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Asymptotic ray theory for transient diffusive electromagnetic fields

Adrianus T. de Hoop,[1] Michael L. Oristaglio, Tarek M. Habashy, and Carlos Torres-Verdín
Schlumberger-Doll Research, Ridgefield, Connecticut

Abstract. We develop an asymptotic ray theory for transient diffusive electromagnetic fields in isotropic media. The formulation is first derived in the time Laplace transform domain by introducing an ansatz procedure, whereby appropriate expansions for the electric and the magnetic field strengths are substituted in the original field equations. We arrive at consistent recurrence formulas for the sequences of field amplitude vectors that multiply appropriately chosen asymptotic sequences of algebraic powers of the Laplace transform parameter. These representations differ for the two types of fields (electric and magnetic fields) and for the two types of sources (electric current source and magnetic current source). The exponential part of the field expressions contains the diffusive equivalent of the eikonal function in the asymptotic ray theory of wave propagation. This function satisfies the diffusive equivalent of the eikonal equation. Next, we derive the transport equations for the vectorial electric and magnetic field amplitudes of the successive orders. Transient field representations within the asymptotic ray approximation are then obtained by carrying out the inverse Laplace transformation to the time domain by inspection. The ray approximation thus obtained is asymptotic for "early times". We consider as an example the case of the electric and the magnetic dipole radiation in a homogeneous medium. Here an exact solution exists, which we show to exhibit the structure of the original ansatz but with a finite number of terms. The asymptotic ray theory for transient diffusive electromagnetic fields is expected to lend itself to important applications in surface, surface-to-borehole, and crosswell transient electromagnetic prospecting.

1. Introduction

The asymptotic ray theory of acoustic and elastodynamic wave fields in inhomogeneous and anisotropic structures has found important applications in geophysical prospecting (for example, in crosswell seismic tomography). This motivates research into the potentialities of ray-asymptotic methods in surface, surface-to-borehole, and crosswell transient electromagnetic prospecting. In the present paper such a theory is developed for the diffusive electromagnetic field equations where the contribution from the electric displacement current is neglected. Starting from these equations in the time Laplace transform domain, an ansatz procedure is introduced, whereby appropriate expansions for the electric and the magnetic field strengths are substituted in the field equations. The algebraic powers of the Laplace transform parameter in the expansions are chosen in such a way that a consistent recurrence scheme results. The exponential part of the field expressions contains the diffusive equivalent of the eikonal function in the asymptotic ray theory of wave propagation. This function satisfies the diffusive equivalent of the eikonal equation. Next, we derive the transport equations for the vectorial electric and magnetic field amplitudes of the successive orders. Subsequently, we transform the results to the time domain and thus obtain the transient field representation within the asymptotic ray approximation. The successive terms in the ray approximation thus obtained form an asymptotic sequence for "early times". As an example, the case of a homogeneous medium is considered as a canonical problem. The exact solution to the point source problem is shown here to have the structure of the ansatz we started with, but it contains a finite number of terms.

[1] Visiting from Laboratory of Electromagnetic Research, Delft University of Technology, Delft, the Netherlands.

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For the scalar diffusion equation an asymptotic ray theory has been developed by *Cohen and Lewis* [1967]. The electromagnetic problem is somewhat more complicated. First, the asymptotic ray representation for the electric field differs from the one for the magnetic field, since the symmetry in the electric and the magnetic field that exists in electromagnetic wave theory [*Born and Wolf*, 1980; *Kline and Kay*, 1965; *Babic and Buldyrev*, 1991] is lost. Furthermore, the representations differ for the two types of sources (electric current sources and magnetic current sources). Finally, different diffusion kernels occur in the different field constituents. As far as the problem of coupling the asymptotic ray representations to a source (for which the point source is a canonical case) is concerned, the difficulty that the successive orders show a spatial singularity of increasing order in the neighborhood of the source [see *Bleistein*, 1984] remains. In this respect it is to be noted that in the case of a homogeneous medium the representation terminates exactly with the term that shows the highest singularity that is compatible with the presence of a point source.

In the analysis, $\mathbf{r}$ is the position vector in an orthogonal, Cartesian reference frame, t is the time, ∇ is the spatial vectorial differentiation operator, and ∂_t denotes differentiation with respect to time. The field quantities are the electric field strength $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and the magnetic field strength $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$. The source quantities are the volume source density of electric current $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ and the volume source density of magnetic current $\mathbf{K} = \mathbf{K}(\mathbf{r}, t)$. The field quantities are taken to be causally related to their excitation by the sources. The medium properties are assumed to be nondispersive and are characterized by the conductivity $\sigma = \sigma(\mathbf{r})$ and the permeability $\mu = \mu(\mathbf{r})$, which are taken to be piecewise continuous functions of position.

All source and field quantities are assumed to be bounded functions of time. They vanish prior to the instant, taken as $t = 0$, at which the sources are switched on. Their time Laplace transforms are thus given by

$$\hat{F}(\mathbf{r}, s) = \int_{t=0}^{\infty} \exp(-st) F(\mathbf{r}, t)\, dt \qquad s \in \mathfrak{R},\ s > 0, \tag{1}$$

where F stands for any of the source or field quantities. Note that we have taken s as real and positive, which is sufficient to guarantee a unique interrelationship between a causal time domain quantity and its Laplace transform. In view of Lerch's theorem [*Widder*, 1946] the uniqueness is even guaranteed for the set of equidistant points on the positive, real s axis $\{s_n = s_0 + nh;\ s_0 \in \mathfrak{R},\ s_0 > 0,\ h \in \mathfrak{R},\ h > 0,\ n = 0, 1, 2, \cdots\}$.

The transient electromagnetic diffusion problem is governed by the electromagnetic field equations in which the electric displacement current is neglected, namely,

$$\nabla \times \mathbf{H} - \sigma\mathbf{E} = \mathbf{J} \tag{2}$$

$$\nabla \times \mathbf{E} + \mu\partial_t\mathbf{H} = -\mathbf{K} \tag{3}$$

with their time Laplace-transform counterparts

$$\nabla \times \hat{\mathbf{H}} - \sigma\hat{\mathbf{E}} = \hat{\mathbf{J}} \tag{4}$$

$$\nabla \times \hat{\mathbf{E}} + s\mu\hat{\mathbf{H}} = -\hat{\mathbf{K}}. \tag{5}$$

The corresponding compatibility relations are

$$\nabla \cdot (\sigma\mathbf{E}) = -\nabla \cdot \mathbf{J} \tag{6}$$

$$\nabla \cdot (\mu\partial_t\mathbf{H}) = -\nabla \cdot \mathbf{K} \tag{7}$$

with their time Laplace transform counterparts

$$\nabla \cdot (\sigma\hat{\mathbf{E}}) = -\nabla \cdot \hat{\mathbf{J}} \tag{8}$$

$$s\nabla \cdot (\mu\hat{\mathbf{H}}) = -\nabla \cdot \hat{\mathbf{K}}. \tag{9}$$

Equations (4) and (5) serve as the starting point for the substitution of a ray asymptotic expansion. From them the diffusive eikonal equation as well as the recurrence relations between the successive vectorial electric and magnetic field amplitudes follow, together with a number of compatibility relations between these amplitudes. The transport equations for these amplitudes, on the other hand, follow more easily from the second-order equations embodied by (4) and (5) and (8) and (9), namely

$$\nabla[(1/\sigma)\nabla \cdot (\sigma\hat{\mathbf{E}})] - \mu\nabla \times [1/\mu(\nabla \times \hat{\mathbf{E}})] - s\sigma\mu\hat{\mathbf{E}} = s\mu\hat{\mathbf{J}} - \nabla[1/\sigma(\nabla \cdot \hat{\mathbf{J}})] + \mu\nabla \times [(1/\mu)\hat{\mathbf{K}}] \tag{10}$$

$$\nabla[(1/\mu)\nabla \cdot (\mu\hat{\mathbf{H}})] - \sigma\nabla \times [1/\sigma(\nabla \times \hat{\mathbf{H}})] - s\sigma\mu\hat{\mathbf{H}} = \sigma\hat{\mathbf{K}} - \nabla[(1/s\mu)(\nabla \cdot \hat{\mathbf{K}})] - \sigma\nabla \times [(1/\sigma)\hat{\mathbf{J}}]. \tag{11}$$

In particular, the cases of a point source of electric current and a point source of magnetic current will be investigated. For these we have

$$\mathbf{J} = J_0(t)\mathbf{a}\delta(\mathbf{r} - \mathbf{r}') \tag{12}$$

$$\mathbf{K} = K_0(t)\mathbf{b}\delta(\mathbf{r} - \mathbf{r}'), \tag{13}$$

where $\mathbf{r}'$ is the position of the point source, $J_0(t)$ and $K_0(t)$ are the source signatures, and $\mathbf{a}$ and $\mathbf{b}$ are unit vectors that specify the orientation of the source. Correspondingly,

$$\hat{\mathbf{J}} = \hat{J}_0(s)\mathbf{a}\delta(\mathbf{r} - \mathbf{r}') \tag{14}$$

$$\hat{\mathbf{K}} = \hat{K}_0(s)\mathbf{b}\delta(\mathbf{r} - \mathbf{r}'). \tag{15}$$

2. Ray Asymptotic Expansions for the Fields Generated by an Electric Current or a Magnetic Current Point Source

For the electromagnetic field generated by a point source of electric current the ray asymptotic expansions are taken as

$$\hat{\mathbf{E}}^J(\mathbf{r}, \mathbf{r}', s) \sim \hat{J}_0(s) \cdot \left[\sum_{m\geq 0} s^{-(m-2)/2}\mathbf{e}_m^J(\mathbf{r}, \mathbf{r}')\right] \exp\left[-s^{1/2}\Psi(\mathbf{r}, \mathbf{r}')\right] \tag{16}$$

$$\hat{\mathbf{H}}^J(\mathbf{r}, \mathbf{r}', s) \sim \hat{J}_0(s) \cdot \left[\sum_{m\geq 0} s^{-(m-1)/2}\mathbf{h}_m^J(\mathbf{r}, \mathbf{r}')\right] \exp\left[-s^{1/2}\Psi(\mathbf{r}, \mathbf{r}')\right], \tag{17}$$

and the expansions for the field generated by a point source of magnetic current are taken as

$$\hat{\mathbf{E}}^K(\mathbf{r}, \mathbf{r}', s) \sim \hat{K}_0(s) \cdot \left[\sum_{m\geq 0} s^{-(m-1)/2}\mathbf{e}_m^K(\mathbf{r}, \mathbf{r}')\right] \exp\left[-s^{1/2}\Psi(\mathbf{r}, \mathbf{r}')\right] \tag{18}$$

$$\hat{\mathbf{H}}^K(\mathbf{r}, \mathbf{r}', s) \sim \hat{K}_0(s) \cdot \left[\sum_{m\geq 0} s^{-m/2}\mathbf{h}_m^K(\mathbf{r}, \mathbf{r}')\right] \exp\left[-s^{1/2}\Psi(\mathbf{r}, \mathbf{r}')\right]. \tag{19}$$

Here $\Psi = \Psi(\mathbf{r}, \mathbf{r}')$ is the diffusion eikonal, and all dependences on s, $\mathbf{r}$, and $\mathbf{r}'$ have been indicated explicitly. Substitution of the expansions in (4) and (5) and equating to zero the terms of equal powers of s leads, for both expansions, to the following recursion relations that hold away from the source and in which the superscripts J and K, as well as the arguments $\mathbf{r}$ and $\mathbf{r}'$, are omitted:

$$\nabla \times \mathbf{h}_{m-1} - \nabla\Psi \times \mathbf{h}_m - \sigma\mathbf{e}_m = \mathbf{0} \quad m = 1, 2, 3, \cdots, \tag{20}$$

$$\nabla \times \mathbf{e}_{m-1} - \nabla\Psi \times \mathbf{e}_m + \mu\mathbf{h}_m = \mathbf{0} \quad m = 1, 2, 3, \cdots, \tag{21}$$

while the terms corresponding to $m = 0$ lead to

$$-\nabla\Psi \times \mathbf{h}_0 - \sigma\mathbf{e}_0 = \mathbf{0}, \tag{22}$$

$$-\nabla\Psi \times \mathbf{e}_0 + \mu\mathbf{h}_0 = \mathbf{0}. \tag{23}$$

To investigate the condition under which a nontrivial solution of (22) and (23) exists, we first scalarly multiply (22) by $\nabla\Psi$, which leads to

$$\mathbf{e}_0 \cdot \nabla\Psi = 0 \tag{24}$$

and scalarly multiply (23) by $\nabla\Psi$, which leads to

$$\mathbf{h}_0 \cdot \nabla\Psi = 0. \tag{25}$$

Elimination of $\mathbf{h}_0$ from (22) and (23) and use of (24) leads to

$$(\nabla\Psi \cdot \nabla\Psi - \sigma\mu)\mathbf{e}_0 = \mathbf{0}. \tag{26}$$

Similarly, elimination of $\mathbf{e}_0$ from (22) and (23) and use of (25) leads to

$$(\nabla\Psi \cdot \nabla\Psi - \sigma\mu)\mathbf{h}_0 = \mathbf{0}. \tag{27}$$

Both equations lead to the condition

$$\nabla\Psi \cdot \nabla\Psi = \sigma\mu. \tag{28}$$

Equation (28) is the diffusive ray counterpart of the eikonal equation in the asymptotic ray theory of wave propagation.

Another property of the zero-order amplitudes follows upon postmultiplying (22) vectorially by $\mathbf{h}_0$. Again, using (25), this leads to

$$\mathbf{e}_0 \times \mathbf{h}_0 = \sigma^{-1}(\mathbf{h}_0 \cdot \mathbf{h}_0)\nabla\Psi. \tag{29}$$

Similarly, premultiplying (23) vectorially by $\mathbf{e}_0$ and using (24), we obtain

$$\mathbf{e}_0 \times \mathbf{h}_0 = \mu^{-1}(\mathbf{e}_0 \cdot \mathbf{e}_0)\nabla\Psi. \tag{30}$$

From (29) and (30) it follows that

$$\sigma \mathbf{e}_0 \cdot \mathbf{e}_0 = \mu \mathbf{h}_0 \cdot \mathbf{h}_0 \tag{31}$$

and hence we can write

$$\mathbf{e}_0 \times \mathbf{h}_0 = (\sigma\mu)^{-1}(\tfrac{1}{2}\sigma \mathbf{e}_0 \cdot \mathbf{e}_0 + \tfrac{1}{2}\mu \mathbf{h}_0 \cdot \mathbf{h}_0)\nabla\Psi. \tag{32}$$

Another useful relation is obtained by taking the gradient of (28). Applying the vector identity that rewrites the gradient of the scalar product of two vector functions, it follows that

$$\nabla(\nabla\Psi \cdot \nabla\Psi) = 2[(\nabla\Psi) \cdot \nabla]\nabla\Psi. \tag{33}$$

Using this, we arrive at

$$[(\nabla\Psi) \cdot \nabla]\nabla\Psi = \tfrac{1}{2}\nabla(\sigma\mu). \tag{34}$$

This result is of importance to the construction of the ray trajectories.

Once $\mathbf{e}_0$ and $\mathbf{h}_0$ have been determined, the recurrence scheme given by (20) and (21) can start. This system of equations is subject to a number of compatibility relations. First, operating on it, as well as on (22) and (23), with the divergence operator ∇, we obtain

$$-\nabla\Psi \cdot (\nabla \times \mathbf{h}_m) + \nabla \cdot (\sigma \mathbf{e}_m) = 0 \qquad m = 0, 1, 2, \cdots, \tag{35}$$

$$\nabla\Psi \cdot (\nabla \times \mathbf{e}_m) + \nabla \cdot (\sigma \mathbf{h}_m) = 0 \qquad m = 0, 1, 2, \cdots. \tag{36}$$

Second, operating on (20) and (21) with $\nabla\Psi$, we obtain

$$-\nabla\Psi \cdot (\nabla \times \mathbf{h}_{m-1}) + \nabla\Psi \cdot (\sigma \mathbf{e}_m) = 0 \qquad m = 1, 2, 3, \cdots, \tag{37}$$

$$\nabla\Psi \cdot (\nabla \times \mathbf{e}_{m-1}) + \nabla\Psi \cdot (\mu \mathbf{h}_m) = 0 \qquad m = 1, 2, 3, \cdots. \tag{38}$$

From (35) and (36) and (37) and (38) it follows that

$$\nabla\Psi \cdot (\sigma \mathbf{e}_m) = \nabla \cdot (\sigma \mathbf{e}_{m-1}) \qquad m = 1, 2, 3, \cdots, \tag{39}$$

$$\nabla\Psi \cdot (\mu \mathbf{h}_m) = \nabla \cdot (\mu \mathbf{h}_{m-1}) \qquad m = 1, 2, 3, \cdots. \tag{40}$$

These relations also follow upon substituting the ray asymptotic expansions into the electromagnetic compatibility relations (8) and (9) away from the sources.

Finally, we have to account for the property that in the determinant of the coefficients of (20) and (21), when considered as a system of equations to be solved for $\mathbf{e}_m$ and $\mathbf{h}_m$, $\nabla\Psi$ satisfies the eikonal equation (28). In view of this, elimination of $\mathbf{e}_m$ from (20) and (21) leads to

$$-\nabla\Psi \times (\nabla \times \mathbf{h}_{m-1}) + (1/\mu)\{[\nabla\Psi \times (\nabla \times \mathbf{e}_{m-1})] \times \nabla\Psi\} = 0 \qquad m = 1, 2, 3, \cdots, \tag{41}$$

and elimination of $\mathbf{h}_m$ to

$$\nabla\Psi \times (\nabla \times \mathbf{e}_{m-1}) + (1/\sigma)\{[\nabla\Psi \times (\nabla \times \mathbf{h}_{m-1})] \times \nabla\Psi\} = 0 \qquad m = 1, 2, 3, \cdots. \tag{42}$$

It is observed that (42) follows from (41) and vice versa.

3. Ray Trajectories

The ray trajectories are constructed from the eikonal equation in the standard manner. They are the curves in space that are the orthogonal trajectories to the family of surfaces where Ψ = const. Therefore, at any point on a surface Ψ = const the unit vector along the normal coincides with the unit vector along the tangent to the ray trajectory passing through that point. Denoting this unit vector by $\boldsymbol{\tau}$, from (28) one obtains

$$\nabla\Psi = (\sigma\mu)^{1/2}\boldsymbol{\tau}, \tag{43}$$

where the property $\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1$ has been used. From this equation it follows that

$$\boldsymbol{\tau} \cdot \nabla\Psi = (\sigma\mu)^{1/2}. \tag{44}$$

Now $\boldsymbol{\tau} \cdot \nabla$ is the spatial directional derivative along the ray trajectory, which, upon introducing the arc length λ along the ray trajectory as a parameter, can also be written as $\mathrm{d}/\mathrm{d}\lambda$. With this notation, (44) becomes

$$\mathrm{d}\Psi/\mathrm{d}\lambda = (\sigma\mu)^{1/2}. \tag{45}$$

Integrating (45) along a ray trajectory from any point $\mathcal{P}$ corresponding to $\lambda = \lambda_{\mathcal{P}}$ to another point $\mathcal{Q}$ corresponding to $\lambda = \lambda_{\mathcal{Q}}$, the relation

$$\Psi_{\mathcal{Q}} - \Psi_{\mathcal{P}} = \int_{\lambda=\lambda_{\mathcal{P}}}^{\lambda_{\mathcal{Q}}} (\mathrm{d}\Psi/\mathrm{d}\lambda)\,\mathrm{d}\lambda = \int_{\lambda=\lambda_{\mathcal{P}}}^{\lambda_{\mathcal{Q}}} (\sigma\mu)^{1/2}\,\mathrm{d}\lambda \tag{46}$$

is obtained, which, in view of the property $(\sigma\mu)^{1/2} > 0$, implies that Ψ increases in the direction in which Ψ increases.

To construct a particular ray trajectory emanating from a point source, Ψ is set equal to zero at the source and a particular starting direction is chosen. The subsequent course of the ray trajectory is then found by solving a second-order ray trajectory differential equation. Let $\boldsymbol{\rho} = \boldsymbol{\rho}(\lambda)$ be the parametric representation of the ray trajectory, then

$$\boldsymbol{\tau} = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\rho} = d\boldsymbol{\rho}/d\lambda. \tag{47}$$

Using (43) and (47) in (34), the relevant differential equation is obtained as

$$(\sigma\mu)^{1/2}(d/d\lambda)[(\sigma\mu)^{1/2}(d\boldsymbol{\rho}/d\lambda)] = \tfrac{1}{2}\nabla(\sigma\mu) \tag{48}$$

or equivalently as

$$d/d\lambda[(\sigma\mu)^{1/2}(d\boldsymbol{\rho}/d\lambda)] = \nabla(\sigma\mu)^{1/2}. \tag{49}$$

Substitution of (43) in (24) and (25) yields

$$\mathbf{e}_0 \cdot \boldsymbol{\tau} = 0 \tag{50}$$

$$\mathbf{h}_0 \cdot \boldsymbol{\tau} = 0. \tag{51}$$

Hence $\mathbf{e}_0$ and $\mathbf{h}_0$ are transverse with respect to the direction of the ray. Furthermore, substitution of (43) into (29), (30), and (32) leads to

$$\begin{aligned} \mathbf{e}_0 \times \mathbf{h}_0 &= (\mu/\sigma)^{1/2}(\mathbf{h}_0 \cdot \mathbf{h}_0)\boldsymbol{\tau} \\ &= (\sigma/\mu)^{1/2}(\mathbf{e}_0 \cdot \mathbf{e}_0)\boldsymbol{\tau} \\ &= (\sigma\mu)^{-1/2}(\tfrac{1}{2}\sigma\mathbf{e}_0 \cdot \mathbf{e}_0 + \tfrac{1}{2}\mu\mathbf{h}_0 \cdot \mathbf{h}_0)\boldsymbol{\tau}, \end{aligned} \tag{52}$$

which shows that $\mathbf{e}_0 \times \mathbf{h}_0$ is oriented along the ray. The results of this section show that for the diffusive case one can employ the standard ray-tracing procedures known to apply to wave phenomena.

4. Amplitude Transport Equations

The amplitude transport equations govern the behavior of the ray asymptotic diffusive field amplitudes $\{\mathbf{e}_m, \mathbf{h}_m;\ m = 0, 1, 2, \cdots\}$ along a ray trajectory. They have the form of a first-order differential equation with $\nabla\Psi\cdot\nabla = (\sigma\mu)^{1/2}(\boldsymbol{\tau}\cdot\nabla) = (\sigma\mu)^{1/2}\, d/d\lambda$ as differential operator. Although the equations can be constructed from the system of simultaneous equations (20) and (21) (see *Kline and Kay* [1965] for the case of electromagnetic waves), they follow much more easily upon substituting the ray asymptotic expansions (16)–(19) in the second-order vector differential equations (10) and (11) for the electric and the magnetic field strengths. Carrying out this procedure and applying several rules from vector calculus, the following result is obtained

$$2(\nabla\Psi \cdot \nabla)\mathbf{h}_0 + \left[\sigma\nabla \cdot \left(\frac{1}{\sigma}\nabla\Psi\right)\right]\mathbf{h}_0 + \left(\frac{\nabla(\sigma\mu)}{\sigma\mu} \cdot \mathbf{h}_0\right)\nabla\Psi = \mathbf{0} \tag{53}$$

$$2(\nabla\Psi \cdot \nabla)\mathbf{e}_0 + \left[\mu\nabla \cdot \left(\frac{1}{\mu}\nabla\Psi\right)\right]\mathbf{e}_0 + \left(\frac{\nabla(\sigma\mu)}{\sigma\mu} \cdot \mathbf{e}_0\right)\nabla\Psi = \mathbf{0} \tag{54}$$

for the leading-order terms in the expansion and

$$\begin{aligned} &2(\nabla\Psi \cdot \nabla)\mathbf{h}_{m-1} + \left[\sigma\nabla \cdot \left(\frac{1}{\sigma}\nabla\Psi\right)\right]\mathbf{h}_{m-1} \\ &+ \left(\frac{\nabla(\sigma\mu)}{\sigma\mu} \cdot \mathbf{h}_{m-1}\right)\nabla\Psi = \nabla\left[\frac{1}{\mu}\nabla \cdot (\mu\mathbf{h}_{m-2})\right] \\ &- \sigma\nabla \times \left(\frac{1}{\sigma}\nabla \times \mathbf{h}_{m-2}\right) \qquad m = 2, 3, 4, \cdots, \end{aligned} \tag{55}$$

$$\begin{aligned} &2(\nabla\Psi \cdot \nabla)\mathbf{e}_{m-1} + \left[\mu\nabla \cdot \left(\frac{1}{\mu}\nabla\Psi\right)\right]\mathbf{e}_{m-1} \\ &+ \left(\frac{\nabla(\sigma\mu)}{\sigma\mu} \cdot \mathbf{e}_{m-1}\right)\nabla\Psi = \nabla\left[\frac{1}{\sigma}\nabla \cdot (\sigma\mathbf{e}_{m-2})\right] \\ &- \mu\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{e}_{m-2}\right) \qquad m = 2, 3, 4, \cdots, \end{aligned} \tag{56}$$

for the higher-order terms.

5. Electric Current and Magnetic Current Point Source Fields in a Homogeneous Medium

So far the formal expansions (16) and (17) and (18) and (19) apply to the source-free diffusive electromagnetic field equations. To couple these expansions to the sources, the cases of an electric and a magnetic dipole in a homogeneous medium are taken here as canonical ones. The field expressions for these cases are exactly known, and they will be shown to have the structure of our ansatz, now, however, with a finite number of terms.

For the diffusive electromagnetic field associated with a point source of electric current with volume density $\hat{\mathbf{J}} = \hat{J}_0(s)\mathbf{a}\delta(\mathbf{r} - \mathbf{r}')$ located at $\mathbf{r} = \mathbf{r}'$ and placed in a homogeneous medium the time Laplace transform expressions are given by

$$\hat{\mathbf{E}}^J = -s\mu\hat{J}_0\mathbf{a}\hat{G} + \sigma^{-1}\nabla[\nabla\cdot(\hat{J}_0\mathbf{a}\hat{G})] \tag{57}$$

$$\hat{\mathbf{H}}^J = \nabla\times(\hat{J}_0\mathbf{a}\hat{G}), \tag{58}$$

where

$$\hat{G} = \frac{\exp\,[-s^{1/2}(\sigma\mu)^{1/2}|\mathbf{R}|]}{4\pi|\mathbf{R}|} \quad \text{with} \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{59}$$

is the time Laplace transform domain Green's function of the scalar diffusion equation. The right-hand sides of (57) and (58) have the form of the expansions (16) and (17) and can be written as

$$\hat{\mathbf{E}}^J = \hat{J}_0(s\mathbf{e}_0^J + s^{1/2}\mathbf{e}_1^J + \mathbf{e}_2^J)\exp\,(-s^{1/2}\Psi) \tag{60}$$

$$\hat{\mathbf{H}}^J = \hat{J}_0(s^{1/2}\mathbf{h}_0^J + \mathbf{h}_1^J)\exp\,(-s^{1/2}\Psi), \tag{61}$$

where the diffusion eikonal is given by

$$\Psi = (\sigma\mu)^{1/2}|\mathbf{R}| \tag{62}$$

and the amplitude factors by

$$\mathbf{e}_0^J = -\mu[\mathbf{a} - (\mathbf{a}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|} \tag{63}$$

$$\mathbf{e}_1^J = -\left(\frac{\mu}{\sigma}\right)^{1/2}[\mathbf{a} - 3(\mathbf{a}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|^2} \tag{64}$$

$$\mathbf{e}_2^J = -\frac{1}{\sigma}[\mathbf{a} - 3(\mathbf{a}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|^3} \tag{65}$$

$$\mathbf{h}_0^J = (\sigma\mu)^{1/2}(\mathbf{a}\times\mathbf{R}^{(1)})\frac{1}{4\pi|\mathbf{R}|} \tag{66}$$

$$\mathbf{h}_1^J = (\mathbf{a}\times\mathbf{R}^{(1)})\frac{1}{4\pi|\mathbf{R}|^2} \tag{67}$$

in which

$$\mathbf{R}^{(1)} = \mathbf{R}/|\mathbf{R}| \tag{68}$$

is the unit vector in the radial direction from the source.

For the diffusive electromagnetic field associated with a point source of magnetic current with volume density $\hat{\mathbf{K}} = \hat{\mathbf{K}}_0(s)\delta(\mathbf{r} - \mathbf{r}')$ located at $\mathbf{r} = \mathbf{r}'$ and in a homogeneous medium the time Laplace transform expressions are given by

$$\hat{\mathbf{H}}^K = -\sigma\hat{K}_0\mathbf{b}\hat{G} + (s\mu)^{-1}\nabla[\nabla\cdot(\hat{K}_0\mathbf{b}\hat{G})], \tag{69}$$

$$\hat{\mathbf{E}}^K = -\nabla\times(\hat{K}_0\mathbf{b}\hat{G}), \tag{70}$$

where, again,

$$\hat{G} = \frac{\exp\,[-s^{1/2}(\sigma\mu)^{1/2}|\mathbf{R}|]}{4\pi|\mathbf{R}|} \quad \text{with} \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{71}$$

is the time Laplace transform domain Green's function of the scalar diffusion equation. The right-hand sides of (69) and (70) have the form of the expansions (18) and (19) and can be written as

$$\hat{\mathbf{H}}^K = \hat{K}_0(\mathbf{h}_0^K + s^{-1/2}\mathbf{h}_1^K + s^{-1}\mathbf{h}_2^K)\exp\,(-s^{1/2}\Psi) \tag{72}$$

$$\hat{\mathbf{E}}^K = \hat{K}_0(s^{1/2}\mathbf{e}_0^K + \mathbf{e}_1^K)\exp\,(-s^{1/2}\Psi) \tag{73}$$

in which the diffusion eikonal is again given by

$$\Psi = (\sigma\mu)^{1/2}|\mathbf{R}| \tag{74}$$

and the amplitude factors by

$$\mathbf{h}_0^K = -\sigma[\mathbf{b} - (\mathbf{b}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|} \tag{75}$$

$$\mathbf{h}_1^K = -\left(\frac{\sigma}{\mu}\right)^{1/2}[\mathbf{b} - 3(\mathbf{b}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|^2} \tag{76}$$

$$\mathbf{h}_2^K = -\frac{1}{\mu}[\mathbf{b} - 3(\mathbf{b}\cdot\mathbf{R}^{(1)})\mathbf{R}^{(1)}]\frac{1}{4\pi|\mathbf{R}|^3} \tag{77}$$

$$\mathbf{e}_0^K = -(\sigma\mu)^{1/2}(\mathbf{b}\times\mathbf{R}^{(1)})\frac{1}{4\pi|\mathbf{R}|} \tag{78}$$

$$\mathbf{e}_1^K = -(\mathbf{b}\times\mathbf{R}^{(1)})\frac{1}{4\pi|\mathbf{R}|^2} \tag{79}$$

in which again

$$\mathbf{R}^{(1)} = \mathbf{R}/|\mathbf{R}| \tag{80}$$

is the unit vector in the radial direction from the source. The results of this section are used to match the limiting behavior of the leading terms in the asymptotic ray expansions to the source strength of a point source of electric or magnetic current.

6. Coupling of the Ray Asymptotic Expansions to a Dipole Source

A procedure to couple the ray asymptotic expansions to an electric or a magnetic dipole source is to let, within a ball of arbitrarily small radius δ around the source point, the spatially singular terms in the expansions coincide with the field expressions applying to a homogeneous medium with conductivity and permeability equal to the values of the inhomogeneous medium at the location of the source. For this purpose the right-hand sides of (63) and (66) are used for the case of an electric dipole source and the right-hand sides of (75) and (78) for the case of a magnetic dipole source, with σ replaced by $\sigma(\mathbf{r}')$ and μ by $\mu(\mathbf{r}')$, while $\mathbf{R}^{(1)}$ is replaced by the initial value of $\boldsymbol{\tau}$ along the chosen ray trajectory leaving the source. Thus the limiting behavior of $\mathbf{e}_0$ and $\mathbf{h}_0$ is determined, and the recurrence relations for the successive terms in the ray asymptotic expansions can start. For this procedure to work we first rely on the property that the exact dipole source solution in the inhomogeneous medium can be decomposed into a singular part that has the described behavior and a regular part that is certainly of order $o(1)$ as $|\mathbf{r} - \mathbf{r}'| \to 0$. Second, we assume that in accordance with the assumed existence of a ray asymptotic expansion as given by our ansatz, the property also holds that the regular part is of order $o(s^{\alpha})$ as $s \to \infty$, where α is the highest power of s that occurs in the singular part of the expansion. For the electric dipole source we accordingly have

$$\hat{\mathbf{E}}^J = (s\hat{\mathbf{e}}_0^J + s^{1/2}\hat{\mathbf{e}}_1^J + \hat{\mathbf{e}}_2^J) \exp(-s^{1/2}\Psi) + \hat{\mathbf{E}}_\infty^J \tag{81}$$

$$\hat{\mathbf{H}}^J = (s^{1/2}\hat{\mathbf{h}}_0^J + \hat{\mathbf{h}}_1^J) \exp(-s^{1/2}\Psi) + \hat{\mathbf{H}}_\infty^J \tag{82}$$

in which the remainders in the expansion satisfy the relations

$$s\hat{\mathbf{E}}_\infty^J = o(1) \qquad |\mathbf{r} - \mathbf{r}'| \to 0 \qquad s \to \infty \tag{83}$$

$$s^{1/2}\hat{\mathbf{H}}_\infty^J = o(1) \qquad |\mathbf{r} - \mathbf{r}'| \to 0 \qquad s \to \infty, \tag{84}$$

while for the magnetic dipole source we have

$$\hat{\mathbf{H}}^K = (\hat{\mathbf{h}}_0^K + s^{-1/2}\hat{\mathbf{h}}_1^K + s^{-1}\hat{\mathbf{h}}_2^K) \exp(-s^{1/2}\Psi) + \hat{\mathbf{H}}_\infty^K \tag{85}$$

$$\hat{\mathbf{E}}^K = (s^{1/2}\hat{\mathbf{e}}_0^K + \hat{\mathbf{e}}_1^K) \exp(-s^{1/2}\Psi) + \hat{\mathbf{E}}_\infty^K \tag{86}$$

in which the remainders in the expansion satisfy the relations

$$\hat{\mathbf{H}}_\infty^K = o(1) \qquad |\mathbf{r} - \mathbf{r}'| \to 0 \qquad s \to \infty \tag{87}$$

$$s^{1/2}\hat{\mathbf{E}}_\infty^J = o(1) \qquad |\mathbf{r} - \mathbf{r}'| \to 0 \qquad s \to \infty. \tag{88}$$

Both the justification of using our ansatz as a ray asymptotic expansion and the analysis of the further nature of the remainders in the expansions would have to follow from a rigorous mathematical analysis, for example, one based on an appropriate integral equation formulation of the problem of determining the field due to a point source in an inhomogeneous medium and subjecting this formulation to a rigorous asymptotic analysis, as is done for ordinary differential equations in *Erdélyi* [1956]. The case of a point source in a one-dimensionally continuously inhomogeneous medium might provide a canonical problem in this context. Here the modified Cagniard method enables one to recast the problem from its differential equation formulation into a system of two coupled integral equations in the direction of variation of the medium properties [*De Hoop*, 1990; *Verweij and De Hoop*, 1990]. This system of integral equations is amenable to a rigorous asymptotic analysis in the sense that in each subsequent step in the asymptotic approximation as $s \to \infty$, the remainder in the expansion can be appropriately estimated [*Verweij*, 1992].

7. Transient Behavior of the Ray Constituents

The transient behavior of the ray constituents is obtained by convolving the source signatures $J_0(t)$ and $K_0(t)$ with the time domain counterparts of the Green's function parts that arise in the different terms of the expansions. The latter are of the type $s^q \exp(-s^{1/2}\Psi)$, with the values $q = 1, 1/2, 0, -1/2$, and -1. These are found as [*Abramowitz and Stegun*, 1965]

$$s \exp(-s^{1/2}\Psi) \Rightarrow \frac{3}{4}(1/\pi)^{1/2}\frac{\Psi}{t^{5/2}} \cdot [(\Psi^2/6t) - 1] \exp(-\Psi^2/4t) H(t), \tag{89}$$

$$s^{1/2} \exp(-s^{1/2}\Psi) \Rightarrow \frac{1}{2}(1/\pi)^{1/2}\frac{1}{t^{3/2}} \cdot [(\Psi^2/2t) - 1] \exp(-\Psi^2/4t) H(t), \tag{90}$$

$$\exp(-s^{1/2}\Psi) \Rightarrow \tfrac{1}{2}(1/\pi)^{1/2}\frac{\Psi}{t^{3/2}} \exp(-\Psi^2/4t) H(t), \tag{91}$$

$$s^{-1/2} \exp(-s^{1/2}\Psi) \Rightarrow (1/\pi t)^{1/2} \exp(-\Psi^2/4t) H(t), \quad (92)$$

$$s^{-1} \exp(-s^{1/2}\Psi) \Rightarrow \left[\tfrac{1}{2}(1/\pi)^{1/2} \cdot \int_{\tau=0}^{t} (\Psi/\tau^{3/2}) \exp(-\Psi^2/4\tau)\, d\tau\right] H(t). \quad (93)$$

The last result can be written alternatively as

$$s^{-1} \exp(-s^{1/2}\Psi) \Rightarrow \cdot \left[(1/\pi t)^{1/2} \int_{\psi=\Psi}^{\infty} \exp(-\psi^2/4t)\, d\psi\right] H(t). \quad (94)$$

The corresponding results readily follow upon substitution.

Although real, positive values of s would suffice for the correspondences (89)–(94) to hold (see section 1), the time domain expressions are most easily obtained by evaluating the corresponding Bromwich inversion integrals. For this evaluation to be carried out the left-hand sides of (89)–(94) must be continued analytically into the complex s plane, in which process the correct (i.e., causal) branch of $s^{1/2}$ is to be taken by introducing a branch cut along the negative real s axis. The time domain expressions for the electric field generated by an electric dipole source located in a homogeneous medium agree with those given by *Ward and Hohmann* [1989].

8. Discussion of the Results

The formal expansions obtained by combining the results of sections 2–7 are presumably asymptotic in the time Laplace transform domain with the product of decreasing half-integer powers of the Laplace transform parameter s and the function exp $(-s^{1/2}\Psi)$ as the asymptotic sequence. Denoting the members of this asymptotic sequence by $\{\hat{\phi}_n(\mathbf{x}, s)\}$, they satisfy the order relation

$$\hat{\phi}_{n+1}(\mathbf{x}, s) = O[s^{-1/2}\hat{\phi}_n(\mathbf{x}, s)]. \quad (95)$$

Hence the necessary condition $\hat{\phi}_{n+1}(\mathbf{x}, s) = o[\hat{\phi}_n(\mathbf{x}, s)]$ for the sequence to be asymptotic [*Erdélyi*, 1956] is met for $s \to \infty$. Transformation of (95) to the time domain (which is an integration operation and hence permissible on an asymptotic sequence) yields

$$\phi_{n+1}(\mathbf{x}, t) = O\left\{\int_{\tau=0}^{t} [\pi(t-\tau)]^{-1/2}\phi_n(\mathbf{x}, \tau)\, d\tau\right\}. \quad (96)$$

Consequently, the sequence $\{\phi_n(\mathbf{x}, t)\}$ satisfies the necessary condition $\phi_{n+1}(\mathbf{x}, t) = o[\phi_n(\mathbf{x}, t)]$ to be asymptotic at early times. The actual magnitudes of the different terms are determined by the maximum values that the right-hand sides of (89)–(94) attain at some positive value of t. The latter value is influenced by the value of the diffusion eikonal Ψ as it has been built up along the relevant ray trajectory.

As the diffusion eikonal equation (28) shows, the ray trajectories in the diffusive approximation are real ray trajectories determined by the profile of the conductivity (in addition to the profile of the permeability). In general, they therefore differ from the (also real) ray trajectories in the lossless medium wave propagation approximation, which trajectories are determined by the profile of the permittivity (in addition to the profile of the permeability). The leading terms in the expansions are polarized in the plane perpendicular to the local unit tangent $\boldsymbol{\tau}$ to the ray trajectory, $\mathbf{e}_0$, $\mathbf{h}_0$, and $\boldsymbol{\tau}$ forming a right-handed vector triad. For these leading terms, (32) serves as a kind of local "energy balance".

9. Conclusions

An asymptotic ray theory for diffusive electromagnetic fields due to a point source of electric or magnetic current in an inhomogeneous, isotropic medium has been developed. Starting in the time Laplace transform domain, an ansatz procedure with a diffusive eikonal function is postulated. The latter leads to recurrence relations in the electric and magnetic field amplitude vectors of the different asymptotic orders. The vectorial nature of the problem induces a number of compatibility relations with regard to the terms occurring in these recurrence relations. From the eikonal equation the differential equation for the ray trajectories follows in the standard manner. The vector diffusion equations for the electric and the magnetic field are used to construct the amplitude transport equations along a ray trajectory. The exact diffusive electromagnetic field from a point source of electric or magnetic current in a homogeneous medium is shown to have the structure of the ansatz. The relevant field expressions are used to match the leading-order amplitudes of the ray asymptotic expansions to the source strengths. The difficulty that the successive orders show singularities of increasing order in the neighborhood of the source is discussed. Finally, the order terms that are compatible with the presence of a spatial delta function in the volume source distributions are transformed from the Laplace domain to the time domain, thus giving the transient behavior of the relevant terms.

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A. T. de Hoop, Laboratory of Electromagnetic Research, Delft University of Technology, P.O. Box 5031, 2600 GA DELFT, the Netherlands. (e-mail: A.T.deHoop@ET.TuDelft.NL)

T. M. Habashy, M. L. Oristaglio, and C. Torres-Verdín, Schlumberger-Doll Research, Old Quarry Road, Ridgefield, CT 06877. (e-mail: habashy@ridgefield.sdr.slb.com; oristaglio@ridgefield.sdr.slb.com; ctorres@ridgefield.sdr.slb.com)

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