73**, 61–67 (1983).
[5]A. T. de Hoop, "A time-domain energy theorem for the scattering of plane acoustic waves in fluids," J. Acoust. Soc. Am. **77**, 11–14 (1985).
[6]A. T. de Hoop, "A time-domain energy theorem for the scattering of plane elastic waves," Wave Motion **7**, 569–577 (1985).
[7]G. S. Kino, "The application of reciprocity theory to scattering of acoustic waves by flaws," J. Appl. Phys. **49**, 3190–3199 (1978).