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The initial-value problems in acoustics, elastodynamics and electromagnetics

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Abstract

The initial-value problems for acoustic waves in fluids, elastic waves in solids and electromagnetic waves are discussed. The governing systems of first-order partial differential equations pertaining to arbitrarily inhomogenous and anisotropic media are taken as point of departure and, correspondingly, the initial values of the pertaining two state quantities (i.e. the two quantities whose product specifies the area density of power flow in each of the wave motions) are prescribed. The initial-value problem thus posed is thought to be more physical (and turns out to be more complicated) than the conventional one associated with the second-order wave equation in one of the two state quantities, where the inital values of this state quantity and its first-order time derivative are prescribed. For the cases of homogeneous, isotropic media, the initial-value problems are solved with the aid of a time Laplace and spatial Fourier transform method that bears resemblance to the modified Cagniard method for solving transient wave propagation problems in layered media.

1. Introduction

The initial-value problems in acoustics, elastodynamics and electromagnetics are customarily formulated (and solved) as initial-value problems associated with a second-order partial differential equation that contains one of the relevant state variables as the wave quantity and solutions are obtained for the time evolution of waves in homogeneous, isotropic media. For acoustic waves in fluids the relevant wave quantity is standardly taken to be the acoustic pressure and the classical Poisson solution of the initial-value problem associated with the scalar wave equation with constant wavespeed provides the solution (see, Baker and Copson [1], Lamb [2]). For elastic waves in solids the wave quantity is standardly taken to be the particle velocity and the solution has been constructed by Love [3]. For electromagnetic waves the vector wave equation for the electric or the magnetic field strength is usually taken as the point of departure and the Poisson solution is applied to each of the Cartesian components (see Jones [4], who also considers the initial-value problem associated with a slightly more general vector wave equation). In each of the three cases, the initial values of the relevant state

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quantity and of its first-order time derivative are taken to specify the further time evolution of the wavefield. Recent developments in this respect for the case of elastic waves in homogeneous, arbitrarily anisotropic solids are covered in an extensive paper by Smit and M.V. de Hoop [5].

From a physical as well as from a mathematical point of view, however, the initial-value problems associated with the coupled first-order systems of partial differential equations pertaining to the three kinds of wave motion seem more fundamental. In them, the initial values of the two state quantities whose product is representative for the area density of power flow (Poynting vector) in the wave motion are to be the prescribed quantities that determine the time evolution of the wavefield. For acoustic waves in fluids the relevant two state quantities are the acoustic pressure and the particle velocity, for elastic waves in solids the particle velocity and the dynamic stress, and for electromagnetic waves the electric field strength and the magnetic field strength. In the present paper, first the general aspects of the initial-value problems thus formulated are disussed for the case of arbitrarily inhomogenous and anisotropic, but linear, time-invariant, instantaneously and locally reacting, media. Here, the time Laplace transformed equations indicate that equivalent volume source densities account for the presence of non-zero initial values. Next, for the case of homogeous media a subsequent spatial Fourier transformation reduces the problem to solving a system of linear, algebraic equations. Finally, the case of homogeneous, isotropic media is worked out in detail with a method that bears some resemblance to the modified Cagniard method for calculating the transient wave motion generated by a source in a layered medium as developed earlier by the present author (see, Cagniard [6], De Hoop [7–9], Achenbach [10], Miklowitz [11], Aki and Richards [12]). An interesting alternative to the latter technique is provided by Wang and Achenbach [13].

To introduce the method, the classical Poisson solution to the initial-value problem associated with the scalar wave equation is rederived in Section 3. Section 4 deals with the initial-value problem associated with acoustic waves in fluids, Section 5 with the one associated with elastic waves in solids and Section 6 with the one associated with electromagnetic waves.

2. The time Laplace and spatial Fourier transforms

The wave motions considered are present in three-dimensional Euclidean space $\mathcal{R}^3$. The position of observation is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The position vector is $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$. The time coordinate is t. The subscript notation for Cartesian vectors and tensors is employed; lower-case latin subscripts are used for this purpose. The summation convention applies. Partial differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol for differentiation with respect to t. The initial values are prescribed at the instant $t = t_0$ and in all space $\boldsymbol{x} \in \mathcal{R}^3$. The time evolution of the wavefields is to be determined in the interval $\{t \in \mathcal{R}; t > t_0\}$. The media in which the waves propagate are assumed to be linear, time invariant, instantaneously and locally reacting. In the general part of the analysis, they may be arbitrarily inhomogeneous and anisotropic. The case of homogeneous, isotropic media is next worked out in detail. The configuration space is unbounded.

Let $u = u(\boldsymbol{x}, t)$ denote any of the wave quantities. The time invariance of the configuration and the causality of the wave motion enable the use of the one-sided Laplace transformation (over the time interval of interest)

$$\hat{u}(\boldsymbol{x}, s) = \int_{t=t_0}^{\infty} \exp(-st) u(\boldsymbol{x}, t)\, \mathrm{d}t, \tag{1}$$

where s is the *complex frequency*. In view of Lerch's theorem of the time Laplace transformation (Widder [14]), $u = u(\boldsymbol{x}, t)$ is for $t > t_0$ uniquely determined by the sequence of equidistant values $\{\hat{u}(\boldsymbol{x}, s_n); s_0 \in \mathcal{R}, s_0 > 0, h \in \mathcal{R}, h > 0, s_n = s_0 + nh, n = 0, 1, 2, \ldots\}$ on the positive real s-axis. Hence, we can without loss of generality take s in Eq. (1) real and positive. For the Laplace transform of the time derivative we obtain through an integration by parts

$$\int_{t=t_0}^{\infty} \exp(-st)\,\partial_t u(\boldsymbol{x},t)\,\mathrm{d}t = -\exp(-st_0)u(\boldsymbol{x},t_0) + s\hat{u}(\boldsymbol{x},s), \tag{2}$$

where we have assumed that $u(\boldsymbol{x},t)$ is of less than exponential growth as $t \to \infty$.

For the case of homogeneous (not necessarily isotropic) media, the spatial Fourier transform provides a useful tool to carry out the further analytical handling of the problem. This transform (over all space) is taken as

$$\tilde{u}(\mathrm{j}\boldsymbol{k},s) = \int_{\boldsymbol{x}\in\mathcal{R}^3} \exp(\mathrm{j}k_m x_m)\hat{u}(\boldsymbol{x},s)\,\mathrm{d}V \quad \text{for } \boldsymbol{k}\in\mathcal{R}^3, \tag{3}$$

where j is the imaginary unit and $\boldsymbol{k}$ is the *angular wavevector*. For the Fourier transform of the spatial derivative we obtain through an application of Gauss' integral theorem (spatial integration by parts)

$$\int_{\boldsymbol{x}\in\mathcal{R}^3} \exp(\mathrm{j}k_m x_m)\,\partial_m\hat{u}(\boldsymbol{x},s)\,\mathrm{d}V = -\mathrm{j}k_m\tilde{u}(\mathrm{j}\boldsymbol{k},s), \tag{4}$$

where it has been assumed that $\hat{u}$ is continuous and that $\hat{u}(\boldsymbol{x},s) = O(|\boldsymbol{x}|^{-p})$, with $p > 3$, as $|\boldsymbol{x}| \to \infty$, which is certainly the case since $\tilde{u}$ shows in fact an exponential decay as $|\boldsymbol{x}| \to \infty$ as long as $\mathrm{Re}(s) > 0$. The transformation inverse to Eq. (3) is given by

$$\hat{u}(\boldsymbol{x},s) = \left(\frac{1}{2\pi}\right)^3 \int_{\boldsymbol{k}\in\mathcal{R}^3} \exp(-\mathrm{j}k_m x_m)\tilde{u}(\boldsymbol{k},s)\,\mathrm{d}V \quad \text{for } \boldsymbol{x}\in\mathcal{R}^3. \tag{5}$$

Just as in the (modified) Cagniard method, the transformation inverse to Eq. (1) will be carried out by inspection.

3. The initial-value problem for the scalar wave equation

As an introduction to the analytical methods employed we discuss in the present section the initial-value problem associated with the three-dimensional scalar wave equation with a constant wavespeed c. The wave function $u = u(\boldsymbol{x},t)$ that is representative for the wave motion satisfies the second-order partial differential equation

$$c^2\partial_m\partial_m u - \partial_t^2 u = 0 \quad \text{for } \boldsymbol{x}\in\mathcal{R}^3 \text{ and } t > t_0 \tag{6}$$

and has the (classical) assumed initial values

$$u(\boldsymbol{x},t_0) = u_0(\boldsymbol{x}) \quad \text{and} \quad \partial_t u(\boldsymbol{x},t_0) = \dot{u}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x}\in\mathcal{R}^3. \tag{7}$$

Taking the time Laplace transform of Eq. (6) and applying Eq. (2) twice, we obtain

$$c^2\partial_m\partial_m\hat{u} - s^2\hat{u} = -\hat{q}^{\mathrm{eq}}, \tag{8}$$

where the equivalent volume source density $\hat{q}^{\text{eq}}$ is given by

$$\hat{q}^{\text{eq}} = [\dot{u}_0(\boldsymbol{x}) + s u_0(\boldsymbol{x})] \exp(-s t_0). \tag{9}$$

Taking the spatial Fourier transform of Eq. (8) and applying Eq. (4) twice, we obtain

$$(c^2 k_m k_m + s^2)\tilde{u} = \tilde{q}^{\text{eq}}. \tag{10}$$

The solution of this equation is given by

$$\tilde{u} = \tilde{G}\tilde{q}^{\text{eq}}, \tag{11}$$

in which

$$\tilde{G} = \frac{1}{c^2 k_m k_m + s^2} \tag{12}$$

is the spectral-domain Green's function of the scalar wave equation. Substituting the expression at the right-hand side of Eq. (12) in the spatial Fourier inversion integral Eq. (5), introducing in $\boldsymbol{k}$-space spherical polar coordinates with $\boldsymbol{x}$ as polar axis and carrying out the resulting integrations (see Appendix A), it is found that

$$\hat{G}(\boldsymbol{x}, s) = \frac{\exp(-s|\boldsymbol{x}|/c)}{4\pi c^2 |\boldsymbol{x}|} \quad \text{for } |\boldsymbol{x}| \neq 0. \tag{13}$$

Using the property that the product of two Fourier spatial transforms yields upon inversion the convolution in configuration space, Eq. (9) and Eqs. (11)–(13) lead to

$$\hat{u}(\boldsymbol{x}, s) = \int_{\boldsymbol{x}' \in \mathcal{R}^3} \frac{\exp[-s(|\boldsymbol{x} - \boldsymbol{x}'|/c + t_0)]}{4\pi c^2 |\boldsymbol{x} - \boldsymbol{x}'|} [\dot{u}_0(\boldsymbol{x}') + s u_0(\boldsymbol{x}')]\, \mathrm{d}V. \tag{14}$$

Introducing in the right-hand side the variables of integration τ and $\boldsymbol{\theta}$, defined through

$$\boldsymbol{x}' = \boldsymbol{x} + c(\tau - t_0)\boldsymbol{\theta}, \tag{15}$$

with $\{\tau \in \mathcal{R}; \tau \geq t_0\}$ and $\boldsymbol{\theta} \in \Omega$, where $\Omega = \{\boldsymbol{\theta} \in \mathcal{R}^3; \theta_m \theta_m = 1\}$ denotes the sphere of unit radius, Eq. (14) can be rewritten as

$$\hat{u}(\boldsymbol{x}, s) = \int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle \dot{u}_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x}, c(\tau - t_0)]}\, \mathrm{d}\tau + s\int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x}, c(\tau - t_0)]}\, \mathrm{d}\tau, \tag{16}$$

in which

$$\langle \{\dot{u}_0, u_0\}(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x}, c(\tau - t_0)]} = \frac{1}{4\pi} \int_{\boldsymbol{\theta} \in \Omega} \{\dot{u}_0, u_0\}[\boldsymbol{x} + c(\tau - t_0)\boldsymbol{\theta}]\, \mathrm{d}\Omega \tag{17}$$

is the spherical mean over the sphere $\mathcal{S}[\boldsymbol{x}, c(\tau - t_0)]$ with center at $\boldsymbol{x}$ and radius $c(\tau - t_0)$. Using Lerch's uniqueness theorem of the one-sided time Laplace transformation, Eq. (16) leads through inversion by inspection to

$$u(\boldsymbol{x},t) = (t-t_0)\langle \dot{u}_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(t-t_0)]} + \partial_t[(t-t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(t-t_0)]}] \quad \text{for } t \geq t_0 \quad \text{and } \boldsymbol{x} \in \mathcal{R}^3. \tag{18}$$

Eq. (18) is the classical Poisson solution to the initial-value problem associated with the scalar wave equation.

4. The initial-value problem for acoustic waves in fluids

In this section the initial-value problem associated with acoustic waves in fluids is discussed. The state quantities $p = p(\boldsymbol{x},t)$ (= acoustic pressure) and $v_r = v_r(\boldsymbol{x},t)$ (= particle velocity) of such a wave, present in a fluid with inertial volume density of mass $\rho_{k,r} = \rho_{k,r}(\boldsymbol{x})$ and compressibility $\kappa = \kappa(\boldsymbol{x})$ and excited by sources with volume density of force $f_k = f_k(\boldsymbol{x},t)$ and volume density of volume injection rate $q = q(\boldsymbol{x},t)$, satisfy the system of first-order partial differential equations

$$\partial_k p + \rho_{k,r}\partial_t v_r = f_k, \tag{19}$$

$$\partial_r v_r + \kappa\partial_t p = q, \tag{20}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3 \text{ and } t > t_0,$$

and have the assumed initial values

$$p(\boldsymbol{x},t_0) \quad \text{and} \quad v_r(\boldsymbol{x},t_0) \quad \text{for } \boldsymbol{x} \in \mathcal{R}^3. \tag{21}$$

Taking the time Laplace transform of Eqs. (19) and (20) and applying Eq. (2), we obtain

$$\partial_k\hat{p} + s\rho_{k,r}\hat{v}_r = \hat{f}_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\exp(-st_0), \tag{22}$$

$$\partial_r\hat{v}_r + s\kappa\hat{p} = \hat{q} + \kappa(\boldsymbol{x})p(\boldsymbol{x},t_0)\exp(-st_0), \tag{23}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3.$$

As the right-hand sides of these equations show, the incorporation of non-zero initial values of the state quantities at $t = t_0$ is equivalent to a change of the volume densities of the actual sources as active for $t > t_0$ into their related equivalent values given by

$$\hat{f}_k^{\text{eq}} = \hat{f}_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\exp(-st_0), \tag{24}$$

$$\hat{q}^{\text{eq}} = \hat{q} + \kappa(\boldsymbol{x})p(\boldsymbol{x},t_0)\exp(-st_0). \tag{25}$$

In the time domain these equivalent source densities are

$$f_k^{\text{eq}} = f_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\delta(t-t_0), \tag{26}$$

$$q^{\text{eq}} = q + \kappa(\boldsymbol{x})p(\boldsymbol{x},t_0)\delta(t-t_0), \tag{27}$$

where $\delta(t-t_0)$ is the Dirac delta distribution operative at $t = t_0$. In view of the superposition principle, the total wave motion is thus the superposition of the one excited by the source distributions as they are active for $t > t_0$ and a contribution from the initial values of the state quantities at $t = t_0$. With the aid of the Green's functions that express the state quantities at position $\boldsymbol{x}$ and instant t in terms of the source distributions at position $\boldsymbol{x}'$ and instant t', the total wave motion can therefore be represented as

$$p(\boldsymbol{x},t)=\int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^s}[G^{pq}(\boldsymbol{x},\boldsymbol{x}',t-t')q(\boldsymbol{x}',t')$$
$$+G_k^{pf}(\boldsymbol{x},\boldsymbol{x}',t-t')f_k(\boldsymbol{x}',t')]\,\mathrm{d}V$$
$$+\int_{\boldsymbol{x}'\in\mathcal{R}^3}[G^{pq}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\kappa(\boldsymbol{x}')p(\boldsymbol{x}',t_0)$$
$$+G_k^{pf}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\rho_{k,r}(\boldsymbol{x}')v_r(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \tag{28}$$

$$v_r(\boldsymbol{x},t)=\int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^s}[G_r^{vq}(\boldsymbol{x},\boldsymbol{x}',t-t')q(\boldsymbol{x}',t')$$
$$+G_{r,k}^{vf}(\boldsymbol{x},\boldsymbol{x}',t-t')f_k(\boldsymbol{x}',t')]\,\mathrm{d}V$$
$$+\int_{\boldsymbol{x}'\in\mathcal{R}^3}[G_r^{vq}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\kappa(\boldsymbol{x}')p(\boldsymbol{x}',t_0)$$
$$+G_{r,k}^{vf}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\rho_{k,r'}(\boldsymbol{x}')v_{r'}(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \tag{29}$$

for $\boldsymbol{x}\in\mathcal{R}^3$ and $t>t_0$,

where $\mathcal{D}^s$ is the spatial support of the source distributions and the superscripts on the symbol G for the Green's functions indicate which mapping from source to field quantity is meant.

In those cases where the Green's functions can be determined analytically, a further reduction of the field representations can be obtained. In particular, this applies to homogeneous media where, owing to the shift invariance in space, the Fourier transformation offers itself as a useful tool. With the aid of Eq. (4) and the property that the acoustic pressure and the particle velocity are, under the present circumstance, continuous functions of position, the Fourier transform of Eqs. (22) and (23) is, omitting the argument $\mathrm{j}\boldsymbol{k}$, obtained as

$$-\mathrm{j}k_k\widetilde{p}+s\rho_{k,r}\widetilde{v}_r=\widetilde{f}_k+\rho_{k,r}\widetilde{v}_r(t_0)\exp(-st_0), \tag{30}$$
$$-\mathrm{j}k_r\widetilde{v}_r+s\kappa\widetilde{p}=\widetilde{q}+\kappa\widetilde{p}(t_0)\exp(-st_0). \tag{31}$$

Once this system of linear algebraic equations has been solved, the inversion to the space-time domain can be carried out with the aid of the modified Cagniard method or any related technique. The simplest case in this category occurs when the medium is isotropic as well as homogeneous. This case is discussed below. In particular, the pure initial-value problem is addressed, i.e. the case where no volume sources are assumed to be active for $t>t_0$.

4.1. Time evolution of an acoustic wavefield in a homogeneous, isotropic medium

For a homogeneous, isotropic fluid we have $\rho_{k,r}=\rho\delta_{k,r}$, where $\delta_{k,r}$ is the Kronecker tensor (symmetrical unit tensor of rank two: $\delta_{i,j}=1$ for $i=j$, $\delta_{i,j}=0$ for $i\neq j$). From Eqs. (30) and (31) we then obtain, putting $\widetilde{f}_k=0$ and $\widetilde{q}=0$,

$$-\mathrm{j}k_k\widetilde{p}+s\rho\widetilde{v}_k=\rho\widetilde{v}_k(t_0)\exp(-st_0), \tag{32}$$
$$-\mathrm{j}k_r\widetilde{v}_r+s\kappa\widetilde{p}=\kappa\widetilde{p}(t_0)\exp(-st_0). \tag{33}$$

Solving $\widetilde{v}_k$ from Eq. (32) as

$$\widetilde{v}_k = s^{-1}\widetilde{v}_k(t_0)\exp(-st_0) + (s\rho)^{-1}\mathrm{j}k_k\widetilde{p} \tag{34}$$

and substituting this expression in Eq. (33), the following equation for $\widetilde{p}$ is obtained:

$$(c^2 k_r k_r + s^2)\widetilde{p} = s\widetilde{p}(t_0)\exp(-st_0) + \mathrm{j}k_r\kappa^{-1}\widetilde{v}_r(t_0)\exp(-st_0), \tag{35}$$

where

$$c = (\rho\kappa)^{-1/2} \tag{36}$$

is the acoustic wavespeed. From this equation $\widetilde{p}$ follows as

$$\widetilde{p} = s\widetilde{\Phi}^p + \kappa^{-1}\mathrm{j}k_r\widetilde{\Phi}^v_r, \tag{37}$$

in which

$$\widetilde{\Phi}^p = \widetilde{G}\widetilde{p}(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{38}$$

$$\widetilde{\Phi}^v_r = \widetilde{G}\widetilde{v}_r(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{39}$$

with

$$\widetilde{G} = \frac{1}{c^2 k_m k_m + s^2}. \tag{40}$$

Upon using the rules for the inverse spatial Fourier transformation, $\widehat{p}$ is from Eq. (37) obtained as

$$\widehat{p} = s\widehat{\Phi}^p - \kappa^{-1}\partial_r\widehat{\Phi}^v_r, \tag{41}$$

in which

$$\widehat{\Phi}^p(\boldsymbol{x}, s) = \int_{\boldsymbol{x}'\in\mathcal{R}^3} \frac{\exp[-s(|\boldsymbol{x}-\boldsymbol{x}'|/c + t_0)]}{4\pi c^2|\boldsymbol{x}-\boldsymbol{x}'|} p(\boldsymbol{x}', t_0)\,\mathrm{d}V, \tag{42}$$

$$\widehat{\Phi}^v_r(\boldsymbol{x}, s) = \int_{\boldsymbol{x}'\in\mathcal{R}^3} \frac{\exp[-s(|\boldsymbol{x}-\boldsymbol{x}'|/c + t_0)]}{4\pi c^2|\boldsymbol{x}-\boldsymbol{x}'|} v_r(\boldsymbol{x}', t_0)\,\mathrm{d}V. \tag{43}$$

Proceeding as in Section 3, the inversion to the time domain yields the result

$$p = \partial_t\Phi^p - \kappa^{-1}\partial_r\Phi^v_r \quad \text{for } t \geq t_0, \tag{44}$$

in which

$$\Phi^p(\boldsymbol{x}, t) = (t - t_0)\langle p(\boldsymbol{x}', t_0)\rangle_{\mathcal{S}[\boldsymbol{x}, c(t-t_0)]}, \tag{45}$$

$$\Phi^v_r(\boldsymbol{x}, t) = (t - t_0)\langle v_r(\boldsymbol{x}', t_0)\rangle_{\mathcal{S}[\boldsymbol{x}, c(t-t_0)]}, \tag{46}$$

for $t \geq t_0$.

Spatial and temporal inversion of Eq. (32) finally yields

$$\begin{aligned} v_r &= v_r(\boldsymbol{x}, t_0) - \rho^{-1}I_t\partial_r p \\ &= v_r(\boldsymbol{x}, t_0) - \rho^{-1}\partial_r\Phi^p + c^2 I_t\partial_r\partial_k\Phi^v_k \\ &\quad \text{for } t \geq t_0, \end{aligned} \tag{47}$$

where I_t denotes time integration from t_0 onward. Eqs. (44)–(47) constitute the solution to the acoustic initial-value problem and govern the pure time evolution of an acoustic wavefield in a homogeneous, isotropic, ideal fluid.

5. The initial-value problem for elastic waves in solids

In this section the initial-value problem associated with elastic waves in solids is discussed. The state quantities $v_r = v_r(\boldsymbol{x},t)$ (= particle velocity) and $\tau_{p,q} = \tau_{p,q}(\boldsymbol{x},t)$ (= dynamic stress) of such a wave, present in a medium with inertial volume density of mass $\rho_{k,r} = \rho_{k,r}(\boldsymbol{x})$ and compliance $S_{i,j,p,q} = S_{i,j,p,q}(\boldsymbol{x})$ and excited by sources with volume density of force $f_k = f_k(\boldsymbol{x},t)$ and volume density of impressed deformation rate $h_{i,j} = h_{i,j}(\boldsymbol{x},t)$, satisfy the system of first-order partial differential equations

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau_{p,q} + \rho_{k,r}\partial_t v_r = f_k, \tag{48}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v_r - S_{i,j,p,q}\partial_t\tau_{p,q} = h_{i,j}, \tag{49}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3 \text{ and } t > t_0,$$

and have the assumed initial values

$$v_r(\boldsymbol{x},t_0) \quad \text{and} \quad \tau_{p,q}(\boldsymbol{x},t_0) \quad \text{for } \boldsymbol{x} \in \mathcal{R}^3. \tag{50}$$

In Eqs. (48) and (49),

$$\Delta^{+}_{i,j,p,q} = \tfrac{1}{2}(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}) \tag{51}$$

is the symmetrical unit tensor of rank four.

Taking the time Laplace transform of Eqs. (48) and (49) and applying Eq. (2), we obtain

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q} + s\rho_{k,r}\hat{v}_r = \hat{f}_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\exp(-st_0), \tag{52}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r - sS_{i,j,p,q}\hat{\tau}_{p,q} = \hat{h}_{i,j} - S_{i,j,p,q}(\boldsymbol{x})\tau_{p,q}(\boldsymbol{x},t_0)\exp(-st_0), \tag{53}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3.$$

As the right-hand sides of these equations show, the incorporation of non-zero initial values of the state quantities at $t = t_0$ is equivalent to a change of the volume densities of the actual sources as they are active for $t > t_0$ into their related equivalent values given by

$$\hat{f}^{\rm eq}_k = \hat{f}_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\exp(-st_0), \tag{54}$$

$$\hat{h}^{\rm eq}_{i,j} = \hat{h}_{i,j} - S_{i,j,p,q}(\boldsymbol{x})\tau_{p,q}(\boldsymbol{x},t_0)\exp(-st_0). \tag{55}$$

In the time domain these equivalent source densities are

$$f^{\rm eq}_k = f_k + \rho_{k,r}(\boldsymbol{x})v_r(\boldsymbol{x},t_0)\delta(t-t_0), \tag{56}$$

$$\hat{h}^{\rm eq}_{i,j} = \hat{h}_{i,j} - S_{i,j,p,q}(\boldsymbol{x})\tau_{p,q}(\boldsymbol{x},t_0)\delta(t-t_0). \tag{57}$$

In view of the superposition principle, the total wave motion is thus the superposition of the one excited by the source distributions as they are active for $t > t_0$ and a contribution from the initial values of the state quantities at $t = t_0$. With the aid of the Green's functions that express the state quantities at position $\boldsymbol{x}$ and instant t in terms of the source distributions at position $\boldsymbol{x}'$ and instant t', the total wave motion can therefore be represented as

$$-\tau_{p,q}(\boldsymbol{x},t)=\int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^s}[G^{\tau f}_{p,q,k}(\boldsymbol{x},\boldsymbol{x}',t-t')f_k(\boldsymbol{x}',t')$$
$$+G^{\tau h}_{p,q,i,j}(\boldsymbol{x},\boldsymbol{x}',t-t')h_{i,j}(\boldsymbol{x}',t')]\,\mathrm{d}V$$
$$+\int_{\boldsymbol{x}'\in\mathcal{R}^3}[G^{\tau f}_{p,q,k}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\rho_{k,r}(\boldsymbol{x}')v_r(\boldsymbol{x}',t_0)$$
$$-G^{\tau h}_{p,q,i,j}(\boldsymbol{x},\boldsymbol{x}',t-t_0)S_{i,j,p',q'}(\boldsymbol{x}')\tau_{p'q'}(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \tag{58}$$

$$v_r(\boldsymbol{x},t)=\int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^s}[G^{vf}_{r,k}(\boldsymbol{x},\boldsymbol{x}',t-t')f_k(\boldsymbol{x}',t')$$
$$+G^{vh}_{r,i,j}(\boldsymbol{x},\boldsymbol{x}',t-t')h_{i,j}(\boldsymbol{x}',t')]\,\mathrm{d}V$$
$$+\int_{\boldsymbol{x}'\in\mathcal{R}^3}[G^{vf}_{r,k}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\rho_{k,r'}(\boldsymbol{x}')v_{r'}(\boldsymbol{x}',t_0)$$
$$-G^{vh}_{r,ij}(\boldsymbol{x},\boldsymbol{x}',t-t_0)S_{i,j,p,q}(\boldsymbol{x}')\tau_{p,q}(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \tag{59}$$

for $\boldsymbol{x}\in\mathcal{R}^3$ and $t>t_0$,

where $\mathcal{D}^s$ is the spatial support of the source distributions and the superscripts on the symbol G for the Green's functions indicate which mapping from source to field quantity is meant, and where the minus sign in the left-hand side of Eq. (58) accounts for the fact that the elastodynamic Poynting vector is the *opposite* of the product of the dynamic stress and the particle velocity.

In those cases where the Green's functions can be determined analytically, a further reduction of the field representations can be obtained. In particular, this applies to homogeneous media where, owing to the shift invariance in space, the Fourier transformation offers itself as a useful tool. With the aid of Eq. (4) and the property that the particle velocity and the dynamic stress are, under the present circumstance, continuous functions of position, the Fourier transform of Eqs. (52) and (53) is, omitting the argument $\mathrm{j}\boldsymbol{k}$, obtained as

$$\Delta^+_{k,m,p,q}\mathrm{j}k_m\tilde{\tau}_{p,q}+s\rho_{k,r}\tilde{v}_r=\tilde{f}_k+\rho_{k,r}\tilde{v}_r(t_0)\exp(-st_0), \tag{60}$$
$$-\Delta^+_{i,j,n,r}\mathrm{j}k_n\tilde{v}_r-sS_{i,j,p,q}\tilde{\tau}_{p,q}=\tilde{h}_{i,j}-S_{i,j,p,q}\tilde{\tau}_{p,q}(t_0)\exp(-st_0). \tag{61}$$

Once this system of linear algebraic equations has been solved, the inversion to the space-time domain can be carried out with the aid of the modified Cagniard method or any related technique. The simplest case in this category occurs when the medium is isotropic as well as homogeneous. This case is discussed below. In particular, the pure initial-value problem is addressed, i.e. the case where no volume sources are assumed to be active for $t>t_0$. Recent results for the general anisotropic case can be found in a paper by Smit and M.V. de Hoop [5].

5.1. *Time evolution of an elastic wavefield in a homogeneous, isotropic medium*

For a homogeneous, isotropic, perfectly elastic solid with volume density of mass ρ and Lamé coefficients λ and μ we have

$$S_{i,j,p,q}=\Lambda\delta_{i,j}\delta_{p,q}+2M\Delta^+_{i,j,p,q}, \tag{62}$$

with

$$\Lambda = -\frac{\lambda}{(3\lambda + 2\mu)2\mu}, \tag{63}$$

$$M = \frac{1}{4\mu}, \tag{64}$$

The corresponding stiffness (i.e., the inverse of the compliance) is given by

$$C_{p,q,i,j} = \lambda\delta_{p,q}\delta_{i,j} + 2\mu\Delta^{+}_{p,q,i,j}. \tag{65}$$

From Eqs. (60) and (61) we then obtain, putting $\widetilde{f}_k = 0$ and $\widetilde{h}_{i,j} = 0$,

$$\Delta^{+}_{k,m,p,q}\mathrm{j}k_m\widetilde{\tau}_{p,q} + s\rho\widetilde{v}_k = \rho\widetilde{v}_k(t_0)\exp(-st_0), \tag{66}$$

$$-\Delta^{+}_{i,j,n,r}\mathrm{j}k_n\widetilde{v}_r - sS_{i,j,p,q}\widetilde{\tau}_{p,q} = -S_{i,j,p,q}\widetilde{\tau}_{p,q}(t_0)\exp(-st_0). \tag{67}$$

Solving $\widetilde{\tau}_{p,q}$ from Eq. (67) as

$$\widetilde{\tau}_{p,q} = s^{-1}\widetilde{\tau}_{p,q}(t_0)\exp(-st_0) - s^{-1}C_{p,q,i,j}(1/2)(\mathrm{j}k_i\widetilde{v}_j + \mathrm{j}k_j\widetilde{v}_i)], \tag{68}$$

and substituting this result in Eq. (66), we arrive at the spectral-domain elastodynamic wave equation for the particle velocity

$$(c_P^2 - c_S^2)k_k k_i\widetilde{v}_i + c_S^2 k_m k_m\widetilde{v}_k + s^2\widetilde{v}_k = \widetilde{Q}_k, \tag{69}$$

in which

$$c_P = [(\lambda + 2\mu)/\rho]^{1/2} \tag{70}$$

is the compressional or P-wave speed,

$$c_S = (\mu/\rho)^{1/2} \tag{71}$$

is the shear or S-wave speed and

$$\widetilde{Q}_k = s\widetilde{v}_k(t_0)\exp(-st_0) - \rho^{-1}\mathrm{j}k_m\Delta^{+}_{k,m,p,q}\widetilde{\tau}_{p,q}(t_0)\exp(-st_0). \tag{72}$$

Application of the operation k_k to Eq. (69) leads to the auxiliary relation

$$k_i\widetilde{v}_i = \frac{k_m\widetilde{Q}_m}{c_P^2 k_m k_m + s^2}. \tag{73}$$

Employing Eq. (73) in Eq. (69) yields

$$(c_S^2 k_m k_m + s^2)\widetilde{v}_k = \widetilde{Q}_k - \frac{(c_P^2 - c_S^2)k_k k_m\widetilde{Q}_m}{c_P^2 k_m k_m + s^2}, \tag{74}$$

from which it follows that

$$\widetilde{v}_k = \frac{\widetilde{Q}_k}{c_S^2 k_m k_m + s^2} - \frac{(c_P^2 - c_S^2)k_k k_m\widetilde{Q}_m}{(c_P^2 k_m k_m + s^2)(c_S^2 k_n k_n + s^2)}. \tag{75}$$

However,

$$\frac{(c_P^2 - c_S^2)}{(c_P^2 k_m k_m + s^2)(c_S^2 k_n k_n + s^2)} = \frac{1}{s^2}\left(\frac{1}{k_m k_m + s^2/c_P^2} - \frac{1}{k_n k_n + s^2/c_S^2}\right). \tag{76}$$

Hence, the expression for $\widetilde{v}_r$ can be written as

$$\widetilde{v}_r = \widetilde{G}_{r,k}\widetilde{Q}_k, \tag{77}$$

in which

$$\widetilde{G}_{r,k} = \widetilde{G}_S\delta_{r,k} - s^{-2}k_rk_k(c_P^2\widetilde{G}_P - c_S^2\widetilde{G}_S), \tag{78}$$

with

$$\widetilde{G}_{P,S} = \frac{1}{c_{P,S}^2k_mk_m + s^2}, \tag{79}$$

is the spectral-domain Green's tensor for the elastodynamic wave equation. Upon introducing the spectral-domain P- and S- wave potentials

$$\widetilde{\Phi}_k^{v;P,S} = \widetilde{G}_{P,S}\widetilde{v}_k(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{80}$$

$$\widetilde{\Phi}_{i,j}^{\tau;P,S} = \widetilde{G}_{P,S}\widetilde{\tau}_{i,j}(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{81}$$

Eq. (77) can be written as

$$\begin{aligned}\widetilde{v}_r = {} & s\widetilde{\Phi}_r^{v;S} - s^{-1}k_rk_k(c_P^2\widetilde{\Phi}_k^{v;P} - c_S^2\widetilde{\Phi}_k^{v;S}) \\ & -\rho^{-1}\mathrm{j}k_m\Delta^+_{r,m,i,j}\widetilde{\Phi}_{i,j}^{\tau;S} + \rho^{-1}s^{-2}\mathrm{j}k_mk_rk_k\Delta^+_{k,m,i,j}(c_P^2\widetilde{\Phi}_{i,j}^{\tau;P} - c_S^2\widetilde{\Phi}_{i,j}^{\tau;S}).\end{aligned} \tag{82}$$

Using the rules for the inverse spatial Fourier transformation, $\widehat{v}_r$ is from Eq. (82) obtained as

$$\begin{aligned}\widehat{v}_r = {} & s\widehat{\Phi}_r^{v;S} + s^{-1}\partial_r\partial_k(c_P^2\widehat{\Phi}_k^{v;P} - c_S^2\widehat{\Phi}_k^{v;S}) \\ & +\rho^{-1}\partial_m\Delta^+_{r,m,i,j}\widehat{\Phi}_{i,j}^{\tau;S} + \rho^{-1}s^{-2}\partial_r\partial_k\partial_m\Delta^+_{k,m,i,j}(c_P^2\widehat{\Phi}_{i,j}^{\tau;P} - c_S^2\widehat{\Phi}_{i,j}^{\tau;S}),\end{aligned} \tag{83}$$

in which

$$\widehat{\Phi}_k^{v;P,S} = \int_{\boldsymbol{x}'\in\mathcal{R}^3} \frac{\exp[-s(|\boldsymbol{x}-\boldsymbol{x}'|/c_{P,S}+t_0)]}{4\pi c_{P,S}^2|\boldsymbol{x}-\boldsymbol{x}'|} v_k(\boldsymbol{x}', t_0)\,\mathrm{d}V, \tag{84}$$

$$\widehat{\Phi}_{i,j}^{\tau;P,S} = \int_{\boldsymbol{x}'\in\mathcal{R}^3} \frac{\exp[-s(|\boldsymbol{x}-\boldsymbol{x}'|/c_{P,S}+t_0)]}{4\pi c_{P,S}^2|\boldsymbol{x}-\boldsymbol{x}'|} \tau_{i,j}(\boldsymbol{x}', t_0)\,\mathrm{d}V. \tag{85}$$

Proceeding as in Section 3, the inversion to the time domain yields the result

$$\begin{aligned}v_r = {} & \partial_t\Phi_r^{v;S} + I_t\partial_r\partial_k(c_P^2\Phi_k^{v;P} - c_S^2\Phi_k^{v;S}) \\ & +\rho^{-1}\partial_m\Delta^+_{r,m,i,j}\Phi_{i,j}^{\tau;S} + \rho^{-1}I^2\partial_r\partial_k\partial_m\Delta^+_{k,m,i,j}(c_P^2\Phi_{i,j}^{\tau;P} - c_S^2\Phi_{i,j}^{\tau;S}) \\ & \text{for } t \geq t_0,\end{aligned} \tag{86}$$

in which

$$\Phi_k^{v;P,S}(\boldsymbol{x}, t) = (t-t_0)\langle v_k(\boldsymbol{x}', t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_{P,S}(t-t_0)]}, \tag{87}$$

$$\Phi_{i,j}^{\tau;P,S}(\boldsymbol{x}, t) = (t-t_0)\langle \tau_{i,j}(\boldsymbol{x}', t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_{P,S}(t-t_0)]}, \tag{88}$$

$$\text{for } t \geq t_0,$$

and where I_t denotes time integration from t_0 onward. Spatial and temporal inversion of Eq. (67) finally yields

$$\tau_{p,q} = \tau_{p,q}(\boldsymbol{x}, t_0) + C_{p,q,i,j} I_t \partial_i v_j. \tag{89}$$

Eqs. (86)–(89) constitute the solution to the elastodynamic initial-value problem and govern the pure time evolution of an elastic wave in a homogeneous, isotropic, perfectly elastic solid. Note that the solution to the elastodynamic initial-value problem posed in this section is more complicated than the one for the elastodynamic wave equation for the particle velocity as given by Love [3].

6. The initial-value problem for electromagnetic waves

In this section the initial-value problem associated with electromagnetic waves is discussed. The state quantities $E_r = E_r(\boldsymbol{x}, t)$ (= electric field strength) and $H_p = H_p(\boldsymbol{x}, t)$ (= magnetic field strength) of such a wave, present in a medium with permittivity $\epsilon_{k,r} = \epsilon_{k,r}(\boldsymbol{x})$ and permeability $\mu_{j,p} = \mu_{j,p}(\boldsymbol{x})$ and excited by sources with volume density of electric current $J_k = J_k(\boldsymbol{x}, t)$ and volume density of magnetic current $K_j = K_j(\boldsymbol{x}, t)$, satisfy the system of first-order partial differential equations (Maxwell's equations)

$$-\varepsilon_{k,m,p} \partial_m H_p + \epsilon_{k,r} \partial_t E_r = -J_k, \tag{90}$$

$$\varepsilon_{j,n,r} \partial_n E_r + \mu_{j,p} \partial_t H_p = -K_j, \tag{91}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3 \text{ and } t > t_0,$$

and have the assumed initial values

$$E_r(\boldsymbol{x}, t_0) \quad \text{and} \quad H_p(\boldsymbol{x}, t_0) \quad \text{for } \boldsymbol{x} \in \mathcal{R}^3. \tag{92}$$

In Eqs. (90) and (91), $\varepsilon_{i,j,k}$ is the completely antisymmetric unit tensor of rank three (Lévi-Civita tensor):

$$\varepsilon_{i,j,k} \begin{cases} = +1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ = -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ = 0 & \text{if } \{i,j,k\} \text{ is no permutation of } \{1,2,3\}. \end{cases} \tag{93}$$

Taking the time Laplace transform of Eqs. (90) and (91) and applying Eq. (2), we obtain

$$-\varepsilon_{k,m,p} \partial_m \widehat{H}_p + s\epsilon_{k,r} \widehat{E}_r = -\widehat{J}_k + \epsilon_{k,r}(\boldsymbol{x}) E_r(\boldsymbol{x}, t_0) \exp(-st_0), \tag{94}$$

$$\varepsilon_{j,n,r} \partial_n \widehat{E}_r + s\mu_{j,p} \widehat{H}_p = -\widehat{K}_j + \mu_{j,p}(\boldsymbol{x}) H_p(\boldsymbol{x}, t_0) \exp(-st_0), \tag{95}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3.$$

As the right-hand sides of these equations show, the incorporation of non-zero initial values of the state quantities at $t = t_0$ is equivalent to a change of the volume densities of the actual sources as they are active for $t > t_0$ into their related equivalent values given by

$$\widehat{J}_k^{\text{eq}} = \widehat{J}_k - \epsilon_{k,r}(\boldsymbol{x}) E_r(\boldsymbol{x}, t_0) \exp(-st_0), \tag{96}$$

$$\widehat{K}_j^{\text{eq}} = \widehat{K}_j - \mu_{j,p}(\boldsymbol{x}) H_p(\boldsymbol{x}, t_0) \exp(-st_0). \tag{97}$$

In the time domain these equivalent source densities are

$$J_k^{\text{eq}} = J_k - \epsilon_{k,r}(\boldsymbol{x}) E_r(\boldsymbol{x}, t_0) \delta(t - t_0), \tag{98}$$

$$K_j^{\text{eq}} = K_j - \mu_{j,p}(\boldsymbol{x}) H_p(\boldsymbol{x}, t_0) \delta(t - t_0). \tag{99}$$

In view of the superposition principle, the total wave motion is thus the superposition of the one excited by the source distributions as they are active for $t > t_0$ and a contribution from the initial values of the state quantities at $t = t_0$. With the aid of the Green's functions that express the state quantities at position $\boldsymbol{x}$ and instant t in terms of the source distributions at position $\boldsymbol{x}'$ and instant t', the total wave motion can therefore be represented as

$$\begin{aligned} E_r(\boldsymbol{x},t) = & \int_{t=t_0}^{\infty} \mathrm{d}t' \int_{\boldsymbol{x}' \in \mathcal{D}^s} [G^{EJ}_{r,k}(\boldsymbol{x},\boldsymbol{x}',t-t') J_k(\boldsymbol{x}',t') \\ & + G^{EK}_{r,j}(\boldsymbol{x},\boldsymbol{x}',t-t') K_j(\boldsymbol{x}',t')]\,\mathrm{d}V \\ & - \int_{\boldsymbol{x}' \in \mathcal{R}^3} [G^{EJ}_{r,k}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\epsilon_{k,r'}(\boldsymbol{x}') E_{r'}(\boldsymbol{x}',t_0) \\ & + G^{EK}_{r,j}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\mu_{j,p}(\boldsymbol{x}') H_p(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \end{aligned} \tag{100}$$

$$\begin{aligned} H_p(\boldsymbol{x},t) = & \int_{t=t_0}^{\infty} \mathrm{d}t' \int_{\boldsymbol{x}' \in \mathcal{D}^s} [G^{HJ}_{p,k}(\boldsymbol{x},\boldsymbol{x}',t-t') J_k(\boldsymbol{x}',t') \\ & + G^{HK}_{p,j}(\boldsymbol{x},\boldsymbol{x}',t-t') K_j(\boldsymbol{x}',t')]\,\mathrm{d}V \\ & - \int_{\boldsymbol{x}' \in \mathcal{R}^3} [G^{HJ}_{p,k}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\epsilon_{k,r}(\boldsymbol{x}') E_r(\boldsymbol{x}',t_0) \\ & + G^{HK}_{p,j}(\boldsymbol{x},\boldsymbol{x}',t-t_0)\mu_{j,p'}(\boldsymbol{x}') H_{p'}(\boldsymbol{x}',t_0)]\,\mathrm{d}V, \end{aligned} \tag{101}$$

$$\text{for } \boldsymbol{x} \in \mathcal{R}^3 \text{ and } t > t_0,$$

where $\mathcal{D}^s$ is the spatial support of the source distributions and the superscripts on the symbol G for the Green's functions indicate which mapping from source to field quantity is meant.

In those cases where the Green's functions can be determined analytically, a further reduction of the field representations can be obtained. In particular, this applies to homogeneous media where, owing to the shift invariance in space, the Fourier transformation offers itself as a useful tool. With the aid of Eq. (4) and the property that the electric field strength and the magnetic field strength are, under the present circumstance, continuous functions of position, the Fourier transform of Eqs. (94) and (95) is, omitting the argument $\mathrm{j}\boldsymbol{k}$, obtained as

$$\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p + s\epsilon_{k,r}\widetilde{E}_r = -\widetilde{J}_k + \epsilon_{k,r}\widetilde{E}_r(t_0)\exp(-st_0), \tag{102}$$

$$-\varepsilon_{j,n,r}\mathrm{j}k_n\widetilde{E}_r + s\mu_{j,p}\widetilde{H}_p = -\widetilde{K}_j + \mu_{j,p}\widetilde{H}_p(t_0)\exp(-st_0). \tag{103}$$

Once this system of linear algebraic equations has been solved, the inversion to the space-time domain can be carried out with the aid of the modified Cagniard method or any related technique. The simplest case in this category occurs when the medium is isotropic as well as homogeneous. This case is discussed below. In particular, the pure initial-value problem is addressed, i.e. the case where no volume sources are assumed to be active for $t > t_0$.

6.1. Time evolution of an electromagnetic wavefield in a homogeneous, isotropic medium

For a homogeneous, isotropic medium we have $\epsilon_{k,r} = \epsilon\delta_{k,r}$ and $\mu_{j,p} = \mu\delta_{j,p}$. From Eqs. (102) and (103) we then obtain, putting $\widetilde{J}_k = 0$ and $\widetilde{H}_p = 0$,

$$\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p + s\epsilon\widetilde{E}_k = \epsilon\widetilde{E}_k(t_0)\exp(-st_0), \tag{104}$$

$$-\varepsilon_{j,n,r}\mathrm{j}k_n\widetilde{E}_r + s\mu\widetilde{H}_j = \mu\widetilde{H}_j(t_0)\exp(-st_0). \tag{105}$$

Solving $\widetilde{H}_j$ from Eq. (105) as

$$\widetilde{H}_j = s^{-1}\widetilde{H}_j(t_0)\exp(-st_0) + (s\mu)^{-1}\varepsilon_{j,n,r}\mathrm{j}k_n\widetilde{E}_r \tag{106}$$

and substituting this result in Eq. (104), we arrive at the spectral-domain equation for the electric field strength

$$-\varepsilon_{k,m,p}\varepsilon_{p,n,r}c^2k_mk_n\widetilde{E}_r + s^2\widetilde{E}_k = s\widetilde{E}_k(t_0)\exp(-st_0) - \epsilon^{-1}\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p(t_0)\exp(-st_0), \tag{107}$$

in which

$$c = (\epsilon\mu)^{-1/2} \tag{108}$$

is the electromagnetic wavespeed. With the aid of the relation

$$\varepsilon_{k,m,p}\varepsilon_{p,n,r} = \delta_{k,n}\delta_{m,r} - \delta_{k,r}\delta_{m,n} \tag{109}$$

Eq. (107) can be rewritten as

$$c^2(-k_kk_r\widetilde{E}_r + k_mk_m\widetilde{E}_k) + s^2\widetilde{E}_k = s\widetilde{E}_k(t_0)\exp(-st_0) - \epsilon^{-1}\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p(t_0)\exp(-st_0). \tag{110}$$

Application of the operation k_k to Eq. (110) leads to the auxiliary relation

$$k_k\widetilde{E}_k = s^{-1}k_k\widetilde{E}_k(t_0)\exp(-st_0). \tag{111}$$

Employing Eq. (111) in Eq. (110), we obtain

$$\begin{aligned}(c^2k_mk_m + s^2)\widetilde{E}_k = {} & s\widetilde{E}_k(t_0)\exp(-st_0) + s^{-1}c^2k_kk_r\widetilde{E}_r(t_0)\exp(-st_0)\\ & -\epsilon^{-1}\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p(t_0)\exp(-st_0),\end{aligned} \tag{112}$$

from which it follows that

$$\widetilde{E}_k = \widetilde{G}[s\widetilde{E}_k(t_0)\exp(-st_0) + s^{-1}c^2k_kk_r\widetilde{E}_r(t_0)\exp(-st_0) - \epsilon^{-1}\varepsilon_{k,m,p}\mathrm{j}k_m\widetilde{H}_p(t_0)\exp(-st_0)], \tag{113}$$

in which

$$\widetilde{G} = \frac{1}{c^2k_mk_m + s^2}. \tag{114}$$

Introducing the spectral-domain potentials

$$\widetilde{\Phi}_k^E = \widetilde{G}\widetilde{E}_k(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{115}$$

$$\widetilde{\Phi}_j^H = \widetilde{G}\widetilde{H}_j(\mathrm{j}\boldsymbol{k}, t_0)\exp(-st_0), \tag{116}$$

Eq. (113) can be written as

$$\widetilde{E}_r = s\widetilde{\Phi}_r^E + s^{-1}c^2k_rk_k\widetilde{\Phi}_k^E - \epsilon^{-1}\varepsilon_{r,m,j}\mathrm{j}k_m\widetilde{\Phi}_j^H. \tag{117}$$

Using the rules for the inverse spatial Fourier transformation, $\widehat{E}_r$ is from Eq. (117) obtained as

$$\widehat{E}_r = s\widehat{\Phi}_r^E - s^{-1}c^2\partial_r\partial_k\widehat{\Phi}_k^E + \epsilon^{-1}\varepsilon_{r,m,j}\partial_m\widehat{\Phi}_j^H, \tag{118}$$

in which

$$\widehat{\Phi}_k^E = \int_{x' \in \mathcal{R}^3} \frac{\exp[-s(|x - x'|/c + t_0)]}{4\pi c^2 |x - x'|} E_k(x', t_0)\, dV, \tag{119}$$

$$\widehat{\Phi}_j^H = \int_{x' \in \mathcal{R}^3} \frac{\exp[-s(|x - x'|/c + t_0)]}{4\pi c^2 |x - x'|} H_j(x', t_0)\, dV. \tag{120}$$

Proceeding as in Section 3, the inversion to the time domain yields the result

$$E_r = \partial_t \Phi_r^E - c^2 I_t \partial_r \partial_k \Phi_k^E + \epsilon^{-1} \varepsilon_{r,m,j} \partial_m \Phi_j^H \quad \text{for } t \geq t_0, \tag{121}$$

in which

$$\Phi_k^E(x, t) = (t - t_0) \langle E_k(x', t_0) \rangle_{\mathcal{S}[x, c(t-t_0)]}, \tag{122}$$

$$\Phi_j^H(x, t) = (t - t_0) \langle H_j(x', t_0) \rangle_{\mathcal{S}[x, c(t-t_0)]}, \quad \text{for } t \geq t_0, \tag{123}$$

and where I_t denotes time integration from t_0 onward.

To arrive at the expression for the magnetic field strength, Eq. (113) is substituted in Eq. (106). This yields

$$\widetilde{H}_p = \widetilde{G}[s\widetilde{H}_p(t_0) \exp(-st_0) + s^{-1} c^2 k_p k_j \widetilde{H}_j(t_0) \exp(-st_0) + \mu^{-1} \varepsilon_{p,n,k} j k_n \widetilde{E}_k]. \tag{124}$$

The right-hand side in this equation has the same structure as the one in Eq. (113). Applying the same procedure as for the electric field strength, we obtain

$$H_p = \partial_t \Phi_p^H - c^2 I_t \partial_p \partial_j \Phi_j^H - \mu^{-1} \varepsilon_{p,n,k} \partial_n \Phi_k^E \quad \text{for } t \geq t_0. \tag{125}$$

Eqs. (121)–(123) and (125) constitute the solution to the electromagnetic initial-value problem and govern the pure time evolution of an electromagnetic wave in a homogeneous, isotropic, lossless medium. Note the symmetry in the expressions for the electric and the magnetic field strengths, a symmetry that is absent in acoustic and elastodynamic wave problems.

7. Conclusion

The initial-value problems in acoustics, elastodynamics and electromagnetics for wave propagation in a lossless medium have been discussed from a general point of view. Their solutions are expressed through the Green's functions that apply to the inhomogeneous and anisotropic media at hand and have the form of a superposition of the contribution from the volume sources as they are active in the the time interval succeeding the instant at which the initial values of the state quantities are given and a contribution from these initial values themselves. The latter expressions govern the pure time evolution of the relevant wave phenomena, once the initial values of the two state quantities (i.e. the wave quantities whose product specifies the power flow density in the wave motion) are given at a certain instant t_0 in all space. The cases of homogeneous, isotropic media are dealt with in detail. From the relevant results it follows that if the spatial support of the initial disturbances is bounded and has a maximum diameter D, the support of the evolved wave phenomenon at the instant t with $t > t_0$ has a support of maximum diameter $D + 2c_{\max}(t - t_0)$, where $c_{\max}$ is the maximum wavespeed of the wave phenomenon involved (in acoustics and electromagnetics this is just *the* wavespeed; in elastodynamics this is the P-wave speed).

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Appendix A. Determination of $\widehat{G}$ from its spatial Fourier transform $\widetilde{G}$

To determine the complex frequency domain scalar Green's function $\widehat{G} = \widehat{G}(\boldsymbol{x}, s)$ from its spatial Fourier transform (cf. Eq. (12))

$$\widetilde{G} = \frac{1}{c^2 k_m k_m + s^2}, \tag{A.1}$$

we evaluate the three-dimensional spatial Fourier integral

$$\widehat{G}(\boldsymbol{x}, s) = \left(\frac{1}{2\pi}\right)^3 \int_{\boldsymbol{k}\in\mathcal{R}^3} \frac{\exp(-\mathrm{j}k_m x_m)}{c^2 k_m k_m + s^2}\,\mathrm{d}V. \tag{A.2}$$

One way to determine $\widehat{G}$ is to apply to the right-hand side of Eq. (A.2) the modified Cagniard method as developed by the present author (see, for example, De Hoop [6]). This method involves, however, the evaluation of a modified Cagniard path of integration in a complex slowness plane, which evaluation is unavoidable when considering wave propagation in layered media, but which can be circumvented in the present case, where the propagation takes place in an unbounded homogeneous medium. In stead a simple change of variables of integration and a subsequent application of the theorem of residues will provide the answer. Following this line, we first introduce in Eq. (A.2) the polar variables of integration $\{k, \theta, \phi\}$, with $0 \le k < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$, around $\boldsymbol{x}$ as polar axis. Then, $k_m x_m = k|\boldsymbol{x}|\cos(\theta)$, $k_m k_m = k^2$ and $\mathrm{d}V = k^2 \sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$. The integrations with respect to ϕ and θ are elementary and lead to

$$\begin{aligned}\widehat{G}(\boldsymbol{x}, s) &= \frac{1}{4\pi^2 \mathrm{j}|\boldsymbol{x}|} \int_{k=0}^{\infty} \frac{\exp(\mathrm{j}k|\boldsymbol{x}|) - \exp(-\mathrm{j}k|\boldsymbol{x}|)}{c^2 k^2 + s^2} k\,\mathrm{d}k \\ &= -\frac{1}{4\pi^2 \mathrm{j} c^2 |\boldsymbol{x}|} \int_{k=-\infty}^{\infty} \frac{\exp(-\mathrm{j}k|\boldsymbol{x}|)}{k^2 + s^2/c^2} k\,\mathrm{d}k.\end{aligned} \tag{A.3}$$

In the last integral on the right-hand side, the integrand, which is an analytic function of k, is continued away from the real axis into the complex k-plane. Next, the path of integration (the real axis) is supplemented by a semi-circle at infinity in the lower half of the k-plane and the residue theorem is applied to the resulting closed contour that encloses the simple pole $k = -\mathrm{j}s/c$ (note that s is real and positive). Since the contribution from the semi-circle at infinity vanishes in view of Jordan's lemma (note that $|\boldsymbol{x}| > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$), the result is

$$\widehat{G}(\boldsymbol{x}, s) = \frac{\exp(-s|\boldsymbol{x}|/c)}{4\pi c^2 |\boldsymbol{x}|} \quad \text{for } |\boldsymbol{x}| \neq 0. \tag{A.4}$$

Eq. (A.4) is used in the main text.

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