Transient Diffusive Electromagnetic Field Computation—A Structured Approach Based on Reciprocity

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Summary. The reciprocity theorem for transient diffusive electromagnetic fields is taken as the point of departure for developing computational methods to model such fields. Mathematically, the theorem is representative of any weak formulation of the field problem. Physically, the theorem describes the interaction between (a discretized version of) the actual field and a suitably chosen computational state. The choice of the computational state determines which type of computational method results from the analysis. It is shown that the finite-element method, the integral-equation method, and the domain-integration method can be viewed as particular cases of discretization of the reciprocity relation. The local field representations of the electric- and the magnetic-field strengths in terms of edge-element expansion functions are worked out in some detail.

The emphasis is on time-domain methods. The relationship with complex frequency-domain methods is indicated and used to symmetrize the basic field equations. This symmetrization expresses the correspondence that exists between transient electromagnetic wavefields in lossless media and transient diffusive electromagnetic fields in conductive media where the electric displacement-current contribution to the field can be neglected in the time window of observation. This aspect is also of importance in numerical modeling.

1 Introduction

The local, pointwise behavior in space-time of transient diffusive electromagnetic (EM) fields is governed by a parabolic system of first-order partial differential equations (Maxwell's equations in the diffusive approximation) that represent the EM phenomena on a local scale. When supplemented with boundary conditions that join the field values on either side of the interfaces where the constitutive parameters jump by finite amounts, and with the requirement of causality in the relationship between the field and its generating sources, the problem has a unique solution. A number of properties of this solution, in particular its analyticity and reciprocity properties, follow from this

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description. The computational handling of the field problem, however, often starts from a weak formulation, where the pointwise, or strong, satisfaction of the equality signs in the equations is replaced with requirements on the equality of certain integrated, or weighted, versions of the differential equations. Such weighted versions can be considered as special cases of the global reciprocity theorem that applies to two different admissible field states that are defined in one and the same domain in configuration space. Conceptually, a computational scheme to evaluate the field then is taken to describe the interaction between (a discretized version of) the actual field state and a suitably chosen computational state. The latter is representative of the method at hand (e.g., finite-element method and its related method of weighted residuals, integral-equation method, domain-integration method). Thus, choosing the reciprocity theorem as the point of departure offers the road to a structured approach to constructing computational schemes for evaluating the field. Besides, the standard source/receiver reciprocity properties (which are also consequences of the reciprocity theorem) can serve as a check on the consistency of the numerical results.

The emphasis is on time-domain methods. The relationship with complex frequency-domain methods is indicated, in particular to symmetrize the diffusive EM field equations in such a manner that the correspondence between transient diffusive EM fields in conductive media and EM wavefields in lossless media becomes manifest.

2 Diffusive EM field

The diffusive EM field under consideration is present in 3-D Euclidean space $\mathcal{R}^3$. The distribution of matter in it is assumed to be time invariant and the materials are assumed to be linear in their EM behavior. Position in the configuration is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal, Cartesian reference frame with the origin $\mathcal{O}$ and the three, mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The corresponding position vector is $\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$. The time coordinate is t. The subscript notation for vectors and tensors is used and the summation convention applies. Differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol for differentiation with respect to t.

The EM constitutive properties of the media in the configuration are characterized by their (electrical) conductivity $\sigma_{k,r} = \sigma_{k,r}(\mathbf{x})$ and their (magnetic) permeability $\mu_{j,p} = \mu_{j,p}(\mathbf{x})$. The constitutive parameters are taken to be positive definite, symmetric tensors of rank two, thus allowing for anisotropy in the medium. The action of the sources that generate the field is characterized by the volume density of (external) electric current $J_k = J_k(\mathbf{x}, t)$ and the volume density of (external) magnetic current $K_j = K_j(\mathbf{x}, t)$. In each subdomain of the configuration where the constitutive coefficients vary continuously with position, the field quantities electric-field strength $E_r = E_r(\mathbf{x}, t)$ and magnetic-field strength $H_p = H_p(\mathbf{x}, t)$ then satisfy the parabolic system of partial differential equations (Ward and Hohmann, 1989)

$$-\epsilon_{k,m,p}\partial_m H_p + \sigma_{k,r}E_r = -J_k, \tag{1}$$

$$\epsilon_{j,n,r}\partial_n E_r + \mu_{j,p}\partial_t H_p = -K_j, \tag{2}$$

where $\epsilon_{k,m,p}$ is the completely antisymmetric unit tensor of rank three (Levi-Civita tensor): $\epsilon_{k,m,p} = +1$ if $\{k, m, p\}$ is an even permutation of $\{1, 2, 3\}$, $\epsilon_{k,m,p} = -1$ if $\{k, m, p\}$ is an odd permutation of $\{1, 2, 3\}$, $\epsilon_{k,m,p} = 0$ in all other cases. The existence of solutions of these field equations requires satisfaction of the compatibility relations

$$\partial_k(\sigma_{k,r}E_r) = -\partial_k J_k, \tag{3}$$

$$\partial_j(\mu_{j,p}\partial_t H_p) = -\partial_j K_j. \tag{4}$$

Across interfaces where $\sigma_{k,r}$ and/or $\mu_{j,p}$ jump by finite amounts, the field quantities are no longer continuously differentiable and the boundary conditions

$$\epsilon_{k,m,p}\nu_m H_p = \text{continuous}, \tag{5}$$

$$\epsilon_{j,n,r}\nu_n E_r = \text{continuous}, \tag{6}$$

should be satisfied. Here, ν_m is the unit vector along the normal to the interface. If the configuration extends to infinity, it is assumed that outside some bounded closed surface $\partial\mathcal{D}_0$ the medium is homogeneous and isotropic. In this domain, denoted by $\mathcal{D}_0$, the (scalar) conductivity has the value σ_0 and the (scalar) permeability the value μ_0. Because the tensor Green's functions for such a medium are analytically known, analytic source-type integral representations for the field quantities in $\mathcal{D}_0$ exist. The latter play a role in the *contrast source* or *scattering* formulation of the field problem.

In the analysis, the *time convolution* operator is needed. For any two space-time functions $F(\mathbf{x}, t)$ and $Q(\mathbf{x}, t)$, this is defined as

$$\mathrm{C}_t(F, Q; \mathbf{x}, t) = \int_{t' \in \mathcal{R}} F(\mathbf{x}, t')Q(\mathbf{x}, t - t')\,dt' \quad \text{for } t \in \mathcal{R}. \tag{7}$$

It has the properties

$$\mathrm{C}_t(F, Q; \mathbf{x}, t) = \mathrm{C}_t(Q, F; \mathbf{x}, t), \tag{8}$$

$$\partial_t \mathrm{C}_t(F, Q; \mathbf{x}, t) = \mathrm{C}_t(\partial_t F, Q; \mathbf{x}, t) = \mathrm{C}_t(F, \partial_t Q; \mathbf{x}, t). \tag{9}$$

For *causal* space-time functions $F(\mathbf{x}, t)$ and $Q(\mathbf{x}, t)$ having the semiinfinite interval $\{t \in \mathcal{R}; t > 0\}$ as their support, $\mathrm{C}_t(F, Q; \mathbf{x}, t)$ is causal as well, with the same support.

The relation between the time-domain quantities and their complex frequency-domain counterparts is given by the time Laplace transformation, which for any space-time function $F(\mathbf{x}, t)$ is

$$\hat{F}(\mathbf{x}, s) = \int_{t \in \mathcal{R}} \exp(-st)F(\mathbf{x}, t)\,dt \quad \text{for } \mathrm{Re}(s) = s_0, \tag{10}$$

where s_0 is some real value of $s \in \mathcal{C}$ for which the integral on the right-hand side is convergent. For *causal*, bounded, space-time functions $F(\mathbf{x}, t)$ having the semi-infinite interval $\{t \in \mathcal{R}; t > 0\}$ as their support, $\hat{F}(\mathbf{x}, s)$ is analytic in the right half $\{s \in \mathcal{C}; \mathrm{Re}(s) > 0\}$ of the complex s-plane.

From Eqs. (7) and (10), the Laplace transform $\hat{\mathrm{C}}_t(F, Q; \mathbf{x}, s)$ of $\mathrm{C}_t(F, Q; \mathbf{x}, t)$ is found as

$$\hat{\mathrm{C}}_t(F, Q; \mathbf{x}, s) = \hat{F}(\mathbf{x}, s)\hat{Q}(\mathbf{x}, s). \tag{11}$$

Further, from Eq. (10) and a subsequent integration by parts, the Laplace transform $\hat{\partial}_t F(\mathbf{x}, s)$ of $\partial_t F(\mathbf{x}, t)$ is

$$\hat{\partial}_t F(\mathbf{x}, s) = s\hat{F}(\mathbf{x}, s). \tag{12}$$

With the aid of this latter rule, the complex frequency-domain field equations are obtained from Eqs. (1) and (2) and (10) and (12):

$$-\epsilon_{k,m,p}\partial_m \hat{H}_p + \sigma_{k,r}\hat{E}_r = -\hat{J}_k, \tag{13}$$

$$\epsilon_{j,n,r}\partial_n \hat{E}_r + s\mu_{j,p}\hat{H}_p = -\hat{K}_j. \tag{14}$$

The complex frequency-domain compatiblity relations are obtained from Eqs. (3) and (4) and (10) and (12):

$$\partial_k(\sigma_{k,r}\hat{E}_r) = -\partial_k \hat{J}_k, \tag{15}$$

$$s\partial_j(\mu_{j,p}\hat{H}_p) = -\partial_j \hat{K}_j. \tag{16}$$

The boundary conditions across interfaces in jumps of the constitutive coefficients are obtained from Eqs. (5) and (6) and (10) and (12):

$$\epsilon_{k,m,p}\nu_m \hat{H}_p = \text{continuous}, \tag{17}$$

$$\epsilon_{j,n,r}\nu_n \hat{E}_r = \text{continuous}. \tag{18}$$

3 Reciprocity theorem

In the reciprocity theorem that is named after H. A. Lorentz, a certain *interaction quantity* is considered that is representative for the interaction between two admissible solutions (states) of the field equations, where the latter are defined in one and the same (proper or improper) subdomain $\mathcal{D}$ of $\mathcal{R}^3$. The domain $\mathcal{D}$ is assumed to be the union of a finite number of subdomains in each of which the field quantities of the two states are continuously differentiable. Furthermore, each of the two states applies to its own medium and has its own volume source distributions. The two states are indicated by the superscripts A and Z, respectively (Fig. 1). The relevant local interaction quantity is $\epsilon_{m,r,p}\partial_m[\mathrm{C}_t(E_r^A, H_p^Z) - \mathrm{C}_t(E_r^Z, H_p^A)]$ (de Hoop, 1987, 1995). Using the standard rules for the spatial differentiation and employing the field equations of the type (1) and (2) for the two states gives

$$\begin{aligned}\epsilon_{m,r,p}\partial_m\big[\mathrm{C}_t\big(E_r^A, H_p^Z\big) - \mathrm{C}_t\big(E_r^Z, H_p^A\big)\big] \\ = -\big(\sigma_{r,k}^Z - \sigma_{k,r}^A\big)\mathrm{C}_t\big(E_r^A, E_k^Z\big) + \big(\mu_{p,j}^Z - \mu_{j,p}^A\big)\partial_t \mathrm{C}_t\big(H_p^A, H_j^Z\big) \\ + \mathrm{C}_t\big(J_k^A, E_k^Z\big) - \mathrm{C}_t\big(K_j^A, H_j^Z\big) - \mathrm{C}_t\big(J_r^Z, E_r^A\big) + \mathrm{C}_t\big(K_p^Z, H_p^A\big).\end{aligned} \tag{19}$$

Equation (19) is the *local form of the EM reciprocity theorem of the time-convolution type.* The first two terms on the right-hand side are representative of the differences (contrasts) in the EM properties of the media in the two states; these terms vanish at

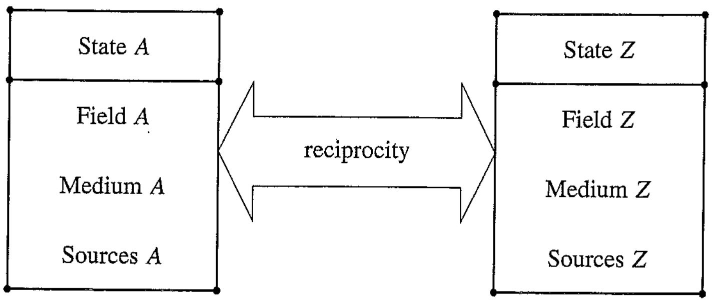

Figure 1. Two admissible states in reciprocity theorem.

those positions where $\sigma^Z_{r,k}(\mathbf{x}) = \sigma^A_{k,r}(\mathbf{x})$ and $\mu^Z_{p,j}(\mathbf{x}) = \mu^A_{j,p}(\mathbf{x})$. At points where these latter conditions hold, the media are denoted as each other's *adjoints*. The last four terms on the right-hand side are representative of the action of the volume sources in the two states; these terms vanish at those positions where the field is source-free.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms in Eq. (19) are continuous. Upon integrating Eq. (19) over each of these subdomains, applying Gauss's integral theorem to the resulting left-hand sides, and adding the results, it follows that

$$\begin{aligned}\epsilon_{m,r,p}\int_{\partial\mathcal{D}} \nu_m\big[\mathrm{C}_t\big(E^A_r, H^Z_p\big) - \mathrm{C}_t\big(E^Z_r, H^A_p\big)\big]\,\mathrm{d}A(\mathbf{x})\\ = \int_{\mathcal{D}} \big[-\big(\sigma^Z_{r,k} - \sigma^A_{k,r}\big)\mathrm{C}_t\big(E^A_r, E^Z_k\big) + \big(\mu^Z_{p,j} - \mu^A_{j,p}\big)\partial_t\mathrm{C}_t\big(H^A_p, H^Z_j\big)\big]\,\mathrm{d}V(\mathbf{x})\\ + \int_{\mathcal{D}} \big[\mathrm{C}_t\big(J^A_k, E^Z_k\big) - \mathrm{C}_t\big(K^A_j, H^Z_j\big) - \mathrm{C}_t\big(J^Z_r, E^A_r\big) + \mathrm{C}_t\big(K^Z_p, H^A_p\big)\big]\,\mathrm{d}V(\mathbf{x}).\end{aligned} \tag{20}$$

Equation (20) is the *global form, for the domain $\mathcal{D}$, of the reciprocity theorem of the time-convolution type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have canceled in view of the boundary conditions (5) and (6). In view of this, in the left-hand side only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains.

The complex frequency-domain versions of the local and the global reciprocity theorems follow from their time-domain counterparts by taking the time Laplace transform. Applying the standard rules given in Section 2, the complex frequency-domain version of the local reciprocity theorem follows from Eq. (19) as

$$\begin{aligned}\epsilon_{m,r,p}\partial_m\big(\hat{E}^A_r\hat{H}^Z_p - \hat{E}^Z_r\hat{H}^A_p\big) = -\big(\sigma^Z_{r,k} - \sigma^A_{k,r}\big)\hat{E}^A_r\hat{E}^Z_k + s\big(\mu^Z_{p,j} - \mu^A_{j,p}\big)\hat{H}^A_p\hat{H}^Z_j\\ + \hat{J}^A_k\hat{E}^Z_k - \hat{K}^A_j\hat{H}^Z_j - \hat{J}^Z_r\hat{E}^A_r + \hat{K}^Z_p\hat{H}^A_p,\end{aligned} \tag{21}$$

and the complex frequency-domain version of the global reciprocity theorem from Eq. (20) as

$$\begin{aligned}\epsilon_{m,r,p}\int_{\partial\mathcal{D}} \nu_m\big(\hat{E}^A_r\hat{H}^Z_p - \hat{E}^Z_r\hat{H}^A_p\big)\,\mathrm{d}A(\mathbf{x})\\ = \int_{\mathcal{D}} \big[-\big(\sigma^Z_{r,k} - \sigma^A_{k,r}\big)\hat{E}^A_r\hat{E}^Z_k + s\big(\mu^Z_{p,j} - \mu^A_{j,p}\big)\hat{H}^A_p\hat{H}^Z_j\big]\,\mathrm{d}V(\mathbf{x})\\ + \int_{\mathcal{D}} \big(\hat{J}^A_k\hat{E}^Z_k - \hat{K}^A_j\hat{H}^Z_j - \hat{J}^Z_r\hat{E}^A_r + \hat{K}^Z_p\hat{H}^A_p\big)\,\mathrm{d}V(\mathbf{x}).\end{aligned} \tag{22}$$

3.1 Limiting case of an unbounded domain

In quite a number of cases, the global reciprocity theorems are applied to an unbounded domain. To handle such cases, the embedding provisions of Section 2 are made and the theorem is applied first to the sphere $\mathcal{S}(\mathcal{O}, \Delta)$ with center at the origin $\mathcal{O}$ of the chosen reference frame and radius Δ, after which the limit $\Delta \to \infty$ is taken. From the source-type field integral representations pertaining to the homogeneous, isotropic embedding, it then follows that the contribution from $\mathcal{S}(\mathcal{O}, \Delta)$ vanishes in the limit $\Delta \to \infty$.

In the above procedure, the EM field equations pertaining to the two states have been taken as the point of departure, and the reciprocity theorems have been derived by operating on the equations in the manner indicated. In the realm of the use of the reciprocity theorems as the basis of a structured approach to the computation of the fields, note that, reversely, a necessary and sufficient condition for the global reciprocity theorem for arbitrary EM states Z satisfying equations of types (1) and (2) and boundary conditions of types (3) and (4) to hold is that the field in state A satisfies equations of types (1) and (2) and boundary conditions of types (3) and (4) as well.

4 Embedding procedure and contrast-source formulations

On many occasions the EM field computation in an entire geophysical configuration is beyond the capabilities because of the storage capacity and the computation times involved. In that case, it is standard practice to select a *target region* of bounded support in which a detailed computation is to be carried out, while the medium in the remaining part of the configuration (the *embedding*) is taken to be so simple that the field in it can be determined with the aid of analytical methods. Examples of such embeddings in $\mathcal{R}^3$ as the configuration space are the homogeneous isotropic embedding, and the embedding consisting of a finite number of parallel homogeneous layers. In these cases, combined time Laplace and spatial Fourier transform techniques provide the analytical tools to determine the field or, in fact, construct the relevant Green's tensors. Once the embedding has been chosen, the problem of computing the field in the target region can be formulated advantageously as a *contrast-source* or *scattering* problem (Hohmann, 1989).

To this end, first the *incident field* $\{E^i_r, H^i_p\}$ is introduced as the field that would be generated by the sources as if they were present in the embedding. Let the constitutive parameters of the embedding be $\sigma^{\mathrm{b}}_{k,r} = \sigma^{\mathrm{b}}_{k,r}(\mathbf{x})$ and $\mu^{\mathrm{b}}_{j,p} = \mu^{\mathrm{b}}_{j,p}(\mathbf{x})$; then, the incident field satisfies the basic field equations

$$-\epsilon_{k,m,p}\partial_m H^i_p + \sigma^{\mathrm{b}}_{k,r}E^i_r = -J_k, \tag{23}$$

$$\epsilon_{j,n,r}\partial_n E^i_r + \mu^{\mathrm{b}}_{j,p}\partial_t H^i_p = -K_j. \tag{24}$$

Next, the *scattered field* $\{E^{\mathrm{s}}_r, H^{\mathrm{s}}_p\}$ is defined as the difference between the total field $\{E_r, H_p\}$ and the incident field $\{E^i_r, H^i_p\}$. Hence, $\{E_r, H_p\} = \{E^i_r + E^{\mathrm{s}}_r, H^i_p + H^{\mathrm{s}}_p\}$. The field equations for the scattered field can be written alternatively as

$$-\epsilon_{k,m,p}\partial_m H^{\mathrm{s}}_p + \sigma_{k,r}E^{\mathrm{s}}_r = -\left(\sigma_{k,r} - \sigma^{\mathrm{b}}_{k,r}\right)E^i_r, \tag{25}$$

$$\epsilon_{j,n,r}\partial_n E^{\mathrm{s}}_r + \mu_{j,p}\partial_t H^{\mathrm{s}}_p = -\left(\mu_{j,p} - \mu^{\mathrm{b}}_{j,p}\right)\partial_t H^i_p, \tag{26}$$

or as

$$-\epsilon_{k,m,p}\partial_m H^{\mathrm{s}}_p + \sigma^{\mathrm{b}}_{k,r}E^{\mathrm{s}}_k = -\left(\sigma_{k,r} - \sigma^{\mathrm{b}}_{k,r}\right)E_r, \tag{27}$$

$$\epsilon_{j,n,r}\partial_n E^{\mathrm{s}}_r + \mu^{\mathrm{b}}_{j,p}\partial_t H^{\mathrm{s}}_p = -\left(\mu_{j,p} - \mu^{\mathrm{b}}_{j,p}\right)\partial_t H_p. \tag{28}$$

In both systems, the right-hand sides only differ from zero in the domain where the constitutive properties of the medium differ from those of the embedding. Further, in none of them do the activating source distributions occur. This has the advantage of a smoother behavior of the right-hand sides of the differential equations, a behavior that is due to the fact that the (incident) field variation is smoother in space than its generating source distributions (Hohmann, 1989). Equations (25) and (26) are typically the point of departure for finite-difference or finite-element computations; Eqs. (27) and (28) are typically the point of departure for integral-equation computations and for the construction of absorbing boundary conditions or Dirichlet-to-Neumann maps.

The source-type integral representations for the incident and the scattered fields are of the type

$$E_r^{\mathrm{i,s}}(\mathbf{x}, t) = \int_{\mathcal{D}^{\mathrm{i,s}}} \left\{ \mathrm{C}_t\left[G_{r,k}^{E,J}(\mathbf{x}, \mathbf{x}', \cdot), J_k^{\mathrm{i,s}}(\mathbf{x}', \cdot)\right] + \mathrm{C}_t\left[G_{r,j}^{E,K}(\mathbf{x}, \mathbf{x}', \cdot), K_j^{\mathrm{i,s}}(\mathbf{x}', \cdot)\right]\right\} \mathrm{d}V(\mathbf{x}'), \tag{29}$$

$$H_p^{\mathrm{i,s}}(\mathbf{x}, t) = \int_{\mathcal{D}^{\mathrm{i,s}}} \left\{ \mathrm{C}_t\left[G_{p,k}^{H,J}(\mathbf{x}, \mathbf{x}', \cdot), J_k^{\mathrm{i,s}}(\mathbf{x}', \cdot)\right] + \mathrm{C}_t\left[G_{p,j}^{H,K}(\mathbf{x}, \mathbf{x}', \cdot), K_j^{\mathrm{i,s}}(\mathbf{x}', \cdot)\right]\right\} \mathrm{d}V(\mathbf{x}'), \tag{30}$$

where $\mathcal{D}^{\mathrm{i}}$ is the support of the volume-source densities

$$J_k^{\mathrm{i}} = J_k, \tag{31}$$

$$K_j^{\mathrm{i}} = K_j, \tag{32}$$

generating the incident field; $\mathcal{D}^{\mathrm{s}}$ is the support of the contrast volume-source densities

$$J_k^{\mathrm{s}} = \left(\sigma_{k,r} - \sigma_{k,r}^{\mathrm{b}}\right) E_r, \tag{33}$$

$$K_j^{\mathrm{s}} = \left(\mu_{j,p} - \mu_{j,p}^{\mathrm{b}}\right) \partial_t H_p, \tag{34}$$

generating the scattered field; and $G_{r,k}^{E,J}$, $G_{r,j}^{E,K}$, $G_{p,k}^{H,J}$, $G_{p,j}^{H,K}$ are the electric-field/electric-current, electric-field/magnetic-current, magnetic-field/electric-current, magnetic-field/ magnetic-current Green's tensors of the homogeneous isotropic embedding.

The complex frequency-domain versions of Eqs. (23) and (34) are found from their time-domain counterparts by replacing the operator ∂_t with the multiplying factor s and replacing the time convolutions with the product of their operands.

5 Computational procedures based on reciprocity

In the structured approach to the development of computational procedures based on reciprocity, the first step consists of selecting, in the global reciprocity theorems derived in Section 3, a finite number of linearly independent computational states for the state Z. The relevant states are indicated by the superscript C and their number is taken to be N^C. Next, state A is taken to be an approximation to the scattered field as introduced in Section 4, in the form of an expansion into a sequence of appropriate, linearly independent, known expansion functions and provided with unknown expansion coefficients. The relevant state is indicated by the superscript s and its field representation contains N^{s} terms. Based on the knowledge (see the end of Section 3) that for any number of arbitrary computational states and with an appropriate expansion containing an infinite number of terms for the scattered state, the application of the reciprocity theorem would lead to the unique, exact solution of the field problem, it is now assumed that the procedure with a finite number of computational states and a finite number of terms in the expansion of the scattered state leads to an approximate solution to the field problem. A quantification of the resulting error can be decided only after having introduced an appropriate error criterion. The latter is beyond the scope of the present analysis, which is focused mainly on the construction of both the computational states and the appropriate

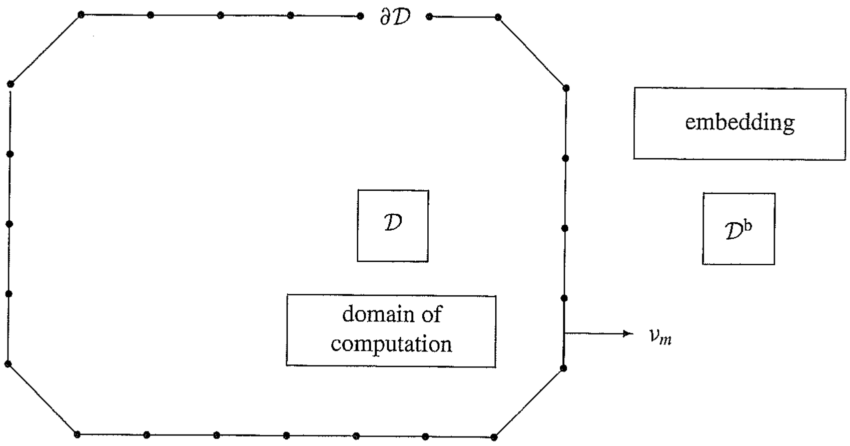

Figure 2. Discretized domain of computation $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ and embedding $\mathcal{D}^{\mathrm{b}}$.

expansion functions. From the beginning, it is clear that, for $N^C < N^s$, the system of linear algebraic equations in the expansion coefficients is underdetermined and hence cannot be solved, whereas, for $N^C = N^s$, the system of linear algebraic equations in the expansion coefficients has, in principle, a unique solution, whereas for $N^C > N^s$, the system of linear algebraic equations in the expansion coefficients is overdetermined and, hence, is amenable to a minimum norm solution in its residual.

The computations generally are carried out on a geometrically discretized version of the configuration. To this end, first the target region or domain of computation $\mathcal{D}$ is selected and discretized (Fig. 2). The boundary surface $\partial\mathcal{D}$ of this domain is taken to be located in the embedding $\mathcal{D}^{\mathrm{b}}$. Its geometric shape is taken such that it can be handled by a *mesh generator*. Typical cases are the discretization into a union of 3-rectangles or 3-simplices (tetrahedra), all of which have vertices, edges, and faces in common (Naber, 1980). The maximum diameter of the elements of the discretized geometry is denoted as its *mesh size*. The mesh size to be chosen depends on the shape of $\partial\mathcal{D}$, as well as on the spatial variations of the constitutive coefficients and the temporal and spatial variations of the volume-source densities and the field values in $\mathcal{D}$.

The mesh size is first adapted to the spatial variations of the known quantities (constitutive coefficients and volume-source densities in forward problems, volume-source densities and measured field values in inverse problems) and later iteratively adapted to the quantities to be computed (field values in forward problems, constitutive coefficients in inverse problems). Coupled to the mesh are, next, the spatial and temporal representations of the discretized known quantities. Finally, the discretized versions of the computational states and the unknown quantities are selected.

To illustrate the procedure, the forward field problem is discussed in more detail below. Discussion of EM inverse-source and inverse-wave-scattering problems can be found in de Hoop (1991).

It is assumed that the incident field has been determined already, for example, by evaluation of the relevant source-type integral representations containing the known Green's tensors of the embedding (see Section 4). In the forward-field computation problem, the constitutive coefficients and the volume-source distributions are given, and the field values are to be computed. As far as the medium properties are concerned, the analysis is concentrated on the case of strongly heterogeneous media where the constitutive coefficients may, in principle, jump from each subdomain of the discretized

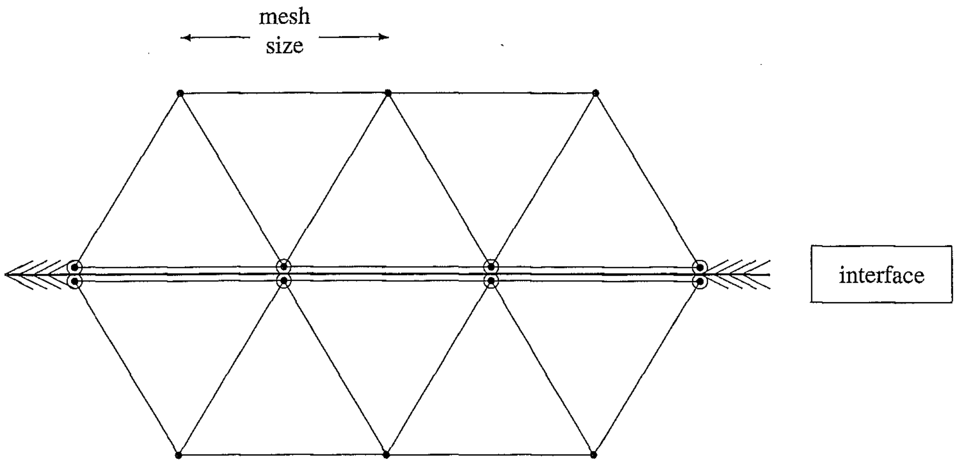

Figure 3. Interface (⋘) and simplicial mesh with multiple nodes (◉) and simple nodes (•).

geometry to any adjacent subdomain. The mesh size is assumed to be chosen so small that *piecewise linear expansions* are accurate enough to locally represent the field values, the constitutive coefficients, and the volume-source densities. A consistent theory then can be developed for a *simplicial mesh* consisting of 3-simplices (tetrahedra) all of which have vertices, edges, and faces in common (Fig. 3).

Consider one of the tetrahedra, Σ say, of the mesh and let $\{x_m(0), x_m(1), x_m(2), x_m(3)\}$ be the position vectors of its vertices. The ordering in the sequence defines the *orientation* of the tetrahedron. Further, let $\{A_m(0), A_m(1), A_m(2), A_m(3)\}$ denote the outwardly oriented vectorial areas of the faces of Σ, where the ordinal number of a face is taken to be the ordinal number of the vertex opposite to it. The position vector $\mathbf{x}$ in Σ then can be expressed in a symmetrical fashion in terms of the *barycentric coordinates* $\{\lambda(0, \mathbf{x}), \lambda(1, \mathbf{x}), \lambda(2, \mathbf{x}), \lambda(3, \mathbf{x})\}$ through

$$x_m = \sum_{I=0}^{3} \lambda(I, \mathbf{x}) x_m(I). \tag{35}$$

Inversely, the barycentric coordinates can be expressed in terms of the position vector via the relation

$$\lambda(I, \mathbf{x}) = 1/4 - (1/3V)(x_m - b_m)A_m(I) \quad \text{for } I = 0, 1, 2, 3, \tag{36}$$

where V is the volume of Σ and

$$b_m = \frac{1}{4}\sum_{I=0}^{3} x_m(I) \tag{37}$$

is the position vector of its *barycenter*. The barycentric coordinates have the property

$$\lambda[I, \mathbf{x}(J)] = \delta(I, J) \quad \text{for } I = 0, 1, 2, 3;\ J = 0, 1, 2, 3, \tag{38}$$

where $\delta(I, J)$ is the Kronecker symbol: $\delta(I, J) = 1$ for $I = J$ and $\delta(I, J) = 0$ for $I \neq J$.

As Eqs. (35) and (38) show, the barycentric coordinates perform a linear interpolation, in the interior of Σ, between the function value 1 at one of the vertices and the function value 0 at the remaining vertices. Consequently, they can be used as the (linear) interpolation functions for any of the quantities occurring in the field computation. As an example, the electric-field strength is considered. This quantity admits the local representation

$$E_r(\mathbf{x}, t) = \sum_{I=0}^{3} A_r^E(I, t)\,\lambda(I, \mathbf{x}) \quad \text{for } \mathbf{x} \in \Sigma, \tag{39}$$

where

$$A_r^E(I, t) = E_r[\mathbf{x}(I), t] \quad \text{for } I = 0, 1, 2, 3. \tag{40}$$

From the *local* representations of type (39), the *global* representations for the domain of computation are constructed. In this process, the values of the constitutive coefficients and the volume-source densities in the interior of the tetrahedron Σ, and hence their limiting values upon approaching (via the interior) the vertices of Σ, have no relation to the values of these quantities in any of the neighbors of Σ. As a consequence, each nodal point of the mesh is, for these quantities, initially considered as a *multiple node*, with multiplicity equal to the number of vertices that meet at that point. Subsequently, the multiple nodes are combined to *simple nodes* in all of those subdomains of the domain of computation where the quantities are known to be continuous. However, for the electric- and the magnetic-field strengths, the situation shows additional features. Here, all components vary continously in space as long as the constitutive coefficients do so (even if the volume-source densities vary only piecewise continuously in space), but across a jump discontinuity in constitutive properties of the medium, the tangential components of the field strengths are to be continuous, whereas their normal components should remain free to jump. A representation that meets these requirements is furnished by the *edge-element representation* (Mur and de Hoop, 1985). In this representation, $A_r^E(I, t) = E_r[\mathbf{x}(I), t]$ is expressed in terms of its projections along the edges that leave the vertex $\mathbf{x}(I)$. Rather than with these projections, we work with the numbers

$$\alpha^E(I, J, t) = E_r[\mathbf{x}(I), t][x_r(J) - x_r(I)] \quad \text{for } I = 0, 1, 2, 3; \quad J = 0, 1, 2, 3, \tag{41}$$

with $\alpha^E(I, I, t) = 0$. In view of the fact that, at the vertex $\mathbf{x}(I)$, the three vectorial edges $\{x_r(J) - x_r(I);\, J \neq I\}$ and the three vectorial faces $\{A_r(K);\, K \neq I\}$ form an (oblique) system of reciprocal base vectors in $\mathcal{R}^3$, the property

$$[x_m(J) - x_m(I)]A_m(K) = -3V[\delta(J, K) - \delta(I, K)] \quad \text{for } I = 0, 1, 2, 3;\; J = 0, 1, 2, 3;\; K = 0, 1, 2, 3 \tag{42}$$

holds. From Eqs. (40)–(42) it follows that

$$E_r[\mathbf{x}(I), t] = -\frac{1}{3V}\sum_{J=0}^{3} \alpha^E(I, J, t)A_r(J) \quad \text{for } I = 0, 1, 2, 3. \tag{43}$$

Because $\alpha(I, I, t) = 0$, we indeed have, through Eq. (41), at each vertex three numbers that, through Eq. (43), represent the expanded electric-field strength. By enforcing the numbers along a particular edge to be the same for all tetrahedra that have this edge in common, the continuity of the tangential components of E_r across edges and faces is guaranteed, and the normal components of E_r across faces are left free to jump. A similar piecewise spatial linear expansion is used for the magnetic-field strength H_p.

The piecewise linear expansions discussed above are used in the context of the different computational methods in existence. These are indicated briefly below.

5.1 Finite-element method

The finite-element method is characterized by taking $\sigma^C_{r,k} = 0$ and $\mu^C_{p,j} = 0$ and choosing either

$$E^C_k \in \{\text{electric-field-strength expansion functions}\}, \tag{44}$$
$$H^C_j = 0, \tag{45}$$
$$J^C_r = 0, \tag{46}$$
$$K^C_p = -\epsilon_{p,n,k}\partial_n E^C_k, \tag{47}$$

or

$$E^C_k = 0, \tag{48}$$
$$H^C_j \in \{\text{magnetic-field-strength expansion functions}\}, \tag{49}$$
$$J^C_r = \epsilon_{r,m,j}\partial_m H^C_j, \tag{50}$$
$$K^C_p = 0. \tag{51}$$

For this method, the choice of the field strengths typifies the computational state.

5.2 Integral-equation method

The integral-equation method is characterized by taking for the constitutive coefficients the values of the embedding, i.e., $\sigma^C_{r,k} = \sigma_0\delta_{r,k}$ and $\mu^C_{p,j} = \mu_0\delta_{p,j}$ and choosing either

$$J^C_r \in \{\text{electric-current volume-source expansion functions}\}, \tag{52}$$
$$K^C_p = 0, \tag{53}$$
$$E^C_k(\mathbf{x}, t) = \int_{\mathcal{D}^J} \{\mathrm{C}_t[G^{E,J}_{k,r}(\mathbf{x}, \mathbf{x}', \cdot), J^C_r(\mathbf{x}', \cdot)]\,\mathrm{d}V(\mathbf{x}'), \tag{54}$$
$$H^C_j(\mathbf{x}, t) = \int_{\mathcal{D}^J} \{\mathrm{C}_t[G^{H,J}_{j,r}(\mathbf{x}, \mathbf{x}', \cdot), J^C_r(\mathbf{x}', \cdot)]\,\mathrm{d}V(\mathbf{x}'), \tag{55}$$

where $\mathcal{D}^J$ is the support of J^C_r, or

$$J^C_r = 0, \tag{56}$$
$$K^C_p \in \{\text{magnetic-current volume-source expansion functions}\}, \tag{57}$$
$$E^C_k(\mathbf{x}, t) = \int_{\mathcal{D}^K} \{\mathrm{C}_t[G^{E,K}_{k,p}(\mathbf{x}, \mathbf{x}', \cdot), K^C_p(\mathbf{x}', \cdot)]\,\mathrm{d}V(\mathbf{x}'), \tag{58}$$
$$H^C_j(\mathbf{x}, t) = \int_{\mathcal{D}^K} \{\mathrm{C}_t[G^{H,K}_{j,p}(\mathbf{x}, \mathbf{x}', \cdot), K^C_p(\mathbf{x}', \cdot)]\,\mathrm{d}V(\mathbf{x}'), \tag{59}$$

where $\mathcal{D}^K$ is the support of K^C_p. For this method, the choice of the volume-source distributions, located in the embedding, typifies the computational state.

5.3 Domain-integration method

The domain-integration method is characterized by taking $\sigma^C_{r,k} = 0$ and $\mu^C_{p,j} = 0$ and choosing either

$$E^C_k = \text{global constant with support } \mathcal{D}, \tag{60}$$
$$H^C_j = 0, \tag{61}$$

$$J_r^C = 0, \quad (62)$$
$$K_p^C = 0, \quad (63)$$

or

$$E_k^C = 0, \quad (64)$$
$$H_j^C = \text{global constant with support } \mathcal{D}, \quad (65)$$
$$J_r^C = 0, \quad (66)$$
$$K_p^C = 0. \quad (67)$$

The value of the constant drops out from the final equations and the latter are equivalent to replacing the field equations with their integrated counterparts over the elementary subdomains of the domain of computation, applying Gauss's integral theorem, and adding the relevant results.

6 Symmetrization of transient diffusive EM field equations

The basic field equations governing the transient diffusive EM field are not symmetric in E_r and H_p, as opposed to their counterparts for transient EM wave propagation in lossless media. Recently, a symmetrization procedure has been developed that shows the interrelation between the transient diffusive EM-field constituents and their suitably defined lossless-medium wavefield counterparts (de Hoop, 1995). The basic idea is to rewrite the time-domain Laplace-transform Eqs. (13) and (14) as

$$-\epsilon_{k,m,p}\partial_m[(s/\alpha)^{1/2}\hat{H}_p] + (\alpha s)^{1/2}[\alpha^{-1}\sigma_{k,r}]\,\hat{E}_r = -(s/\alpha)^{1/2}\hat{J}_k, \quad (68)$$
$$\epsilon_{j,n,r}\partial_n\hat{E}_r + (\alpha s)^{1/2}\mu_{j,p}[(s/\alpha)^{1/2}\hat{H}_p] = -\hat{K}_j, \quad (69)$$

where α is an arbitrary constant. Equations (70) and (71) resemble the time Laplace-transform EM field equations for wavefields in a lossless medium with permittivity $\alpha^{-1}\sigma_{k,r}$, permeability $\mu_{j,p}$, electric-field strength $\hat{E}_r$, magnetic-field strength $(s/\alpha)^{1/2}$ $\hat{H}_p$, volume-source density of electric current $(s/\alpha)^{1/2}\hat{J}_k$, and volume-source density of magnetic current $\hat{K}_j$, but with s replaced with $(\alpha s)^{1/2}$. The Schouten-Van der Pol theorem for the time Laplace transform [Schouten (1934), (1961); Van der Pol (1934), (1960); see also Van der Pol and Bremmer (1950)] provides the tool to establish the relevant interrelation, which for computational purposes can be used to construct, by a simple time-like integration routine, transient diffusive EM field values from their computed wavefield counterparts in a lossless medium once the latter have been determined with the aid of standard software for computing wavefields. Details are given by de Hoop (1996).

7 Conclusions

A structured approach, with reciprocity as the basic principle, has been developed to construct schemes for the computation of transient diffusive EM-fields. It is shown that the known algorithms concerning the finite-element, integral-equation, and domain-integration techniques all can be viewed as particular choices for the computational state with which the interaction of (the approximating expansion of) the actual field to be computed is set equal to zero. It is believed that the approach also can lead to additional types of algorithms.

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