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Transient diffusive electromagnetic fields in stratified media: Calculation of the two-dimensional E-polarized field

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Abstract. An analytic method is presented for calculating transient diffusive electromagnetic fields in stratified media. The method is inspired by the modified Cagniard method for calculating transient wave fields in media of this kind. Closed-form time domain expressions are constructed for the fields generated by a localized source. The expressions have the form of a superposition of diffusive generalized-ray constituents. The case of the two-dimensional E-polarized field generated by a line source of electric current is worked out in detail.

1. Introduction

Electromagnetic fields in conductive media are, as far as the electric properties of the medium in which they occur are concerned, dependent on the medium's conductive and dielectric properties. Depending on the scale on which the fields (and their sources) vary with time, either of the two constitutive aspects may be predominant. When the dielectric properties dominate, the electric displacement current yields a larger contribution to the field than the electric conduction current, and the field is wavelike in nature. On the other hand, when the conductive properties dominate, the electric conduction current yields a larger contribution than the electric displacement current, and the field is diffusive in nature. Now diffusion always leads to an instantaneous response to the action of the sources involved, which property conflicts with the universal property that electromagnetic fields can only propagate with a finite wave speed. Hence the diffusive approximation only applies to those cases where the bulk of the phenomena takes place after the (smaller) onset of the wave has passed. In several areas of application the approximation is perfectly legitimate. Examples are eddy current distribution in electrical machinery, transformers, and other electrical appliances [*Tegopoulos and Kriezis*, 1985]; transient electromagnetic methods for geophysical prospecting in conductive subsurface structures of the Earth [*Nabighian*, 1989]; and pulsed eddy-current quantitative nondestructive inspection techniques for detecting corrosion and holes in steel pipelines in, for example, the petrochemical industry [*Dunbar*, 1988].

Computationally, the configurations arising in the technical applications are of such a degree of complexity that the only way of analyzing their (diffusive) electromagnetic behavior is through the use of appropriate numerical algorithms. Any serious use of such algorithms requires as benchmarks certain canonical problems whose solutions can be obtained by analytical methods. The configuration consisting of parallel layers with different electric conduction as well as different magnetic properties provides such a canonical problem. In the present contribution the calculation of the transient diffusive electromagnetic field will be carried out with the aid of a method that is inspired by the modified Cagniard method for analyzing transient waves in lossless layered media [*De Hoop*, 1960, 1979, 1988a; *Achenbach*, 1973; *Langenberg*, 1974]. The result is in the form of a superposition of "diffusive generalized-ray constituents." For the simpler case of a two-media configuration the method has been reported by *De Hoop and Oristaglio* [1988]. Another feature of interest in relation to the subject of investigation is the general correspondence principle that relates transient wave fields in a configuration with dielectric media to transient diffusive fields in a conductive medium whose electrical conductivity follows the spatial profile of the permittivity [*De Hoop*, 1996a; see also *De Hoop*, 1996b].

The method will be elucidated for the two-dimensional E-polarized field generated by an electric-current line source. Several applications of the technique to three-dimensional configurations are discussed by *Combee* [1991].

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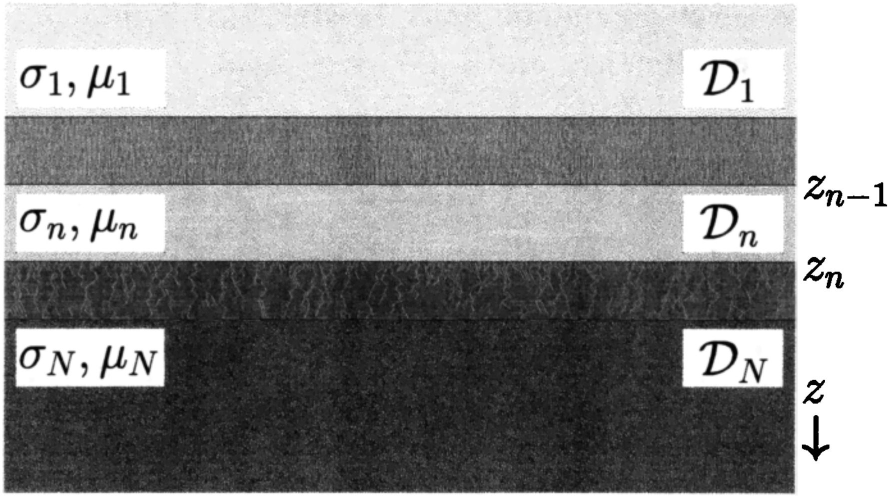

Figure 1. Layered-medium configuration.

2. Formulation of the Problem

The configuration in which the transient diffusive electromagnetic fields are analyzed consists of a finite number of parallel layers in between two half-spaces. Position in the configuration is specified by the coordinates $\{x, y, z\}$ with respect to an orthogonal, Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. In accordance with geophysical convention, the z axis is taken along the "vertical," with z increasing in the downward direction, while the planes parallel to the x, y plane are denoted as "horizontal." The position vector is $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$. The time coordinate is t. The vectorial spatial differentiation operator is $\boldsymbol{\nabla} = \mathbf{i}_x\partial_x + \mathbf{i}_y\partial_y + \mathbf{i}_z\partial_z$; differentiation with respect to time is denoted by ∂_t.

The upper half-space occupies the domain $\mathcal{D}_1 = \{\mathbf{r} \in \mathcal{R}^3, -\infty < z < z_1\}$, the $N - 2$ layers occupy the domains $\mathcal{D}_n = \{\mathbf{r} \in \mathcal{R}^3, z_{n-1} < z < z_n\}$, with $n = 2, \cdots, N - 1$, and the lower half-space occupies the domain $\mathcal{D}_N = \{\mathbf{r} \in \mathcal{R}^3, z_{N-1} < z < \infty\}$ (Figure 1). The electromagnetic constitutive properties of $\mathcal{D}_n$ are characterized by the electrical conductivity σ_n and the magnetic permeability μ_n.

The electromagnetic field quantities in $\mathcal{D}_n$ are the electric field strength $\mathbf{E}_n$ and the magnetic field strength $\mathbf{H}_n$. The electromagnetic source quantities in $\mathcal{D}_n$ are the volume density of electric current $\mathbf{J}_n$ and the volume density of magnetic current $\mathbf{K}_n$. The source and field quantities satisfy the diffusive electromagnetic field equations

$$\boldsymbol{\nabla} \times \mathbf{H}_n - \sigma_n\mathbf{E}_n = \mathbf{J}_n \qquad \mathbf{r} \in \mathcal{D}_n, \tag{1}$$

$$\boldsymbol{\nabla} \times \mathbf{E}_n + \mu_n\partial_t\mathbf{H}_n = -\mathbf{K}_n \qquad \mathbf{r} \in \mathcal{D}_n. \tag{2}$$

Across source-free interfaces between the different subdomains the tangential components of $\mathbf{E}$ and $\mathbf{H}$ should be continuous, and hence

$$\lim_{z \uparrow z_n} \{E_{x;n}, E_{y;n}, H_{x;n}, H_{y;n}\} = \lim_{z \downarrow z_n} \{E_{x;n+1}, E_{y;n+1}, H_{x;n+1}, H_{y;n+1}\} \qquad n = 1, \cdots, N-1. \tag{3}$$

Furthermore, the field should be causally related to the action of its sources.

The configuration is time invariant as well as spatially shift invariant in the x, y directions. These invariances are fundamental to the method of solution employed since they allow for the adequate use of the relevant time Laplace and spatial Fourier transformations.

3. Transform Domain Expressions for the Two-Dimensional *E*-Polarized Field Components Generated by an Electric-Current Line Source

In this section, the modified Cagniard method for diffusive electromagnetic fields will be illustrated for the case of a two-dimensional E-polarized field that is generated by a line source of electric current located at the interface $\{z = z_S\}$, where $z_S \in \{z_1, \cdots, z_{N-1}\}$. The volume source density of the line source of electric current is taken as

$$\mathbf{J} = I_S(t)\delta(x, z - z_S)\mathbf{i}_y, \tag{4}$$

where $I_S = I_S(t)$ is the "source signature" and $\delta(x, z - z_S)$ is the two-dimensional Dirac distribution operative at $\{x = 0, z = z_S\}$. The field is then independent of y. Its nonvanishing field components satisfy the field equations (see equations (1) and (2)) (note that we have taken the line source to be located at an interface)

$$\partial_z H_{x;n} - \partial_x H_{z;n} - \sigma_n E_{y;n} = 0 \qquad \mathbf{r} \in \mathcal{D}_n, \tag{5}$$

$$\partial_x E_{y;n} + \mu_n \partial_t H_{z;n} = 0 \qquad \mathbf{r} \in \mathcal{D}_n, \tag{6}$$

$$-\partial_z E_{y;n} + \mu_n \partial_t H_{x;n} = 0 \qquad \mathbf{r} \in \mathcal{D}_n, \tag{7}$$

$$n = 1, \cdots, N,$$

and the interface boundary and excitation conditions (see equation (3))

$$\lim_{z \downarrow z_n} E_{y;n+1} - \lim_{z \uparrow z_n} E_{y;n} = 0, \tag{8}$$

$$\lim_{z \downarrow z_n} H_{x;n+1} - \lim_{z \uparrow z_n} H_{x;n} = I_S(t)\delta(x)\delta_{n,S}, \tag{9}$$

$$n = 1, \cdots, N - 1,$$

where $\delta_{n,S}$ is the Kronecker symbol: $\delta_{n,S} = 1$ for $n = S$, and $\delta_{n,S} = 0$ for $n \neq S$.

First, a one-sided, causal, time Laplace transformation of the type

$$\hat{E}_{y;n}(x, z, s) = \int_{t=0}^{\infty} \exp(-st) E_{y;n}(x, z, t)\, dt \tag{10}$$

is carried out, in which the transform parameter s is taken to be real-valued and positive (which, according to Lerch's theorem [*Widder*, 1946] is sufficient for the uniqueness of the mathematical interrelation between a causal time function and its complex frequency (s-) domain counterpart) and $t = 0$ is taken as the instant at which the source is switched on in a configuration with zero initial field. Then, $\hat{\partial}_t = s$. Next, for the complex frequency domain field components, scaled horizontal spatial Fourier representations of the type

$$\hat{E}_{y;n}(x, z, s) = \frac{s^{1/2}}{2\pi} \int_{\alpha=-\infty}^{\infty} \exp(-\mathrm{j}s^{1/2}\alpha x) \tilde{E}_{y;n}(\mathrm{j}\alpha, z, s)\, d\alpha \tag{11}$$

are used, in which $s^{1/2}\alpha$ is the (real-valued) transform parameter and j is the imaginary unit. With this, $\tilde{\partial}_x = -\mathrm{j}s^{1/2}\alpha$. Using representations of this type in the source-free diffusive Maxwell equations, it is found that in each of the subdomains $\mathcal{D}_n$ of the configuration the field can be decomposed into a "downdiffusing" and an "updiffusing" constituent. For the electric field strength these constituents are written as

$$\tilde{E}_{y;n} = \tilde{E}^{+}_{y;n} + \tilde{E}^{-}_{y;n}, \tag{12}$$

with the downdiffusing constituent

$$\tilde{E}^{+}_{y;n} = \tilde{A}^{+}_{n} \exp[-s^{1/2}\gamma_n(z - z_{n-1})] \tag{13}$$

$$z_{n-1} < z < z_n,$$

and the updiffusing constituent

$$\tilde{E}^{-}_{y;n} = \tilde{A}^{-}_{n} \exp[-s^{1/2}\gamma_n(z_n - z)] \tag{14}$$

$$z_{n-1} < z < z_n.$$

In these expressions, $\tilde{A}^{+}_n$ and $\tilde{A}^{-}_n$ are the "amplitudes" of the constituents as they originate at the interfaces $\{z = z_{n-1}\}$ and $\{z = z_n\}$ of $\mathcal{D}_n$, respectively, and $\gamma_n = \gamma_n(\mathrm{j}\alpha)$ is the "vertical diffusive propagator," given by

$$\gamma_n = (D_n^{-1} + \alpha^2)^{1/2}, \tag{15}$$

where

$$D_n = (\sigma_n \mu_n)^{-1} \tag{16}$$

is the electrical diffusion coefficient of the medium in $\mathcal{D}_n$. The corresponding horizontal magnetic field components follow as

$$\tilde{H}_{x;n} = \tilde{H}^{+}_{x;n} + \tilde{H}^{-}_{x;n}, \tag{17}$$

with

$$\tilde{H}^{\pm}_{x;n} = \mp s^{-1/2}\tilde{Y}_n \tilde{E}^{\pm}_{y;n} \qquad z_{n-1} < z < z_n, \tag{18}$$

in which

$$\tilde{Y}_n = \tilde{Y}_n(\mathrm{j}\alpha) = \gamma_n(\mathrm{j}\alpha)/\mu_n. \tag{19}$$

The vertical components follow as

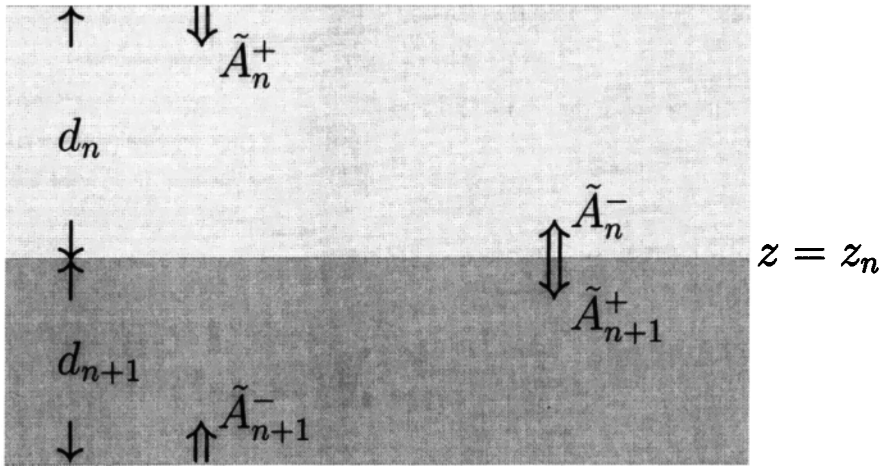

Figure 2. Boundary conditions at source-free interface $\{z = z_n \neq z_S\}$.

$$\tilde{H}_{z;n} = \tilde{H}_{z;n}^+ + \tilde{H}_{z;n}^-, \tag{20}$$

with

$$\tilde{H}_{z;n}^{\pm} = s^{-1/2}(\mathrm{j}\alpha/\mu_n)\tilde{E}_{y;n}^{\pm} \qquad z_{n-1} < z < z_n. \tag{21}$$

Causality requires that the field be bounded as $z \to -\infty$ and $z \to \infty$, with the consequence that $\tilde{A}_1^+ = 0$ and $\tilde{A}_N^- = 0$.

Substitution of (12)–(14) and (17) and (18) in the interface boundary and excitation conditions (8)–(9) leads to the system of equations (Figures 2 and 3)

$$\tilde{A}_{n+1}^+ + \tilde{A}_{n+1}^- \exp(-s^{1/2}\gamma_{n+1}d_{n+1}) = \tilde{A}_n^+ \exp(-s^{1/2}\gamma_n d_n) + \tilde{A}_n^-, \tag{22}$$

$$-\tilde{Y}_{n+1}\tilde{A}_{n+1}^+ + \tilde{Y}_{n+1}\tilde{A}_{n+1}^- \exp(-s^{1/2}\gamma_{n+1}d_{n+1}) = s^{1/2}\hat{I}_S\delta_{n,S} - \tilde{Y}_n\tilde{A}_n^+ \exp(-s^{1/2}\gamma_n d_n) + \tilde{Y}_n\tilde{A}_n^-, \tag{23}$$

$$n = 1, \cdots, N-1,$$

in which

$$d_n = z_{n+1} - z_n \qquad n = 2, \cdots, N-1 \tag{24}$$

is the thickness of the layer occupying the domain $\mathfrak{D}_n$.

Solving (22) and (23) for $\tilde{A}_n^-$ and $\tilde{A}_{n+1}^+$ (i.e., for the amplitudes of the constituents diffusing away from the interface $\{z = z_n\}$), we obtain

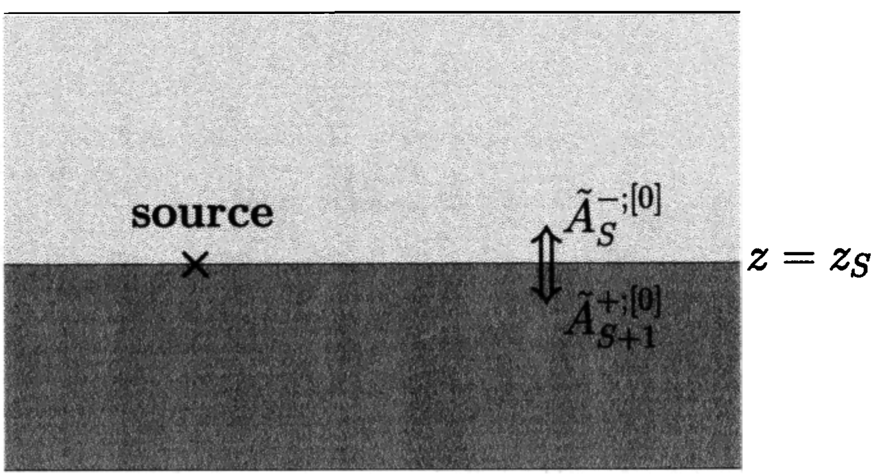

Figure 3. Excitation conditions at interface $\{z = z_S\}$ containing a source (starting value of iterative solution).

$$\bar{A}_n^- = -\frac{s^{1/2}\hat{I}_S}{\bar{Y}_n + \bar{Y}_{n+1}}\,\delta_{n,S} + \bar{S}_n^{-,+}\bar{A}_n^+ \exp(-s^{1/2}\gamma_n d_n) + \bar{S}_n^{-,-}\bar{A}_{n+1}^- \exp(-s^{1/2}\gamma_{n+1} d_{n+1}), \tag{25}$$

$$\bar{A}_{n+1}^+ = -\frac{s^{1/2}\hat{I}_S}{\bar{Y}_n + \bar{Y}_{n+1}}\,\delta_{n,S} + \bar{S}_n^{+,+}\bar{A}_n^+ \exp(-s^{1/2}\gamma_n d_n) + \bar{S}_n^{+,-}\bar{A}_{n+1}^- \exp(-s^{1/2}\gamma_{n+1} d_{n+1}) \tag{26}$$

$$n = 1, \cdots, N-1,$$

in which

$$\bar{S}_n^{-,+} = \frac{\bar{Y}_n - \bar{Y}_{n+1}}{\bar{Y}_n + \bar{Y}_{n+1}}, \qquad \bar{S}_n^{-,-} = \frac{2\bar{Y}_{n+1}}{\bar{Y}_n + \bar{Y}_{n+1}},$$
$$\bar{S}_n^{+,+} = \frac{2\bar{Y}_n}{\bar{Y}_n + \bar{Y}_{n+1}}, \qquad \bar{S}_n^{+,-} = \frac{\bar{Y}_{n+1} - \bar{Y}_n}{\bar{Y}_n + \bar{Y}_{n+1}}, \tag{27}$$

are the scattering coefficients (reflection and transmission coefficients) of the interface $\{z = z_n\}$ in between $\mathcal{D}_n$ and $\mathcal{D}_{n+1}$.

To generate the diffusive equivalents of the generalized ray constituents, (25) and (26) are employed iteratively according to the scheme

$$\bar{A}_n^{-;[i]} = \bar{S}_n^{-,+}\bar{A}_n^{+;[i-1]} \exp(-s^{1/2}\gamma_n d_n) + \bar{S}_n^{-,-}\bar{A}_{n+1}^{-;[i-1]} \exp(-s^{1/2}\gamma_{n+1} d_{n+1}), \tag{28}$$

$$\bar{A}_{n+1}^{+;[i]} = \bar{S}_n^{+,+}\bar{A}_n^{+;[i-1]} \exp(-s^{1/2}\gamma_n d_n) + \bar{S}_n^{+,-}\bar{A}_{n+1}^{-;[i-1]} \exp(-s^{1/2}\gamma_{n+1} d_{n+1}), \tag{29}$$

$$i = 1, 2, 3, \cdots; n = 1, \cdots, N-1,$$

starting from

$$\bar{A}_n^{-;[0]} = -\frac{s^{1/2}\hat{I}_S}{\bar{Y}_n + \bar{Y}_{n+1}}\,\delta_{n,S}, \tag{30}$$

$$\bar{A}_{n+1}^{+;[0]} = -\frac{s^{1/2}\hat{I}_S}{\bar{Y}_n + \bar{Y}_{n+1}}\,\delta_{n,S}, \tag{31}$$

and keeping $\bar{A}_1^{+;[i]} = 0$ and $\bar{A}_N^{-;[i]} = 0$ for all $i = 0, 1, 2, \cdots$. This procedure can be proved to be convergent. (Note in this respect that s, γ_n, and d_n are all real-valued and positive.)

Operating in this manner, an expression for $\bar{E}_{y;n}$ is obtained in the form of a superposition of "diffusive generalized-ray constituents" (for generalized-ray constituents in wave propagation, see *Wiggins and Helmberger* [1974] and *De Hoop* [1988b]) of the type

$$\bar{E}_{y;n}^{\pm} = \sum_{i=0}^{\infty} \bar{E}_{y;n}^{\pm;[i]}, \tag{32}$$

with

$$\bar{E}_{y;n}^{\pm;[i]} = -\frac{s^{1/2}\hat{I}_S}{\bar{Y}_S + \bar{Y}_{S+1}}\,\bar{\Pi}_n^{\pm;[i]}(\mathrm{j}\alpha) \exp\left(-s^{1/2} \sum_{\nu}^{[i]\pm} \gamma_\nu h_\nu\right), \tag{33}$$

in which the amplitude factor $\bar{\Pi}_n^{\pm;[i]}$ arises out of the application of the iterative procedure of (28)–(31) and (13) and (14) as the product of i scattering (reflection or transmission) coefficients, while the argument of the diffusive propagation factor exp $(-s^{1/2}\ \Sigma_\nu^{\pm;[i]}\ \gamma_\nu h_\nu)$ arises out of this iterative procedure as the summation of i contributions of the type $\gamma_\nu h_\nu$ over the layers that contribute to the ith iterate. Note in this respect that the starting terms of the iterative procedure, given by (30) and (31), contain neither a scattering coefficient nor a propagation factor, while each iteration in (28) and (29) leads to the multiplication of the amplitude factor by one of the interface scattering coefficients as well as to the addition of one term to the summation in the propagation factor.

The corresponding expressions for the magnetic field components are, in view of (18) and (21), of the type

$$\bar{H}_{x,z;n}^{\pm} = \sum_{i=0}^{\infty} \bar{H}_{x,z;n}^{\pm;[i]}, \tag{34}$$

with

$$\bar{H}_{x;n}^{\pm;[i]} = \pm\hat{I}_S \frac{\bar{Y}_n}{\bar{Y}_S + \bar{Y}_{S+1}}\,\bar{\Pi}_n^{\pm;[i]}(\mathrm{j}\alpha) \cdot \exp\left(-s^{1/2} \sum_{\nu}^{[i]\pm} \gamma_\nu h_\nu\right), \tag{35}$$

$$\bar{H}_{z;n}^{\pm;[i]} = -\hat{I}_S \frac{\mathrm{j}\alpha/\mu_n}{\bar{Y}_S + \bar{Y}_{S+1}}\,\bar{\Pi}_n^{\pm;[i]}(\mathrm{j}\alpha) \cdot \exp\left(-s^{1/2} \sum_{\nu}^{[i]\pm} \gamma_\nu h_\nu\right). \tag{36}$$

In the next section, the transform domain expressions for the field components will be used to construct their space-time equivalents.

4. Time Domain Expressions for the Two-Dimensional *E*-Polarized Field Components

The transformation back to the time domain will be carried out with the aid of the modified Cagniard method as developed by the author [*De Hoop*, 1960; *De Hoop and Oristaglio*, 1988]; this method will be applied to each diffusive generalized-ray constituent separately. To this end, the complex frequency domain representation of an electric field diffusive generalized-ray constituent is, from (11) and (33), omitting the subscript n and the superscript $\pm;[i]$, and introducing in the spatial Fourier representation $p = \mathrm{j}\alpha$ as the variable of integration, written as

$$\hat{w}_E(x, \{h_\nu\}, s) = \frac{s^{1/2}\hat{\imath}_S(s)}{2\pi \mathrm{j}} \int_{p=-\mathrm{j}\infty}^{\mathrm{j}\infty} \tilde{a}_E(p) \cdot \exp\left[-s^{1/2}\left(px + \sum_\nu \gamma_\nu(p)h_\nu\right)\right] dp, \tag{37}$$

and a magnetic field diffusive generalized-ray constituent as

$$\hat{w}_H(x, \{h_\nu\}, s) = \frac{\hat{\imath}_S(s)}{2\pi \mathrm{j}} \int_{p=-\mathrm{j}\infty}^{\mathrm{j}\infty} \tilde{a}_H(p) \cdot \exp\left[-s^{1/2}\left(px + \sum_\nu \gamma_\nu(p)h_\nu\right)\right] dp, \tag{38}$$

in which (see equation (15))

$$\gamma_n(p) = (D_n^{-1} - p^2)^{1/2}. \tag{39}$$

In the further application of our method we need the asymptotic relations

$$\tilde{a}_E(p) = O(p^{-1}) \qquad |p| \to \infty \tag{40}$$

$$\tilde{a}_H(p) = O(1) \qquad |p| \to \infty, \tag{41}$$

which follow from (33), (35), and (36). Starting from (37) and (38), the integrands are continued analytically into the complex p plane, away from the imaginary p axis, and the integration is carried out along a path (the modified Cagniard path) on which

$$px + \sum_\nu \gamma_\nu(p)h_\nu = \tau, \tag{42}$$

with τ real-valued and positive. In the continuation, we keep $\mathrm{Re}(\gamma_n) > 0$, everywhere in the complex p plane, which implies that branch cuts are introduced along $\{p \in \mathscr{C};\ \mathrm{Im}(p) = 0, D_n^{-1/2} < |\mathrm{Re}(p)| < \infty\}$. Candidates for the modified Cagniard path are the part of the real p axis in between the branch points $p = -\min_\nu \{D_\nu^{-1/2}\}$ and $p = \min_\nu \{D_\nu^{-1/2}\}$ that are closest to $p = 0$ and the path $\{p \in \mathscr{C}; p = p^B\} \cup \{p \in \mathscr{C}; p = p^{B*}\}$, where $p = p^B(x, \{h_\nu\}, \tau)$ is the complex part of the solution to (42) that is located in the upper part of the complex p plane and that goes to infinity as $\tau \to \infty$; the asterisk denotes complex conjugate. On the latter part, τ varies from its minimum value T_B to ∞, where $\tau = T_B$ is to be solved from the relations

$$x - p\sum_\nu \frac{h_\nu}{\gamma_\nu(p)} = 0, \quad px + \sum_\nu \gamma_\nu(p)h_\nu = T_B, \tag{43}$$

the first one of which follows from (42) upon differentiation with respect to p and putting $\partial_p\tau = 0$. Furthermore, the modified Cagniard path must originate from the original path of integration $\{p \in \mathscr{C}; \mathrm{Re}(p) = 0\}$ through a continuous deformation without passing singularities of the integrand in order that Cauchy's theorem can be applied. To this end, circular arcs at infinity are to join the two paths. The behavior of $\tilde{a}_E(p)$ (see equation (40)) and $\tilde{a}_H(p)$ (see equation (41)) as $|p| \to \infty$ ensures that in view of Jordan's lemma the contributions from the joining circular arcs always vanish in the process of path deformation in (37), while in (38) this contribution only vanishes as long as $\Sigma_\nu\ \gamma_\nu(p)h_\nu \neq 0$. In the case $\Sigma_\nu\ \gamma_\nu(p)h_\nu = 0$, the transformation of (38) back to the time domain requires special treatment. For the moment, we continue with the case where Jordan's lemma applies. Then, the integration along the imaginary p axis can be replaced by an integration along the path $\{p \in \mathscr{C}; p = p^B\} \cup \{p \in \mathscr{C}; p = p^{B*}\}$ as long as $\tilde{a}_E(p)$ and $\tilde{a}_H(p)$ do not contain any γ_n other than the ones that occur in the exponential part of the integrand. Now $\tilde{a}_E(p)$ and/or $\tilde{a}_H(p)$ do contain such a γ_n, γ_H, say, in case they are composed of an excitation coefficient or a reflection coefficient pertaining to a medium into which the diffusive generalized-ray constituent under consideration is not transmitted. When this happens, and if $D_H^{-1/2} < \min_\nu \{D_\nu^{-1/2}\}$, the complex part of the modified Cagniard path must be supplemented by a loop around the branch cut associated with γ_H. This loop, as well as the complex part of the modified Cagniard path, is parametrized as $\{p \in \mathscr{C}; p = p^H\} \cup \{p \in \mathscr{C}; p = p^{H*}\}$, where $p = p^H(x, \{\gamma_\nu\}, \tau)$ is the solution with an imaginary part +j0 that results from solving (42) in the interval $T_H < \tau < T_B$, where T_H follows from

$$x/D_H^{1/2} + \sum_\nu (D_\nu^{-1} - D_H^{-1})^{1/2} h_\nu = T_H. \quad (44)$$

Taking the contributions from $\{p \in \mathscr{C}; p = p^H\}$ and $\{p \in \mathscr{C}; p = p^{H*}\}$ together, as well as the contributions from $\{p \in \mathscr{C}; p = p^B\}$ and $\{p \in \mathscr{C}; p = p^{B*}\}$, applying Schwarz's reflection principle of complex function theory, and introducing τ as the variable of integration, we thus arrive at

$$\hat{w}_{E,H} = \hat{w}_{E,H}^B + \hat{w}_{E,H}^H, \quad (45)$$

in which

$$\hat{w}_{E,H}^B = \hat{I}_S(s)\hat{g}_{E,H}^B(x, \{h_\nu\}, s) \quad (46)$$

and

$$\hat{w}_{E,H}^H = \hat{I}_S(s)\hat{g}_{E,H}^H(x, \{h_\nu\}, s), \quad (47)$$

with

$$\hat{g}_E^B = \frac{s^{1/2}}{\pi}\int_{\tau=T_B}^{\infty} \exp(-s^{1/2}\tau)\,\mathrm{Im}\left[\tilde{a}_E(p^B)\frac{\partial p^B}{\partial \tau}\right] d\tau, \quad (48)$$

$$\hat{g}_E^H = \frac{s^{1/2}}{\pi}\int_{\tau=T_H}^{T_B} \exp(-s^{1/2}\tau)\,\mathrm{Im}\left[\tilde{a}_E(p^H)\frac{\partial p^H}{\partial \tau}\right] d\tau, \quad (49)$$

and

$$\hat{g}_H^B = \frac{1}{\pi}\int_{\tau=T_B}^{\infty} \exp(-s^{1/2}\tau)\,\mathrm{Im}\left[\tilde{a}_H(p^B)\frac{\partial p^B}{\partial \tau}\right] d\tau, \quad (50)$$

$$\hat{g}_H^H = \frac{1}{\pi}\int_{\tau=T_H}^{T_B} \exp(-s^{1/2})\,\mathrm{Im}\left[\tilde{a}_H(p^H)\frac{\partial p^H}{\partial \tau}\right] d\tau. \quad (51)$$

In view of Lerch's theorem on the uniqueness of the one-sided Laplace transform with respect to time of a causal quantity [*Widder*, 1946], the time domain equivalents of (45)–(47) follow, upon using the convolution theorem, as

$$w_{E,H} = w_{E,H}^B + w_{E,H}^H, \quad (52)$$

in which

$$w_{E,H}^B = I_S(t) \overset{(t)}{*} g_{E,H}^B(x, \{h_\nu\}, t) \quad (53)$$

$$w_{E,H}^H = I_S(t) \overset{(t)}{*} g_{E,H}^H(x, \{h_\nu\}, t) \quad (54)$$

where $\overset{(t)}{*}$ denotes time convolution. The time domain equivalents of (48)–(51) are obtained as

$$g_E^B(x, \{h_\nu\}, t) = \frac{1}{\pi}\int_{\tau=T_B}^{\infty} k^{EJ}(t, \tau)\,\mathrm{Im}\left[\tilde{a}_E(p^B)\frac{\partial p^B}{\partial \tau}\right] d\tau, \quad (55)$$

$$g_E^H(x, \{h_\nu\}, t) = \frac{1}{\pi}\int_{\tau=T_H}^{T_B} k^{EJ}(t, \tau)\,\mathrm{Im}\left[\tilde{a}_E(p^H)\frac{\partial p^H}{\partial \tau}\right] d\tau, \quad (56)$$

and

$$g_H^B(x, \{h_\nu\}, t) = \frac{1}{\pi}\int_{\tau=T_B}^{\infty} k^{HJ}(t, \tau)\,\mathrm{Im}\left[\tilde{a}_H(p^B)\frac{\partial p^B}{\partial \tau}\right] d\tau, \quad (57)$$

$$g_H^H(x, \{h_\nu\}, t) = \frac{1}{\pi}\int_{\tau=T_H}^{T_B} k^{HJ}(t, \tau)\,\mathrm{Im}\left[\tilde{a}_H(p^H)\frac{\partial p^H}{\partial \tau}\right] d\tau, \quad (58)$$

in which the electric-field/electric-current and magnetic-field/electric-current diffusive electromagnetic kernel functions are given by [*Abramowitz and Stegun*, 1965] (Figures 4 and 5)

$$k^{EJ}(t, \tau) = \frac{1}{2\pi^{1/2}}\frac{1}{t^{3/2}}\left(\frac{\tau^2}{2t} - 1\right)\exp\left(-\frac{\tau^2}{4t}\right)H(t), \quad (59)$$

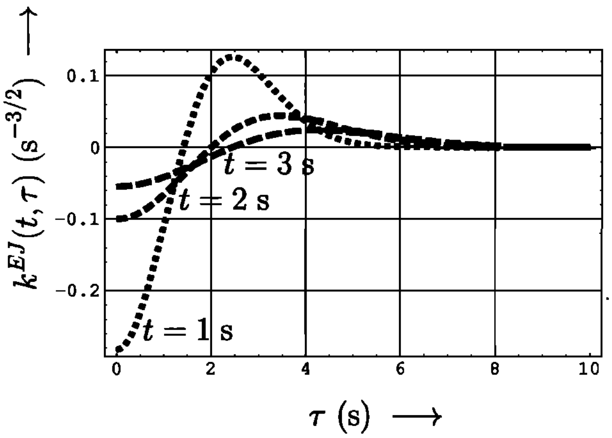

Figure 4. Diffusive electromagnetic (EM) kernel function associated with electric field due to electric-current source.

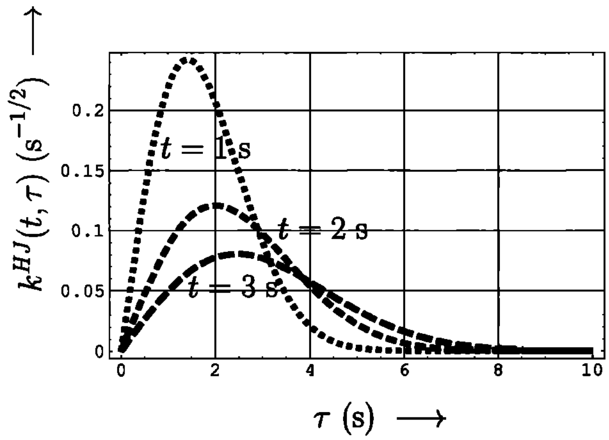

Figure 5. Diffusive EM kernel function associated with magnetic field due to electric-current source.

$$k^{HJ}(t, \tau) = \frac{1}{2\pi^{1/2}} \frac{\tau}{t^{3/2}} \exp\left(-\frac{\tau^2}{4t}\right) H(t), \qquad (60)$$

where $H(t)$ denotes the Heaviside unit step function: $H(t) = \{0, 1/2, 1\}$ for $\{t < 0, t = 0, t > 0\}$.

With this, the major ingredients for constructing the total space-time field expressions are at our disposal. The only exception left is the case where $\Sigma_\nu \gamma_\nu(p) h_\nu = 0$ in the expression for the magnetic field components. Evidently, this condition can only arise if $h_\nu = 0$ for all ν, i.e., at the interface where the exciting line source is located and for the zero-order iterate only. For this case, (17)–(21) and (30) and (31) lead to

$$\tilde{H}^{-;[0]}_{x;S} \overset{!}{=} -\hat{I}_S \frac{\tilde{Y}_S}{\tilde{Y}_S + \tilde{Y}_{S+1}}, \qquad (61)$$

$$\tilde{H}^{+;[0]}_{x;S+1} = \hat{I}_S \frac{\tilde{Y}_{S+1}}{\tilde{Y}_S + \tilde{Y}_{S+1}}, \qquad (62)$$

and

$$\tilde{H}^{-;[0]}_{z;S} = -\hat{I}_S \frac{j\alpha/\mu_S}{\tilde{Y}_S + \tilde{Y}_{S+1}}, \qquad (63)$$

$$\tilde{H}^{+;[0]}_{z;S+1} = -\hat{I}_S \frac{j\alpha/\mu_{S+1}}{\tilde{Y}_S + \tilde{Y}_{S+1}}, \qquad (64)$$

and the modified Cagniard path reduces to a loop around the branch cuts associated with γ_S and γ_{S+1}. By using the asymptotic relation $\gamma_n(j\alpha) = \pm\alpha + O(\alpha^{-1})$ as $\alpha \to \pm\infty$, we extract in the right-hand sides of (61)–(64) the asymptotic behavior as $\alpha \to \pm\infty$ and rewrite these equations as

$$\tilde{H}^{-;[0]}_{x;S} = -\hat{I}_S\left\{\frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} + \left[\frac{\tilde{Y}_S}{\tilde{Y}_S + \tilde{Y}_{S+1}} - \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}}\right]\right\}, \qquad (65)$$

$$\tilde{H}^{+;[0]}_{x;S+1} = \hat{I}_S\left\{\frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} + \left[\frac{\tilde{Y}_{S+1}}{\tilde{Y}_S + \tilde{Y}_{S+1}} - \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}}\right]\right\}, \qquad (66)$$

and

$$\tilde{H}^{-;[0]}_{z;S} = -\hat{I}_S\left\{\frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha) + \left[\frac{j\alpha/\mu_S}{\tilde{Y}_S + \tilde{Y}_{S+1}} - \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha)\right]\right\}, \qquad (67)$$

$$\tilde{H}^{-;[0]}_{z;S+1} = -\hat{I}_S\left\{\frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha) + \left[\frac{j\alpha/\mu_{S+1}}{\tilde{Y}_S + \tilde{Y}_{S+1}} - \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha)\right]\right\}. \qquad (68)$$

The terms in brackets satisfy, uniformly in $\arg(p)$, the conditions for the application of Jordan's lemma (so that for them the integral along the original path of integration can be replaced by one along the modified Cagniard path) and represent the contributions to the magnetic field that are bounded at the position of the line source. Transformation back to the space-time domain of the extracted first terms yields

$$-\hat{I}_S \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \to -I_S(t) \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \delta(x), \qquad (69)$$

$$\hat{I}_S \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \to I_S(t) \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \delta(x), \qquad (70)$$

and

$$-\hat{I}_S \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha) \to -I_S(t) \frac{\mu_S^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \frac{1}{x}, \qquad (71)$$

$$-\hat{I}_S \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \mathrm{j}\,\mathrm{sign}(\alpha) \to -I_S(t) \frac{\mu_{S+1}^{-1}}{\mu_S^{-1} + \mu_{S+1}^{-1}} \frac{1}{x}. \qquad (72)$$

These terms represent the contributions to the magnetic field that are unbounded at the position of the line source.

5. The Modified Cagniard Path, Its Properties, and Its Determination

The central issue in the application of the modified Cagniard method is the construction of the (candidates for the) modified Cagniard path defined by (42). In case the summation in the left-hand side of this equation consists of a single term only, an analytic expression for p as a function of τ is readily obtained. When the relevant summation contains several terms, however, numerical techniques for constructing p as a function of τ have to be employed. (For the case where the summation consists of two terms, Cardano's formula for the roots of an algebraic equation of degree four does provide a possible analytic representation for the modified Cagniard path, but in practice this formula, which provides the four roots to the equation that results upon squaring (42) twice, turns out to be difficult to handle as far as extracting the single solution satisfying (42) is concerned.) Subject to certain precautions, the numerical techniques required are straightforward. The main aspects are briefly discussed below.

As (28)–(36) show, E_y and H_x are even functions of x, while H_z is an odd function of x. In view of this property, we can restrict out computations to the domain $\{0 \le x < \infty, -\infty < z < \infty\}$.

5.1. Case of a Single Vertical Diffusive Propagator Term

For the case of a single vertical diffusive propagator term, (42) reduces to

$$px + \gamma_1(p)h_1 = \tau, \tag{73}$$

with τ real-valued and positive. From this equation, the head-wave equivalent of the modified Cagniard path is found as

$$p^H = \frac{x}{x^2 + h_1^2}\tau - \frac{h_1}{x^2 + h_1^2}(T_B^2 - \tau^2)^{1/2} \tag{74}$$

$$T_H < \tau < T_B,$$

where (see equation (44))

$$T_H = x/D_H^{1/2} + (D_1^{-1} - D_H^{-1})^{1/2}h_1 \tag{75}$$

$$T_B = (x^2 + h_1^2)^{1/2}/D_1^{1/2}, \tag{76}$$

while the body-wave equivalent of the path is found as

$$p^B = \frac{x}{x^2 + h_1^2}\tau + \mathrm{j}\frac{h_1}{x^2 + h_1^2}(\tau^2 - T_B^2)^{1/2} \tag{77}$$

$$T_B < \tau < \infty.$$

5.2. Case of Multiple Vertical Diffusive Propagator Terms

For the case of multiple vertical diffusive propagator terms the first step to be carried out is to determine the value T_B of τ for which the path corresponding to body-wave equivalent intersects the real p axis. On this path, τ reaches a minimum, while on the path corresponding to the head-wave equivalent τ reaches a maximum. The relevant value $p_{B;0}$ of p is found by applying a root-finding procedure to the first equation in (43). Furthermore, the relation

$$\partial_\tau^2 p = -\sum_\nu \frac{D_\nu^{-1}h_\nu}{\gamma_\nu^3(p)}, \tag{78}$$

which follows upon differentiating (42) twice with respect to τ, shows that $\partial_p^2\,\tau < 0$ for all values of p in the interval $\{p \in \mathscr{C}; 0 < \mathrm{Re}\,(p) < D_{\max}^{-1/2}, \mathrm{Im}\,(p) = 0\}$, where $D_{\max} = \max_n\{D_n\}$. This has the consequence that the first equation in (43) has, in the indicated interval, a single root only.

The circumstance that the first equation in (43) is unbounded at $p = D_{\max}^{-1/2}$ makes Newton's root-finding procedure unsuitable for the present application. The regula falsi method, however, does work, provided that two starting values can be produced, in one of which we have $\partial_p\tau < 0$, while at the other $\partial_p\tau > 0$. Since

$$x - p\sum_\nu \frac{h_\nu}{\gamma_\nu(p)} < x - \frac{p}{\gamma_{\max}(p)}\sum_\nu h_\nu \tag{79}$$

on the interval $\{p \in \mathscr{C}; 0 \le \mathrm{Re}(p) < D_{\max}^{-1}, \mathrm{Im}(p) = 0\}$, where

$$\gamma_{\max} = (D_{\max}^{-1} - p^2)^{1/2}, \tag{80}$$

the value

$$p_{\inf} = D_{\max}^{-1/2}\frac{x}{\left(x^2 + \sum_\nu h_\nu^2\right)^{1/2}} \tag{81}$$

makes $\partial_p\tau < 0$. Furthermore, by observing that

$$x - p\sum_\nu \frac{h_\nu}{\gamma_\nu(p)} < x - p\frac{h_{\max}}{\gamma_{\max}} \tag{82}$$

on the interval $\{p \in \mathscr{C}; 0 \le \mathrm{Re}(p) < D_{\max}^{-1}, \mathrm{Im}(p) = 0\}$, where $h_{\max}$ is the vertical diffusion

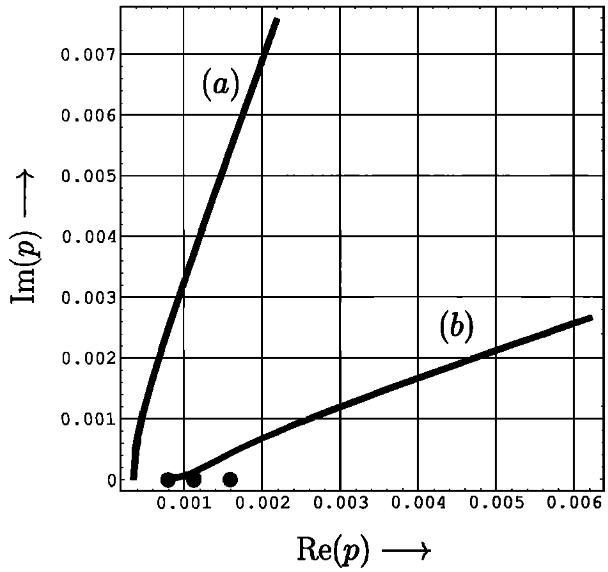

Figure 6. Modified Cagniard paths for a three-layer medium with $h_1 = 1$ m, $h_2 = 2$ m, $h_3 = 4$ m; $\sigma_1 = 0.5$ S/m, $\sigma_2 = 1.0$ S/m, $\sigma_3 = 2.0$ S/m; $\mu_1 = \mu_2 = \mu_3 = \mu_0 = 4\pi \times 10^{-7}$ H/m, (curve a) $x = 2$ m (diffusive propagation normal to the direction of stratification is predominant), (curve b) $x = 16$ m (diffusive propagation parallel to the direction of stratification is predominant). The modified Cagniard paths are continuous deformations of the original path of integration (the imaginary p axis).

propagator path in the layer with electrical diffusion coefficient $D_{\max}$, the value

$$p_{\sup} = D_{\max}^{-1/2} \frac{x}{(x^2 + h_{\max}^2)^{1/2}} \tag{83}$$

makes $\partial_p \tau > 0$. Taking $p_{\inf}$ and $p_{\sup}$ as the two starting values in the regula falsi method, the procedure is always convergent.

Once $p_{B;0}$ and T_B have been determined, Newton's method can, away from $\tau = T_B$, be employed with the (single) starting value arising from the asymptotic relationship

$$p \sim \frac{\tau}{x - \mathrm{j} \sum_\nu h_\nu} \qquad \tau \to \infty. \tag{84}$$

Figure 6 shows some typical modified Cagniard paths for the case where the vertical diffusive propagation takes place in three media with different electrical diffusion coefficients. In one case, the diffusive propagation normal to the direction of stratification is predominant; in the other case the diffusive propagation parallel to the direction of stratification predominates. In the latter case, the presence of the different branch points shows up more pronouncedly (through changes in the tangent to the modified Cagniard path) than in the former.

6. Conclusion

An analytic generalized-ray space-time formalism for the calculation of transient diffusive electromagnetic fields in stratified media has been developed. It is based on an adaptation of the modified Cagniard method that serves to construct the generalized-ray space-time formalism of wave propagation in stratified media. The case of a two-dimensional E-polarized field generated by an impulsive line source of electric current is worked out in detail. The computation of the relevant modified Cagniard paths in the scaled complex plane associated with the spatial Fourier transformation parallel to the medium's interfaces and the amplitude factors of the generalized-ray constituents requires only elementary numerical algorithms, although certain precautions have to be taken. For the application of the iterative procedure out of which the amplitude factors and the diffusive propagation factors of the successive diffusive generalized ray constituents result, a symbolic manipulation applications program can profitably be employed.

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