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Wave-field reciprocity and optimization in remote sensing

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A unified approach to local optimization techniques and wave-field reciprocity as applied to constructing solutions to remote sensing, imaging and inversion problems in acoustic, elastic and electromagnetic wave theory is presented. The starting point is a system of linear, first-order partial differential equations in space-time of the class to which the indicated wave phenomena gives rise. For this system, three types of remote sensing problems—the inverse-source, the inverse-scattering, and the inverse-transducted-wave-field problems—are formulated, and the construction of their solutions via local optimization techniques is discussed. Emphasis is placed on iterative algorithms that are based on a guaranteed decrease in the mismatch between modelled and observed data at each update of the medium. Subsequently, the wave-field reciprocity theorems of the time-convolution and the time-correlation types are derived and their occurrence in the optimization procedures is discussed. Also, attention is paid to approximate methods, in particular to the Rayleigh–Gans–Born approximation. Approximations of this sort provide the means to invoke the method of preconditioning in the process of inverting the operator equations. 'Exotic media' (for example, chiral media in electromagnetics) are included in the analysis.

Keywords: reciprocity; remote sensing; optimization; inversion; imaging

1. Introduction

This paper deals with the construction of 'solutions' to remote sensing wave problems of the inverse-source, inverse-scattering and inverse-transducted-field types, with the aid of local optimization techniques, and the role that wave-field reciprocity theorems play in these techniques. A unified scheme that applies to acoustic waves in fluids, elastic waves in solids, and electromagnetic waves in vacuum and matter is presented. The known techniques—as developed in, for example, acoustic medical tomography, geophysical prospecting (through seismic and electromagnetic methods), sonic and ultrasonic borehole sensing techniques, evaluation of fossil energy reservoirs, and quantitative non-destructive techniques for evaluating mechanical structures—can all be considered as particular cases of the general scheme.

The analysis is presented in the space-time domain and applies to waves in configurations containing linear, passive, time-invariant, causally responding, locally reacting media, leaving room for arbitrary inhomogeneity, anisotropy and relaxation

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effects. 'Exotic media' (for example, chiral media in electromagnetics) are included. The starting point is a system of linear, first-order, partial differential equations that couple the spatial derivatives of the relevant field strengths and the time derivatives of the corresponding field fluxes to the action of volume source distributions. The further relationship between the field fluxes and the field strengths is expressed via the constitutive relations of the medium through which the wave field passes.

For the required mathematical tools to be applicable, certain restrictions must be put on the geometry of the configuration in which the wave motion is present. These are specified in § 2. Section 3 contains the wave equations in their general form and the constitutive relations, together with the corresponding interface boundary conditions of the continuity type, while Appendix A lists the wave field and source arrays as well as the spatial differentiation arrays that apply to acoustic, elastodynamic and electromagnetic wave fields. In § 4, the remote sensing problem is formulated. A distinction is made between the (multiple-receiver) inverse-source problem, where an unknown volume source distribution in a known medium is to be reconstructed from the field strengths measured at a collection of receivers, the (multiple-source/multiple-receiver) inverse-scattering problem, where, in a certain target region (scattering domain), the contrast in constitutive parameters with respect to the ones of a given embedding is to be reconstructed from the field strengths measured at a collection of receivers, while a collection of sources irradiates the configuration, and the (multiple-source) inverse-transducted-wave-field problem, where, through transduction of the wave field in a certain target region (for example, from elastic to optical in a translucent part of an elastic solid), the constitutive parameters in the target region are to be reconstructed from the (indirectly observed) wave field in that region that is generated by a collection of exterior sources.

In the inverse-source problem, the measurement set-up generates a *data equation* that interrelates the collection of measured data to the unknown volume source distribution (the *model quantity*). In the inverse-scattering problem, the measurement set-up generates a *data equation* that interrelates the measured data with the unknown contrast volume source density in the scattering region (the *model quantity* in the data equation) and an *object equation* that interrelates the contrast volume source density and the contrast in constitutive parameters in the scattering region. In the object equation, the contrast volume source density and the contrast in constitutive parameters can alternately play the role of data and model quantity. In the inverse-transducted-wave-field problem, the measurement set-up generates a *data equation* that interrelates the total wave field observed in the target region and the constitutive parameters in that region.

The optimization approach to 'solving' the inverse-source and inverse-scattering problems relies on the use of appropriate *mismatch* or *error criteria*, which quantify the mismatch between the data that would be generated by an assumed model and the actual data. The mismatch or error criterion (also denoted as the 'cost function' or 'penalty function' in optimization theory) must meet some elementary requirements (such as positive definiteness), but is otherwise, to a large extent, free. The structure of the criteria that we employ leaves room for generating an iterative solution procedure to the relevant operator equations, which has the property that successive updates of the model lead to a guaranteed decrease in the mismatch of the data. The relevant procedure is directly related to the method of steepest descent. The general framework of the method is presented in Appendix B.

Section 5 contains the application of the solution procedure to the data equation in the inverse-source problem; § 6 contains the application to the data equation and the object equation in the inverse-scattering problem, while § 7 contains the application to the data equation in the inverse-transducted-field problem.

In carrying out the inversion procedures, a *wave-field continuation* away from the actual physical receivers (through which the wave field is measured) may prove to be profitable, not only from a computational point of view, but also from a conceptual one. The relevant expressions are provided by the *wave-field reciprocity theorems*. Through them, wave fields and their sources in one subdomain of the configuration space are quantitatively interrelated to the wave fields and their sources in subdomains elsewhere in the configuration. As far as the time coordinate is concerned, there exist two types of wave-field reciprocity theorems, namely one of the *time-convolution type* and one of the *time-correlation type*. They are discussed in §§ 8 and 10, respectively. At this point, it is remarked that for the reciprocity theorems to hold, the spatial differential operator array in the general wave equation must have certain symmetry properties. The governing wave equations in acoustics, elastodynamics and electromagnetics do satisfy these requirements (see Appendix A). Furthermore, for the reciprocity theorem of the time-convolution type to hold, the medium's constitutive relation must have the form of a time convolution (which is indeed the case for the class of media considered in § 3), while the chosen mismatch criteria in the optimization approach (which have the form of a distance function in a Riemannian space) are compatible with the application of the reciprocity theorem of the time-correlation type. The interrelations that the reciprocity theorems provide are particularly useful if, at certain locations in the configuration space, the wave field can either quantitatively be judged on physical grounds or can be approximated on the basis of a mathematical analysis. The wave-field extrapolation or 'modelling' formulae themselves are analysed in § 9, as well as Huygens's principle and Oseen's extinction theorem. The adjoint wave field, or 'reverse-time' extrapolation formulae, are introduced in § 11.

On almost all occasions, the achievements of measurement set-ups and reconstruction algorithms in inverse scattering are tested in the first-order, low-contrast, *Rayleigh–Gans–Born approximation*. Here, a low-contrast approximation is carried through in the object equation, thus relating the observed data directly, and in a linear fashion, to the contrast in constitutive parameters in the contrasting target region. In this approximation, the iterative procedure of Appendix B can be directly employed to reconstruct, from the observed data, the contrast in constitutive properties of the medium in the target region. In § 12, the relevant approximation is discussed.

In a number of cases, the iterative procedures for solving the relevant data and/or object equations turn out to require too many iterations or to lead to an unacceptably large limiting error, or both. Within the framework of the outlined procedure, two options remain for possible improvement of the situation. Firstly, the chosen error criterion might not be well enough adapted to the measurement set-up. (For example, receivers that carry little information about the model to be reconstructed are given relatively too large weighting coefficients.) Here, the remedy can be sought in choosing a different set of weighting coefficients in the mismatch criterion. Secondly, the data and/or object equations themselves can be the source of discomfort. In this case, the *method of preconditioning* can provide a way out. In this method, the data and/or object equations are, prior to their being subjected to an iterative solution procedure, replaced by ones that arise upon applying to them an appropriate homogeneous, linear *preconditioning operator*. Section 13 and Appendix C briefly discuss this aspect. In particular, useful preconditioning operators can be constructed through the use of the geometrical ray representations of the Green's functions in combination with the Rayleigh–Gans–Born approximation.

Throughout the analysis, appropriate operator and inner product notations are employed that are designed to facilitate the construction of the accompanying computer codes. Here, too, the uniformity in approach to acoustic, electromagnetic and elastodynamic inverse-source and inverse-scattering theory is evident. One field of application where this uniformity is expected to be of importance is in what is denoted, in geophysical exploration, as *fully integrated reservoir evaluation and monitoring*, where sonic, electromagnetic and seismic measurements are all combined to arrive at a (fossil energy reservoir) model that is the optimum that one can reconstruct from the observed data.

As regards the previous literature on the subject, the following material is mentioned. A 'minimum-norm' gradient technique with a guaranteed decreasing error in the data fit can be obtained from the work of Lions (1968). An overview of the original work in the mathematics of iterative solutions to optimization problems is to be found in Kantorovich & Akilov (1982). The concept of data 'fitting' with a view to resolution analysis has been introduced by Backus & Gilbert (1970). It has been further exploited and developed into optimization procedures in a large number of papers, including the ones by Bamberger *et al.* (1979), Tarantola & Valette (1982), Lailly (1983), Kolb *et al.* (1986), Kennett & Williamson (1988) and Nolet (1981). For the particular case of electromagnetic waves, we refer to the papers by Kleinman & van den Berg (1992, 1993) and van den Berg & Kleinman (1997) . For a general overview of 'weak' inverse-scattering theory, we refer to Parker (1977).

The observation of Tarantola (1984), that inversion can be expressed in terms of more conventional image-processing methods (see Claerbout 1971, 1992; Schultz & Sherwood 1980; Berkhout 1982*a*, *b*; Wapenaar & Berkhout 1989; Berryhill 1984), is a recurring motif in the literature that applies to almost all common approaches to the inverse problem. Conversely, first steps towards transforming imaging into (linearized) inversion, can be found in the papers by Keys & Weglein (1983), Stolt & Weglein (1985) and Bleistein (1987).

The potentialities of inversion techniques relying on the use of the geometrical ray representation of the Green's functions have been extensively explored, for the acoustic case, in the papers by Beylkin (1985) and Miller *et al.* (1987), and have led to a generalized Radon transform formulation of the remote sensing problem. For the formulation of generalized Radon transform inversion for the elastodynamic case, we refer to De Hoop *et al.* (1999), and, for its 'minimum-norm' analogue, we refer to De Hoop & Brandsberg-Dahl (2000).

Preconditioning methods have received attention in the thesis of Sevink (1996) and the paper by Jin *et al.* (1992).

The reciprocity theorems that we formulate are generalizations of the ones presented by Rayleigh (1873), for acoustic waves in fluids; Betti (1871–1873) and Rayleigh (1873), for elastic waves in solids; and Lorentz (1896), for electromagnetic waves. For a modern treatment and clarification of these theorems, see De Hoop (1988). Wave-field computations in the *direct* scattering problem based on the reciprocity theorems have been discussed by De Hoop & De Hoop (1996); local optimization in the *inverse* scattering problem has been put in the framework of reciprocity theorems by De Hoop (1996).

2. Description of the configuration

The configuration for which the remote sensing problem is formulated consists of a bounded domain $\mathcal{D} \subset \mathbb{R}^3$, with piecewise smooth closed boundary surface $\partial\mathcal{D}$, in which a linear, passive, time-invariant, causally responding, locally reacting and possibly inhomogeneous and anisotropic medium is present that is either instantaneously reacting or shows relaxation effects. The complement of closure $\mathcal{D} \cup \partial\mathcal{D}$ in $\mathbb{R}^3$ is denoted as $\mathcal{D}^\infty$. In $\mathcal{D}^\infty$, a linear, passive, locally reacting, homogeneous and isotropic medium is present that is either instantaneously reacting or shows, at most, relaxation effects of the linear-friction or linear-hysteresis type. For such a medium, the Green's function (point-source solution) to the pertaining system of partial differential equations can be constructed with the aid of analytical methods, and, in particular, its causality can be established. The domain $\mathcal{D}$ consists of a finite number N^{D} ($N^{\mathrm{D}} \geqslant 1$) of subdomains $\{\mathcal{D}_n;\ n = 1, \dots, N^{\mathrm{D}}\}$, with piecewise smooth closed boundary surfaces $\{\partial\mathcal{D}_n;\ n = 1, \dots, N^{\mathrm{D}}\}$, in which the constitutive parameters of the medium vary continuously with position. Across the interfaces between any two adjacent subdomains, the constitutive parameters may jump by finite amounts. These conditions put on the geometry of the configuration, and on the variation with position of the constitutive parameters, ensure the existence and the uniqueness of the causal wave field generated by volume sources with bounded spatial support, whose source strengths are bounded functions of time and vary piecewise continuously in space. Further, the conditions are sufficient for Gauss's divergence theorem to be applicable. The domain $\mathcal{D}^\infty$ is occasionally denoted as the *embedding*.

The position in the configuration is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a Cartesian reference frame with the origin $\mathcal{O}$ and the three, mutually perpendicular, base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, each of unit length. In the given order, the base vectors form a right-handed system. The corresponding position vector is $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$. The time coordinate is t. Partial differentiation with respect to x_m is denoted by ∂_m; partial differentiation with respect to t is denoted by ∂_t. The gradient is written as $\nabla = \boldsymbol{i}_1\partial_1 + \boldsymbol{i}_2\partial_2 + \boldsymbol{i}_3\partial_3$.

3. The field equations, constitutive relations and interface boundary conditions

The wave fields under consideration have in common that their physical magnitudes are described by two sets of field quantities. The first set is denoted as the set of *field strengths* and is loosely related to what, in classical thermodynamics, are denoted as *intensive quantities*; the second set is denoted as the set of *field fluxes* and is loosely related to what, in classical thermodynamics, are denoted as *extensive quantities*. The field equations couple the first-order spatial derivatives of the field strengths to the first-order time derivatives of the field fluxes and the acting volume sources that generate the field. The structure of the relevant equations is such that the resulting physical phenomenon is a wave motion in which the power flux density is associated with a certain combination of the field-strength quantities, and the flux density of momentum transfer is associated with a certain combination of the field fluxes. For *acoustic waves in fluids*, the 'field strengths' are the *acoustic pressure* and the *particle velocity*, the 'field fluxes' are the *mass flow density* and the *cubic dilatation*, while the volume source quantities are the *volume source density of force* and the *volume source density of fluid injection rate* (De Hoop 1995, §§ 2.3 and 2.4). For *elastic waves in solids*, the 'field strengths' are the *particle velocity* and the *dynamic stress*, the 'field fluxes' are the *mass flow density* and the *deformation*, while the volume source quantities are the *volume source density of force* and the *volume source density of deformation rate* (De Hoop 1995, §§ 10.3 and 10.4). For *electromagnetic waves*, the 'field strengths' are the *electric field strength* and the *magnetic field strength*, the 'field fluxes' are the *electric flux density* and the *magnetic flux density*, while the volume source quantities are the *volume source density of electric current* and the *volume source density of magnetic current* (De Hoop 1995, § 18.3). For our analysis, the two sets of field quantities and the source quantities are arranged as one-dimensional arrays that contain, in a particular order, the components of the relevant quantities with respect to the chosen background Cartesian reference frame. The ordering of the components induces a particular *signature array* (i.e. a two-dimensional array with numbers +1 and −1 on its main diagonal only) that also plays a role in the wave-field reciprocity theorem of the time-convolution type. The relevant arrays are given in Appendix A. Upper-case subscripts are used to indicate the elements of these arrays; for repeated subscripts, the summation convention applies. The subscript range is four for acoustic waves, 12 for elastic waves and six for electromagnetic waves.

Let $F_P = F_P(\boldsymbol{x}, t)$ denote the array of the field strengths, $\Phi_I = \Phi_I(\boldsymbol{x}, t)$ the array of the field fluxes and $Q_I = Q_I(\boldsymbol{x}, t)$ the array of volume source densities; then the system of first-order partial differential equations associated with the wave motion is

$$\mathcal{D}_{IP} F_P + \partial_t \Phi_I = Q_I. \tag{3.1}$$

Here, $\mathcal{D}_{IP} = \mathcal{D}_{IP}(\nabla)$ is the pertaining two-dimensional array of first-order spatial partial differentiation operators; its elements are listed in Appendix A. Equation (3.1) is denoted as the *field equation*.

The medium in which the wave motion is present is assumed to be linear, time-invariant, causally responding and locally reacting in its acoustic, elastodynamic or electromagnetic behaviour. Leaving room for arbitrary inhomogeneity, anisotropy and relaxation effects, its constitutive relation is written as

$$\Phi_I(\boldsymbol{x}, t) = \int_{t'=0}^{\infty} X_{IP}(\boldsymbol{x}, t') F_P(\boldsymbol{x}, t - t') \, \mathrm{d}t', \tag{3.2}$$

where the two-dimensional array $X_{IP}(\boldsymbol{x}, t)$ contains the medium's constitutive relaxation functions and the lower limit of integration takes into account the causal response of the medium. Using the symbol $\overset{(t)}{*}$ for time convolution evaluated at time t, equation (3.2) can also be written as

$$\Phi_I(\boldsymbol{x}, t) = X_{IP}(\boldsymbol{x}, t) \overset{(t)}{*} F_P(\boldsymbol{x}, t), \quad \text{with } X_{IP} = 0 \text{ for } t < 0. \tag{3.3}$$

For an instantaneously reacting medium,

$$X_{IP}(\boldsymbol{x}, t) = M_{IP}(\boldsymbol{x}) \delta(t), \tag{3.4}$$

and, hence,

$$\Phi_I(\boldsymbol{x},t) = M_{IP}(\boldsymbol{x})F_P(\boldsymbol{x},t), \tag{3.5}$$

where $M_{IP} = M_{IP}(\boldsymbol{x})$ is the two-dimensional array of the medium's constitutive coefficients.

No further restrictions than the ones already formulated are put on the array of constitutive parameters. This implies that 'exotic media' (for example, chiral media in electromagnetics) are included in the analysis. The block diagonal subarrays of the array of constitutive parameters are representative of the properties of conventional media; their block off-diagonal subarrays are representative of 'exotic effects'.

In each of the subdomains $\{\mathcal{D}_n;\ n = 1,\dots,N^{\mathrm{D}}\}$ and $\mathcal{D}^{\infty}$, where the constitutive parameters change continuously with position, the field strengths are continuously differentiable with respect to the spatial coordinates, and equation (3.1) holds. Across interfaces of (jump) discontinuity in constitutive parameters, the field quantities show finite jumps, but are subject to the interface boundary conditions

$$N_{IP}F_P = \text{continuous across interface}, \tag{3.6}$$

where N_{IP} is the *two-dimensional unit normal array* that arises upon replacing ∇ in D_{IP} with $\boldsymbol{n}$, where $\boldsymbol{n}$ is the local unit vector along the normal to the interface. In components, with $\boldsymbol{n} = n_1\boldsymbol{i}_1 + n_3\boldsymbol{i}_3 + n_3\boldsymbol{i}_3$, we have

$$N_{IP} = n_1 D_{IP}(\boldsymbol{i}_1) + n_2 D_{IP}(\boldsymbol{i}_2) + n_3 D_{IP}(\boldsymbol{i}_3). \tag{3.7}$$

4. The remote sensing problem

In the remote sensing problem, one distinguishes between the *inverse-source problem*, the *inverse-scattering* or *inverse-constituency problem*, and the *inverse-transducted-field problem*. The three types of problem differ in formulation as well as in handling. Their formulations will be given separately below; their handling will be discussed in subsequent sections.

(a) *The inverse-source problem*

In the inverse-source problem, a volume source with known or guessed bounded support $\mathcal{D}^{\mathrm{T}} \subset \mathbb{R}^3$, and unknown volume source density $Q_I^{\mathrm{T}} = Q_I^{\mathrm{T}}(\boldsymbol{x},t)$ emits—in a linear, time-invariant, causally responding, locally reacting medium with known constitutive properties, which we represent by its constitutive relaxation functions $X_{IP} = X_{IP}(\boldsymbol{x},t)$—a wave field that is recorded at a set of N^{R} ($N^{\mathrm{R}} \geqslant 1$) linear, time-invariant, causal volume receivers with known, bounded, disjoint supports $\{\mathcal{D}_n^{\mathrm{R}} \subset \mathbb{R}^3;\ n = 1,\dots,N^{\mathrm{R}}\}$, and known receiving characteristics $\{R_{P;m};\ m = 1,\dots,N^{\mathrm{R}}\}$. The set of *recorded data* is

$$d_m^{\mathrm{T}}(t) = \int_{\mathcal{D}_m^{\mathrm{R}}} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} F_P^{\mathrm{T}}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}), \quad \text{for } m = 1,\dots,N^{\mathrm{R}}, \tag{4.1}$$

where $F_P^{\mathrm{T}} = F_P^{\mathrm{T}}(\boldsymbol{x},t)$ is the field strength of the wave field emitted by the source. The right-hand side expresses that the receiver interacts, throughout its spatial support, with the field strength of the emitted wave field in a linear, time-invariant manner.

To relate the wave field recorded at the receivers to the sources that generate them, we introduce the two-dimensional array of Green's functions as the causal solution to the system of field equations

$$\mathsf{D}_{IP} G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) + \partial_t X_{IP} \overset{(t)}{*} G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) = \delta^+_{II'} \delta(\boldsymbol{x} - \boldsymbol{x}', t), \tag{4.2}$$

with point-source excitation at $\boldsymbol{x} = \boldsymbol{x}'$. The array $\delta^+_{II'}$ is the two-dimensional Kronecker array: $\delta^+_{II'} = 1$ for $I = I'$, $\delta^+_{II'} = 0$ for $I \neq I'$; $\delta(\boldsymbol{x} - \boldsymbol{x}', t)$ is the four-dimensional Dirac distribution operative at $\boldsymbol{x} = \boldsymbol{x}'$ and $t = 0$. Then, applying the superposition principle, we can write

$$F^{\mathrm{T}}_P(\boldsymbol{x}, t) = \int_{\mathcal{D}^{\mathrm{T}}} G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q^{\mathrm{T}}_{I'}(\boldsymbol{x}', t) \, \mathrm{d}V(\boldsymbol{x}'). \tag{4.3}$$

The aim of the thus-formulated *multiple-receiver inverse-source problem* is to reconstruct, as far and as accurately as possible, the quantity Q^{T}_I from the recorded dataset $\{d^{\mathrm{T}}_m(t);\ m = 1, \dots, N^{\mathrm{R}}\}$. A definition of what is meant by 'as accurately as possible', and a method of reconstruction, are discussed in § 5.

(b) *The inverse-scattering problem*

In the inverse-scattering or inverse-constituency problem, the aim is to reconstruct, in a certain, known or guessed, *target region* of bounded support $\mathcal{D}^{\mathrm{s}} \subset \mathbb{R}^3$, the constitutive parameters, by irradiating the target region with a finite number N^{T} ($N^{\mathrm{T}} \geqslant 1$) of disjoint sources with known supports $\{\mathcal{D}^{\mathrm{T}}_n;\ n = 1, \dots, N^{\mathrm{T}}\}$ and known volume source densities $\{Q^{\mathrm{T}}_{I;n} = Q^{\mathrm{T}}_{I;n}(\boldsymbol{x}, t);\ n = 1, \dots, N^{\mathrm{T}}\}$, and recording the set of generated wave fields at a finite set of N^{R} ($N^{\mathrm{R}} \geqslant 1$) disjoint receivers with known supports $\{\mathcal{D}^{\mathrm{R}}_m;\ m = 1, \dots, N^{\mathrm{R}}\}$ and known receiving characteristics $\{R_{P;m} = R_{P;m}(\boldsymbol{x}, t);\ m = 1, \dots, N^{\mathrm{R}}\}$. The set of *recorded data* is

$$d^{\mathrm{T}}_{m,n}(t) = \int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x}, t) \overset{(t)}{*} F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t) \, \mathrm{d}V(\boldsymbol{x}),$$
$$\text{for } m = 1, \dots, N^{\mathrm{R}}, \quad n = 1, \dots, N^{\mathrm{T}}, \tag{4.4}$$

where $F^{\mathrm{T}}_{P;n} = F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t)$ is the field strength of the wave field emitted by the source with source strength $Q^{\mathrm{T}}_{I;n}$.

Formulated in this manner, the reconstruction process would involve the constitutive parameters of the entire universe. To restrict the reconstruction process to the target region $\mathcal{D}^{\mathrm{s}}$, it is assumed that only in $\mathcal{D}^{\mathrm{s}}$ will the constitutive parameters show a *contrast* with the ones of the configuration in which the target region is *embedded*, which implies that the target region is considered as a *scattering region* in an embedding with known constitutive parameters. Let the constitutive parameters of the embedding (or *background*) be represented by its array of relaxation functions $X^{\mathrm{b}}_{IP} = X^{\mathrm{b}}_{IP}(\boldsymbol{x}, t)$ and let $G^{\mathrm{b}}_{PI'} = G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t)$ be the array of Green's functions of this background; then $G^{\mathrm{b}}_{PI'} = G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t)$ is the causal solution to the system of equations

$$\mathsf{D}_{IP} G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) + \partial_t X^{\mathrm{b}}_{IP} \overset{(t)}{*} G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) = \delta^+_{II'} \delta(\boldsymbol{x} - \boldsymbol{x}', t). \tag{4.5}$$

We now introduce the set of *incident wave fields* $\{F^{\mathrm{i}}_{P;n} = F^{\mathrm{i}}_{P;n}(\boldsymbol{x},t);\ n = 1,\dots,N^{\mathrm{T}}\}$ as the wave fields that the irradiating sources would emit in the known embedding, i.e.

$$F^{\mathrm{i}}_{P;n}(\boldsymbol{x},t) = \int_{\mathcal{D}^{\mathrm{T}}_n} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t) \overset{(t)}{*} Q^{\mathrm{T}}_{I';n}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}'), \quad \text{for } n = 1,\dots,N^{\mathrm{T}}. \tag{4.6}$$

The set of recorded data that would result from this wave field is

$$d^{\mathrm{i}}_{m,n}(t) = \int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} F^{\mathrm{i}}_{P;n}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}), \qquad \text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}. \tag{4.7}$$

Next, the differences between the actual wave field $\{F^{\mathrm{T}}_{P;n};\ n = 1,\dots,N^{\mathrm{T}}\}$ and the incident wave field $\{F^{\mathrm{i}}_{P;n};\ n = 1,\dots,N^{\mathrm{T}}\}$ are introduced as the *scattered wave field* $\{F^{\mathrm{s}}_{P;n} = F^{\mathrm{s}}_{P;n}(\boldsymbol{x},t);\ n = 1,\dots,N^{\mathrm{T}}\}$, where

$$F^{\mathrm{s}}_{P;n}(\boldsymbol{x},t) = F^{\mathrm{T}}_{P;n}(\boldsymbol{x},t) - F^{\mathrm{i}}_{P;n}(\boldsymbol{x},t), \quad \text{for } n = 1,\dots,N^{\mathrm{T}}. \tag{4.8}$$

In addition, the *scattered-field dataset* is introduced as

$$d^{\mathrm{s}}_{m,n}(t) = d^{\mathrm{T}}_{m,n}(t) - d^{\mathrm{i}}_{m,n}(t), \quad \text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}. \tag{4.9}$$

From equations (4.4) and (4.6)–(4.9) this can be expressed as

$$d^{\mathrm{s}}_{m,n}(t) = \int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} F^{\mathrm{s}}_{P;n}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}), \qquad \text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}. \tag{4.10}$$

The field equations pertaining to the total wave field $\{F^{\mathrm{T}}_{P;n};\ n = 1,\dots,N^{\mathrm{T}}\}$, namely

$$\mathsf{D}_{IP}F^{\mathrm{T}}_{P;n} + \partial_t X_{IP} \overset{(t)}{*} F^{\mathrm{T}}_{P;n} = Q^{\mathrm{T}}_{I;n}, \quad \text{for } n = 1,\dots,N^{\mathrm{T}}, \tag{4.11}$$

and the field equations pertaining to the incident wave field $\{F^{\mathrm{i}}_{P;n};\ n = 1,\dots,N^{\mathrm{T}}\}$, namely

$$\mathsf{D}_{IP}F^{\mathrm{i}}_{P;n} + \partial_t X^{\mathrm{b}}_{IP} \overset{(t)}{*} F^{\mathrm{i}}_{P;n} = Q^{\mathrm{T}}_{I;n}, \quad \text{for } n = 1,\dots,N^{\mathrm{T}}, \tag{4.12}$$

are now used to arrive at field equations for the scattered wave field $\{F^{\mathrm{s}}_{P;n};\ n = 1,\dots,N^{\mathrm{T}}\}$. Two alternative forms for these field equations exist. One form considers the scattered wave field present in a medium with constitutive parameters of the embedding, and is given by

$$\mathsf{D}_{IP}F^{\mathrm{s}}_{P;n} + \partial_t X^{\mathrm{b}}_{IP} \overset{(t)}{*} F^{\mathrm{s}}_{P;n} = Q^{\mathrm{s}}_{I;n}, \quad \text{for } n = 1,\dots,N^{\mathrm{T}}, \tag{4.13}$$

in which the *contrast volume source density* $Q^{\mathrm{s}}_{I;n} = Q^{\mathrm{s}}_{I;n}(\boldsymbol{x},t)$ is found as

$$\begin{aligned} Q^{\mathrm{s}}_{I;n} &= -\partial_t C^{X}_{IP} \overset{(t)}{*} F^{\mathrm{T}}_{P;n} \\ &= -\partial_t C^{X}_{IP} \overset{(t)}{*} (F^{\mathrm{i}}_{P;n} + F^{\mathrm{s}}_{P;n}), \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \text{ and } n = 1,\dots,N^{\mathrm{T}}, \end{aligned} \tag{4.14}$$

and where

$$C^X_{IP} = X_{IP} - X^{\mathrm{b}}_{IP} \tag{4.15}$$

is the contrast in constitutive relaxation functions between the 'scattering object' and the embedding. The support of $Q^{\mathrm{s}}_{I;n}$ is $\mathcal{D}^{\mathrm{s}}$ for all $n = 1, \dots, N^{\mathrm{T}}$. On account of equations (4.5) and (4.13), an integral representation for the evaluation of $F^{\mathrm{s}}_{P;n}$ is furnished by

$$F^{\mathrm{s}}_{P;n}(\boldsymbol{x}, t) = \int_{\mathcal{D}^{\mathrm{s}}} G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}', t) \, \mathrm{d}V(\boldsymbol{x}'), \quad \text{for } n = 1, \dots, N^{\mathrm{T}}. \tag{4.16}$$

Substitution of equation (4.16) into equation (4.14) leads to

$$Q^{\mathrm{s}}_{I;n} = -\partial_t C^X_{IP} \overset{(t)}{*} F^{\mathrm{i}}_{P;n} - \partial_t C^X_{IP} \overset{(t)}{*} \int_{\mathcal{D}^{\mathrm{s}}} G^{\mathrm{b}}_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}', t) \, \mathrm{d}V(\boldsymbol{x}'),$$
$$\text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \text{ and } n = 1, \dots, N^{\mathrm{T}}. \tag{4.17}$$

The other form of the system of field equations for the scattered wave field considers the scattered wave field to be present in the actual configuration and is given by

$$\mathcal{D}_{IP} F^{\mathrm{s}}_{P;n} + \partial_t X_{IP} \overset{(t)}{*} F^{\mathrm{s}}_{P;n} = -\partial_t C^X_{IP} \overset{(t)}{*} F^{\mathrm{i}}_{P;n}, \quad \text{for } n = 1, \dots, N^{\mathrm{T}}. \tag{4.18}$$

This system is of use in the computation of $F^{\mathrm{s}}_{P;n}$, for given values of the constitutive parameters, with the aid of numerical algorithms that apply to partial differential equations, such as finite-element, finite-difference and domain-integration methods.

Given the chosen embedding, the aim of the thus-formulated *multiple-source/multiple-receiver inverse-scattering problem* is to reconstruct, from the scattered-field dataset $\{d^{\mathrm{s}}_{m,n}(t);\ m = 1, \dots, N^{\mathrm{R}};\ n = 1, \dots, N^{\mathrm{T}}\}$, as far and as accurately as possible, the contrast C^X_{IP} in constitutive parameters pertaining to the domain $\mathcal{D}^{\mathrm{s}}$ and therewith the constitutive parameters of the medium in $\mathcal{D}^{\mathrm{s}}$. A definition of what is meant by 'as accurately as possible' and a method of reconstruction are discussed in § 6.

(c) *The inverse-transducted-wave-field problem*

In the inverse-transducted-wave-field problem, it is assumed that, in a certain, source-free target region $\mathcal{D}^{\mathrm{Tr}} \in \mathbb{R}^3$, the total wave field $\{F^{\mathrm{T}}_{P;n};\ n = 1, \dots, N^{\mathrm{T}}\}$ generated by N^{T} exterior sources is, through wave-field transduction, accessible to measurement. Such a case occurs, for example, when, in a translucent part of a solid material, wave-field transduction from elastic to optical can be accomplished. (In this respect, we refer the reader also to the heat-pulse method in the imaging of phonons in crystalline solids; see Wolfe (1998).) In this case, the field equation itself with the constitutive relation substituted, namely

$$\mathcal{D}_{IP} F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t) = -X_{IP}(\boldsymbol{x}, t) \overset{(t)}{*} \partial_t F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t), \quad \text{for } n = 1, \dots, N^{\mathrm{T}}, \tag{4.19}$$

serves as the data equation. As before, the aim of the thus-constructed *multiple-source inverse-transducted-wave-field problem* is to reconstruct, from equation (4.19), as far and as accurately as possible, the constitutive parameters X_{IP} in the target region. A definition of what is meant by 'as accurately as possible', and a method of reconstruction, are discussed in § 7.

5. An iterative optimization technique for handling the inverse-source problem

Any iterative optimization technique for handling the inverse-source problem, i.e. reconstructing the unknown volume source density Q_I^{T} as accurately as possible from the given dataset $\{d_m^{\mathrm{T}};\ m = 1,\ldots,N^{\mathrm{R}}\}$, as generated by equation (4.1), starts with defining an appropriate error criterion that quantifies the mismatch between this dataset and the one corresponding to assumed values of the volume source density. In accordance with the chosen error criterion, an iterative process is developed that starts with a certain initial value of the volume source density and aims at reducing, at each step of the process, the mismatch as defined by the error criterion ('improvement condition').

At this point, it is stressed that the generation of the dataset via measurement of the response of the receivers is dictated by the physics of the problem. The choice of a mismatch or error criterion is, however, largely a subjective one, and adaptable to the circumstances that occur in the actual measurement or field situation.

The fundamentals of the iterative solution technique are given in Appendix B. For the results of this appendix to be applicable, an appropriate inner product in the data space and the pertaining direct modelling operator have to be introduced. For the inner product in the data space, which we denote by $\langle\cdot,\cdot\rangle_{\mathrm{d}}$, we take the expression

$$\langle u_m^{\mathrm{T}}, v_m^{\mathrm{T}}\rangle_{\mathrm{d}} = \int_{\mathcal{T}} \left[\sum_{m,m'=1}^{N^{\mathrm{R}}} w_{m,m'}^{\mathrm{T}} u_m^{\mathrm{T}}(t) v_{m'}^{\mathrm{T}}(t)\right] \mathrm{d}t, \tag{5.1}$$

where $\mathcal{T}$ is the time window used for the reconstruction procedure, while

$$\{w_{m,m'}^{\mathrm{T}};\ m,m' = 1,\ldots,N^{\mathrm{R}}\}$$

is a set of weighting coefficients. This set has the symmetry property $w_{m,m'}^{\mathrm{T}} = w_{m',m}^{\mathrm{T}}$, while the expression $\|u_m^{\mathrm{T}}\|_{\mathrm{d}}$, defined through

$$\|u_m^{\mathrm{T}}\|_{\mathrm{d}}^2 = \langle u_m^{\mathrm{T}}, u_m^{\mathrm{T}}\rangle_{\mathrm{d}}, \tag{5.2}$$

is assumed to have the standard properties of a norm. The inner product in equation (5.1) has the form of a sum of *time correlations* evaluated at zero time shift.

The pertaining modelling operator,

$$\boldsymbol{\Omega}_{I';m}^{\mathrm{T}} : Q_{I'}^{\mathrm{T}}(\boldsymbol{x}', t) \longmapsto d_m^{\mathrm{T}}(t), \tag{5.3}$$

is inferred from equations (4.1) and (4.3), is linear, and is given by

$$\boldsymbol{\Omega}_{I';m}^{\mathrm{T}}[Q_{I'}^{\mathrm{T}}](t) = \int_{\mathcal{D}_m^{\mathrm{R}}} R_{P;m}(\boldsymbol{x}, t) \overset{(t)}{*} \left[\int_{\mathcal{D}^{\mathrm{T}}} G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q_{I'}^{\mathrm{T}}(\boldsymbol{x}', t)\,\mathrm{d}V(\boldsymbol{x}')\right] \mathrm{d}V(\boldsymbol{x}),$$
$$\text{for } m = 1,\ldots,N^{\mathrm{R}}. \tag{5.4}$$

Note that $Q_{I'}^{\mathrm{T}} = Q_{I'}^{\mathrm{T}}(\boldsymbol{x}', t)$ has the support $\mathcal{D}^{\mathrm{T}}$, while

$$d_m^{\mathrm{T}}(t) = \boldsymbol{\Omega}_{I';m}^{\mathrm{T}}[Q_{I'}^{\mathrm{T}}(\boldsymbol{x}', t)], \quad \text{for } m = 1,\ldots,N^{\mathrm{R}}, \tag{5.5}$$

is defined for $t \in \mathbb{R}$.

The inner product in the model space (of volume source densities) that allows for the construction of the operator adjoint to the modelling operator follows from equations (5.1) and (5.4). It is denoted by $\langle\cdot,\cdot\rangle_Q$ and is given by

$$\langle U^{\mathrm{T}}_{I'}, V^{\mathrm{T}}_{I'}\rangle_Q = \int_{t=-\infty}^{\infty}\left[\int_{\mathcal{D}^{\mathrm{T}}} U^{\mathrm{T}}_{I'}(\boldsymbol{x}',t)V^{\mathrm{T}}_{I'}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}')\right]\mathrm{d}t. \tag{5.6}$$

This inner product satisfies the requirements set out in Appendix B. Upon interchanging order of integration, the operator

$$\boldsymbol{\Omega}^{\mathrm{T}*}_{I';m} : d^{\mathrm{T}}_m(t) \longmapsto Q^{\mathrm{T}}_{I'}(\boldsymbol{x}',t), \tag{5.7}$$

adjoint to $\boldsymbol{\Omega}^{\mathrm{T}}_{I';m}$, is then found to be

$$\begin{aligned}
&\boldsymbol{\Omega}^{\mathrm{T}*}_{I';m}[d^{\mathrm{T}}_m](\boldsymbol{x}',t)\\
&\quad= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{T}}_{m,m'}\left[\int_{\mathcal{D}^{\mathrm{R}}_{m'}} R_{P;m'}(\boldsymbol{x},t) \overset{(-t)}{*} G_{PI'}(\boldsymbol{x},\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x})\right] \overset{(t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{T}}_m)(t)\\
&\quad= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{T}}_{m,m'}\left[\int_{\mathcal{D}^{\mathrm{R}}_{m'}} R_{P;m'}(\boldsymbol{x},t) \overset{(t)}{*} G_{PI'}(\boldsymbol{x},\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x})\right] \overset{(-t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{T}}_m)(-t),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \boldsymbol{x}' \in \mathcal{D}^{\mathrm{T}},
\end{aligned} \tag{5.8}$$

in which $\chi_{\mathcal{T}}(t)$ is the characteristic function of the set $\mathcal{T}$, i.e.

$$\chi_{\mathcal{T}}(t) = \{1, \tfrac{1}{2}, 0\}, \quad \text{for } t \in \{\mathcal{T}, \partial\mathcal{T}, \mathcal{T}'\}, \tag{5.9}$$

where $\partial\mathcal{T}$ is the boundary of $\mathcal{T}$ and $\mathcal{T}'$ is the complement of $\mathcal{T} \cup \partial\mathcal{T}$ in $\mathbb{R}$. The first representation of the adjoint operator (in the second line of equation (5.8)) has the interpretation of *reverse-time* wave-field extrapolation (Esmersoy & Oristaglio 1988; Esmersoy & Miller 1989); the second representation of the adjoint operator (in the third line) has the interpretation of wave-field extrapolation composed with a *time-reversed mirror* applied to the data (Derode *et al.* 1995).

With these preliminaries, the application of the iterative solution technique of Appendix B is straightforward. In this procedure, the successive data residuals are

$$r^{\mathrm{T};[i]}_m = d^{\mathrm{T}}_m - \boldsymbol{\Omega}^{\mathrm{T}}_{I';m}[Q^{\mathrm{T};[i]}_{I'}], \quad \text{for } m = 1,\dots,N^{\mathrm{R}} \text{ and } i = 0,1,2,\dots, \tag{5.10}$$

with the starting value $Q^{\mathrm{T};[0]}_{I'} = 0$, while

$$\epsilon^{[i]}_d = \|r^{\mathrm{T};[i]}_m\|^2_{\mathrm{d}}, \quad \text{for } i = 0,1,2,\dots, \tag{5.11}$$

and

$$\bar{\epsilon}^{[i]}_{\mathrm{d}} = \frac{\epsilon^{[i]}_{\mathrm{d}}}{\|d^{\mathrm{T}}_m\|^2_{\mathrm{d}}}, \quad \text{for } i = 0,1,2,\dots, \tag{5.12}$$

are the relevant errors and normalized errors.

6. An iterative optimization technique for handling the inverse-scattering problem

Any iterative optimization technique for handling the inverse-scattering problem, i.e. reconstructing, as accurately as possible, the unknown contrast in constitutive parameter functions C^X_{IP} from the scattered dataset

$$\{d^{\mathrm{s}}_{m,n};\ m=1,\dots,N^{\mathrm{R}};\ n=1,\dots,N^{\mathrm{T}}\},$$

as generated by equations (4.10), (4.16) and (4.17), starts with defining appropriate error criteria that quantify the mismatch in the satisfaction of the equality signs in equation (4.10) (the *data equation*) and in equation (4.17) (the *object equation*).

As far as the *data equation* is concerned, the error criterion quantifies the mismatch between the actual observations and the scattered dataset constructed from assumed values of the contrast volume source densities $\{Q^{\mathrm{s}}_{I';n};\ n=1,\dots,N^{\mathrm{T}}\}$. This part of the analysis can be considered as a set of inverse-source problems for each of the irradiating sources separately, and runs along the lines presented in § 5: with the Green's functions $G_{PI'}$ replaced by the ones associated with the embedding $G^{\mathrm{b}}_{PI'}$. The *object equation* is, for a given incident wave field, nonlinear in the contrast in constitutive parameters, in view of the fact that the scattered wave field does also depend on this contrast. However, upon first using the data equation to reconstruct the sequence of contrast volume source densities, and using these reconstructed values in the object equation, the latter can be envisaged as an operator equation that is linear in the contrast in constitutive parameters. In it, the reconstructed contrast volume source densities are the 'data', while the contrast in constitutive parameters is the 'model' quantity.

The fundamentals of the iterative solution technique are given in Appendix B. For the results of this appendix to be applicable, the pertaining modelling operators and appropriate inner products in the relevant data and model spaces have to be introduced.

For the inner products (for each of the irradiating sources) in the scattered data space, which we denote by $\langle\cdot,\cdot\rangle_{\mathrm{s};n}$, we take the expressions

$$\langle u^{\mathrm{s}}_{m,n}, v^{\mathrm{s}}_{m',n}\rangle_{\mathrm{s};n} = \int_{\mathcal{T}}\left[\sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{s}}_{m,m';n}u^{\mathrm{s}}_{m,n}(t)v^{\mathrm{s}}_{m',n}(t)\right]\mathrm{d}t, \quad \text{for } n=1,\dots,N^{\mathrm{T}}, \tag{6.1}$$

where $\mathcal{T}$ is the time window used for the reconstruction procedure, while, for each $n=1,\dots,N^{\mathrm{T}}$, $\{w^{\mathrm{s}}_{m,m';n};\ m,m'=1,\dots,N^{\mathrm{R}}\}$ is a set of weighting coefficients. This set has the symmetry property $w^{\mathrm{s}}_{m,m';n}=w^{\mathrm{s}}_{m',m;n}$, while the expression $\|u^{\mathrm{s}}_{m,n}\|_{\mathrm{s};n}$, defined through

$$\|u^{\mathrm{s}}_{m,n}\|^2_{\mathrm{s};n} = \langle u^{\mathrm{s}}_{m,n}, u^{\mathrm{s}}_{m',n}\rangle_{\mathrm{s};n}, \quad \text{for } n=1,\dots,N^{\mathrm{T}}, \tag{6.2}$$

is assumed to have the standard properties of a norm. The pertaining direct modelling operators,

$$\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n} : Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t) \longmapsto d^{\mathrm{s}}_{m,n}(t), \quad \text{for } m=1,\dots,N^{\mathrm{R}}, \quad n=1,\dots,N^{\mathrm{T}}, \tag{6.3}$$

are inferred from equations (4.10) and (4.16). They are given through

$$\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}[Q^{\mathrm{s}}_{I';n}](t) = \int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} \left[\int_{\mathcal{D}^{\mathrm{T}}} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t) \overset{(t)}{*} Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}') \right] \mathrm{d}V(\boldsymbol{x}),$$
$$\text{for } m=1,\dots,N^{\mathrm{R}}, \quad n=1,\dots,N^{\mathrm{T}}. \tag{6.4}$$

Note that $Q^{\mathrm{s}}_{I';n} = Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t)$ has the support $\mathcal{D}^{\mathrm{s}}$, while

$$d^{\mathrm{s}}_{m,n}(t) = \boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}[Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t)], \quad \text{for } m=1,\dots,N^{\mathrm{R}}, \quad n=1,\dots,N^{\mathrm{T}}, \tag{6.5}$$

has support $\mathbb{R}$. From equations (6.1) and (6.4), the inner product in the model space (of contrast volume source densities) that allows for the construction of the operator adjoint to the direct modelling operator follows. Denoting this latter inner product by $\langle\cdot,\cdot\rangle_Q$, we obtain

$$\langle U^{\mathrm{s}}_{I';n}, V^{\mathrm{s}}_{I';n}\rangle_{Q;n} = \int_{t=-\infty}^{\infty} \left[\int_{\mathcal{D}^{\mathrm{s}}} U^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t) V^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}') \right] \mathrm{d}t,$$
$$\text{for } n=1,\dots,N^{\mathrm{T}}. \tag{6.6}$$

This inner product clearly satisfies the requirements set out in Appendix B. The corresponding norm $\|U^{\mathrm{s}}_{I';n}\|_{Q;n}$ is given by

$$\|U^{\mathrm{s}}_{I';n}\|_{Q;n} = \langle U^{\mathrm{s}}_{I';n}, U^{\mathrm{s}}_{I';n}\rangle_{Q;n}, \quad \text{for } n=1,\dots,N^{\mathrm{T}}. \tag{6.7}$$

In analogy with the inverse-source problem, the operator

$$\boldsymbol{\Omega}^{\mathrm{s}*}_{I';m,n} : d^{\mathrm{s}}_{m,n}(t) \longmapsto Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}',t), \tag{6.8}$$

adjoint to $\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}$, is then found to be

$$\boldsymbol{\Omega}^{\mathrm{s}*}_{I';m,n}[d^{\mathrm{s}}_{m,n}](\boldsymbol{x}',t)$$
$$= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{s}}_{m,m';n} \left[\int_{\mathcal{D}^{\mathrm{R}}_{m'}} R_{P;m'}(\boldsymbol{x},t) \overset{(-t)}{*} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}) \right] \overset{(t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{s}}_{m,n})(t),$$
$$\text{for } \boldsymbol{x}' \in \mathcal{D}^{\mathrm{s}} \text{ and } n=1,\dots,N^{\mathrm{T}}, \tag{6.9}$$

in which $\chi_{\mathcal{T}}(t)$ is the characteristic function of the set $\mathcal{T}$. Rewriting the adjoint to $\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}$ in the form

$$\boldsymbol{\Omega}^{\mathrm{s}*}_{I';m,n}[d^{\mathrm{s}}_{m,n}](\boldsymbol{x}',t)$$
$$= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{s}}_{m,m';n} \left[\int_{\mathcal{D}^{\mathrm{R}}_{m'}} R_{P;m'}(\boldsymbol{x},t) \overset{(t)}{*} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}) \right] \overset{(-t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{s}}_{m,n})(-t),$$
$$\text{for } \boldsymbol{x}' \in \mathcal{D}^{\mathrm{s}} \text{ and } n=1,\dots,N^{\mathrm{T}}, \tag{6.10}$$

we can interpret the action of the adjoint as: time reversing the data, followed by propagating the results as if they were sources ('illumination') into the configuration (Berkhout & Rietveld 1995). Thorbecke (1997) refers to an operation of this kind as 'focusing in receivers'. The outcome ('common focal point gather') is parametrized by (the 'focal point') $\boldsymbol{x}' \in \mathcal{D}^{\mathrm{s}}$, and is a function of time t and source position $n = 1, \dots, N^{\mathrm{T}}$.

With these preliminaries, the iterative solution technique of Appendix B can be used to construct, from the scattered dataset, the contrast volume source densities associated with each of the irradiating sources. In this procedure, the successive residual scattered datasets are

$$r^{\mathrm{s};[i]}_{m,n} = d^{\mathrm{s}}_{m,n} - \boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}[Q^{\mathrm{s};[i]}_{I';n}],$$
$$\text{for } m = 1, \dots, N^{\mathrm{R}}, \quad n = 1, \dots, N^{\mathrm{T}}, \quad i = 0, 1, 2, \dots, \tag{6.11}$$

with the starting value $Q^{\mathrm{s};[0]}_{I',n} = 0$, while

$$\epsilon^{[i]}_{\mathrm{s};n} = \|r^{\mathrm{s};[i]}_{m,n}\|^2_{\mathrm{s};n}, \quad \text{for } n = 1, \dots, N^{\mathrm{T}}, \quad i = 0, 1, 2, \dots, \tag{6.12}$$

and

$$\bar{\epsilon}^{[i]}_{\mathrm{s};n} = \frac{\epsilon^{[i]}_{\mathrm{s};n}}{\|d^{\mathrm{s}}_{m,n}\|^2_{\mathrm{s};n}}, \quad \text{for } n = 1, \dots, N^{\mathrm{T}}, \quad i = 0, 1, 2, \dots, \tag{6.13}$$

are the relevant errors and normalized errors.

Now that a sequence of contrast volume source densities

$$\{Q^{\mathrm{s}}_{I';n} = Q^{\mathrm{s}}_{I';n}(\boldsymbol{x}, t);\ n = 1, \dots, N^{\mathrm{T}}\}$$

has been constructed, the corresponding scattered wave field

$$\{F^{\mathrm{s}}_{P;n} = F^{\mathrm{s}}_{P;n}(\boldsymbol{x}, t);\ n = 1, \dots, N^{\mathrm{T}}\},$$

and, hence, the total wave field

$$\{F^{\mathrm{T}}_{P;n} = F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t);\ n = 1, \dots, N^{\mathrm{T}}\},$$

can be computed (see equations (4.6), (4.8) and (4.16)), in particular in $\mathcal{D}^{\mathrm{s}}$. With these, the object equation (4.14) can be written as

$$Q^{\mathrm{s}}_{I;n}(\boldsymbol{x}, t) = -\partial_t C^X_{IP}(\boldsymbol{x}, t) \overset{(t)}{*} F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t), \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \text{ and } n = 1, \dots, N^{\mathrm{T}}. \tag{6.14}$$

In this equation, we consider the contrast volume source densities as the 'data' and the contrast in constitutive parameters as the 'model' quantity. To cast the equation in an operator form, we introduce the operator

$$\boldsymbol{\Lambda}^Q_{P;n} : C^X_{IP}(\boldsymbol{x}, t) \longmapsto Q^{\mathrm{s}}_{I;n}(\boldsymbol{x}, t), \quad \text{for } n = 1, \dots, N^{\mathrm{T}}. \tag{6.15}$$

Combining equations (6.14) and (6.15) it is found that

$$\boldsymbol{\Lambda}^Q_{P;n}[C^X_{IP}](\boldsymbol{x}, t) = -\partial_t C^X_{IP}(\boldsymbol{x}, t) \overset{(t)}{*} F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t),$$
$$\text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \text{ and } n = 1, \dots, N^{\mathrm{T}}. \tag{6.16}$$

Note that both

$$Q^{\mathrm{s}}_{I;n} = \boldsymbol{\Lambda}^Q_{P;n}[C^X_{IP}], \quad \text{for } n = 1, \dots, N^{\mathrm{T}}, \tag{6.17}$$

and $C^X_{IP} = C^X_{IP}(\boldsymbol{x}, t)$ have spatial support $\mathcal{D}^{\mathrm{s}}$.

For the inner product in the relevant data space (of reconstructed contrast volume source densities), which we denote by $\langle\cdot,\cdot\rangle_Q$, we take the expression

$$\langle u^Q_{I;n}, v^Q_{I;n'}\rangle_Q = \int_{t=-\infty}^{\infty}\int_{\mathcal{D}^{\mathrm{s}}}\left[\sum_{n,n'=1}^{N^{\mathrm{T}}} w^Q_{n,n'} u^Q_{I;n}(\boldsymbol{x},t) v^Q_{I;n'}(\boldsymbol{x},t)\right]\mathrm{d}V(\boldsymbol{x})\mathrm{d}t, \tag{6.18}$$

where $\{w^Q_{n,n'}; n,n'=1,\dots,N^{\mathrm{T}}\}$ is a set of weighting coefficients. This set has the symmetry property $w^Q_{n,n'} = w^Q_{n',n}$, while the expression $\|u^Q_{I;n}\|_Q$, defined through

$$\|u^Q_{I;n}\|^2_Q = \langle u^Q_{I;n}, u^Q_{I;n'}\rangle_Q, \tag{6.19}$$

is assumed to have the standard properties of a norm. The weighting coefficients are representative of the algorithm that is used to combine the results from the different irradiating sources to arrive at a value of the contrast in constitutive parameters that is independent of these sources. From equations (6.16) and (6.18), the inner product in the model space (of contrasts in constitutive properties) that allows for the construction of the operator adjoint to the direct modelling operator follows. Denoting the latter inner product by $\langle\cdot,\cdot\rangle_X$, we obtain

$$\langle U^X_{IP}, V^X_{IP}\rangle_X = \int_{t=-\infty}^{\infty}\left[\int_{\mathcal{D}^{\mathrm{s}}} U^X_{IP}(\boldsymbol{x},t) V^X_{IP}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x})\right]\mathrm{d}t. \tag{6.20}$$

This inner product satisfies the requirements of Appendix B. The corresponding norm $\|U^X_{IP}\|_X$ follows as

$$\|U^X_{IP}\|^2_X = \langle U^X_{IP}, U^X_{IP}\rangle_X. \tag{6.21}$$

The operator

$$\boldsymbol{\Lambda}^{Q*}_{P;n} : Q^{\mathrm{s}}_{I;n} \longmapsto C^X_{IP}, \tag{6.22}$$

adjoint to $\boldsymbol{\Lambda}^Q_{P;n}$, is then found to be

$$\begin{aligned}\boldsymbol{\Lambda}^{Q*}_{P;n}[Q^{\mathrm{s}}_{I;n}](\boldsymbol{x},t) &= \sum_{n,n'=1}^{N^{\mathrm{T}}} w^Q_{n,n'}\partial_t Q^{\mathrm{s}}_{I;n}(\boldsymbol{x},t) \overset{(t)}{*} F^{\mathrm{T}}_{P;n'}(\boldsymbol{x},-t)\\ &= \sum_{n,n'=1}^{N^{\mathrm{T}}} w^Q_{n,n'} F^{\mathrm{T}}_{P;n'}(\boldsymbol{x},t) \overset{(-t)}{*} \partial_t Q^{\mathrm{s}}_{I;n}(\boldsymbol{x},-t).\end{aligned} \tag{6.23}$$

This operation contains time correlations, and may be annotated by 'focusing in sources'. The time correlation reflects what is known as the imaging condition (Claerbout 1971; Esmersoy & Oristaglio 1988).

With these preliminaries, the iterative solution technique of Appendix B can be used to reconstruct, from the reconstructed sequence of contrast volume source densities, the contrast in constitutive parameters. In this procedure, the successive contrast volume source density (data) residuals are

$$r^{Q;[i]}_{I;n} = Q^{\mathrm{s}}_{I;n} - \boldsymbol{\Lambda}^Q_{P;n}[C^{X;[i]}_{IP}], \quad \text{for } n = 1,\dots,N^{\mathrm{T}}, \quad i = 0,1,2,\dots, \tag{6.24}$$

with the starting value $C^{X;[0]}_{IP'} = 0$, while

$$\epsilon^{[i]}_Q = \|r^{Q;[i]}_{I;n}\|^2_Q, \quad \text{for } i = 0, 1, 2, \ldots, \tag{6.25}$$

and

$$\bar{\epsilon}^{[i]}_Q = \frac{\epsilon^{[i]}_Q}{\|Q^{\mathrm{s}}_{I;n}\|^2_Q}, \quad \text{for } i = 0, 1, 2, \ldots, \tag{6.26}$$

are the relevant errors and normalized errors.

7. An iterative optimization technique for handling the inverse-transducted-wave-field problem

Occasionally, it happens that in a certain bounded, source-free subdomain of the configuration, the total wave field generated by a number of exterior sources is, via transduction of the pertaining wave field, accessible to measurement. For example, such a case arises when, in a translucent (part of a) solid, the field quantities associated with an exteriorly generated elastic wave motion are measured via optical detection techniques. On this occasion, optimization methods for reconstructing the elastic-wave constitutive parameters can be directly applied to the pertaining source-free elastic wave-field equations. A method of this kind finds application in the quantitative non-destructive evaluation of materials.

Let $\mathcal{D}^{\mathrm{Tr}}$ be the subdomain in which transduction of the elastic wave field is feasible. Then, observation leads to the data equation (cf. equation (4.19))

$$\mathsf{D}_{IP'} F^{\mathrm{T}}_{P';n}(\boldsymbol{x}, t) = -\partial_t F^{\mathrm{T}}_{P;n}(\boldsymbol{x}, t) \overset{(t)}{*} X_{IP}(\boldsymbol{x}, t),$$
$$\text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{Tr}}, \quad t \in \mathcal{T}, \quad n = 1, \ldots, N^{\mathrm{T}}, \tag{7.1}$$

where $\mathcal{T}$ is the time window used for the reconstruction procedure and N^{T} is the number of irradiating sources. In this relation, $\{\mathsf{D}_{IP'} F_{P';n};\ n = 1, \ldots, N^{\mathrm{T}}\}$ is the sequence of 'data' and X_{IP} is the 'model' quantity. For the inner product in the data space, which we denote by $\langle \cdot, \cdot \rangle_F$, we take the expression

$$\langle u^{\mathrm{T}}_{I;n}, v^{\mathrm{T}}_{I;n} \rangle_F = \int_{\mathcal{T}} \int_{\mathcal{D}^{\mathrm{Tr}}} \left[\sum_{n,n'=1}^{N^{\mathrm{T}}} w^F_{n,n'} u^{\mathrm{T}}_{I;n}(\boldsymbol{x}, t) v^{\mathrm{T}}_{I;n'}(\boldsymbol{x}, t) \right] \mathrm{d}V(\boldsymbol{x}) \mathrm{d}t, \tag{7.2}$$

where $\{w^F_{n,n'};\ n = 1, \ldots, N^{\mathrm{T}};\ n' = 1, \ldots, N^{\mathrm{T}}\}$ is a set of weighting coefficients. This set has the symmetry property $w^F_{n,n'} = w^F_{n',n}$, while the expression $\|u^{\mathrm{T}}_{I;n}\|_F$ defined through

$$\|u^{\mathrm{T}}_{I;n}\|^2_F = \langle u^{\mathrm{T}}_{I;n}, u^{\mathrm{T}}_{I;n} \rangle_F, \tag{7.3}$$

is assumed to have the standard properties of a norm. The weighting coefficients are representative for the algorithm that is used to combine the results corresponding to the different irradiating sources to arrive at a value of the constitutive parameters that is independent of these sources. To cast the data equation (7.1) into an operator form, we introduce the operator

$$\boldsymbol{\Omega}^F_{P;n} : X_{IP}(\boldsymbol{x}, t) \longmapsto \mathsf{D}_{IP'} F^{\mathrm{T}}_{P';n}(\boldsymbol{x}, t), \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{Tr}} \text{ and } n = 1, \ldots, N^{\mathrm{T}}. \tag{7.4}$$

Combining equations (7.1) and (7.4), it is found that

$$\boldsymbol{\Omega}^F_{P;n}[X_{IP}] = -\partial_t F_{P;n} \overset{(t)}{*} X_{IP}, \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{Tr}} \text{ and } n = 1, \dots, N^{\mathrm{T}}. \tag{7.5}$$

From equations (7.2) and (7.5), the inner product in the model space (of constitutive parameters) follows. Denoting the latter inner product by $\langle \cdot, \cdot \rangle_X$, we obtain

$$\langle U^X_{IP}, V^X_{IP} \rangle_X = \int_{t=-\infty}^{\infty} \left[\int_{\mathcal{D}^{\mathrm{Tr}}} U^X_{IP}(\boldsymbol{x}, t) V^X_{IP}(\boldsymbol{x}, t) \, \mathrm{d}V(\boldsymbol{x}) \right] \mathrm{d}t. \tag{7.6}$$

This inner product evidently satisfies the requirements of Appendix B. The corresponding norm $\|U^X_{IP}\|_X$ follows as

$$\|U^X_{IP}\|^2_X = \langle U^X_{IP}, U^X_{IP} \rangle_X. \tag{7.7}$$

The operator

$$\boldsymbol{\Omega}^{F*}_{P;n} : \mathsf{D}_{IP'} F^{\mathrm{T}}_{P';n} \longmapsto X_{IP}, \tag{7.8}$$

adjoint to $\boldsymbol{\Omega}^F_{P;n}$, is then found to be

$$\boldsymbol{\Omega}^{F*}_{P;n}[\mathsf{D}_{IP'} F_{P';n}] = \sum_{n,n'=1}^{N^{\mathrm{T}}} w^F_{n,n'} [\chi_{\mathcal{T}}(t) \mathsf{D}_{IP'} F^{\mathrm{T}}_{P';n}(\boldsymbol{x}, t)] \overset{(t)}{*} \partial_t F^{\mathrm{T}}_{P;n'}(\boldsymbol{x}, -t). \tag{7.9}$$

With these preliminaries, the iterative solution technique of Appendix B can be used to reconstruct, from the sequence of wave-field data, the constitutive parameters. In this procedure, the successive wave-field data residuals are

$$r^{F;[i]}_{I;n} = \mathsf{D}_{IP'} F_{P';n} - \boldsymbol{\Omega}^F_{P;n}[X^{[i]}_{IP}], \quad \text{for } i = 0, 1, 2, \dots, \tag{7.10}$$

with the starting value $X^{[0]}_{IP} = 0$, while

$$\epsilon^{[i]}_F = \|r^{F;[i]}_{I;n}\|^2_F, \qquad \text{for } i = 0, 1, 2, \dots, \tag{7.11}$$

and

$$\bar{\epsilon}^{[i]}_F = \frac{\epsilon^{[i]}_F}{\|\mathsf{D}_{IP'} F^{\mathrm{T}}_{P';n}\|^2_F}, \qquad \text{for } i = 0, 1, 2, \dots, \tag{7.12}$$

are the relevant errors and normalized errors.

8. The wave-field reciprocity theorem of the time-convolution type

In general, the remote sensing problem is concerned with reconstructing, from certain recorded wave-field data, certain properties of the configuration elsewhere in space. The underlying physical phenomenon can be regarded as an interaction between the sensors ('receivers') that register the wave field and the (remote, known or unknown) sources ('transmitters') that generate the recorded field. This type of interaction is subject to the principle of reciprocity. Therefore, reciprocity theorems can be expected to be somewhere at the basis of the remote sensing problem. As far as their structure in time is concerned, two types of interaction show up in time-invariant configurations, namely time convolution and time correlation. In the present section, the wave-field reciprocity theorem of the time-convolution type will be discussed; the wave-field reciprocity theorem of the time-correlation type is the subject of § 9.

A reciprocity theorem quantitatively describes the interaction between two different 'states' that could occur in one and the same geometrical configuration in space. For the moment, this geometrical configuration consists of the bounded domain $\mathcal{D} \subset \mathbb{R}^3$ with its piecewise smooth closed boundary surface $\partial\mathcal{D}$. The unit vector along the outward normal to $\partial\mathcal{D}$ is denoted by $\boldsymbol{n}$. Each of the two states is characterized by its field strength, its field flux, its volume source density, and its constitutive relaxation function. The configuration and the constitutive parameters are assumed to be time invariant and the constitutive parameters are assumed to vary piecewise continuously in space. The superscripts A and Z are used to indicate the two states. Then, at each interior point of a subdomain of the configuration where the constitutive parameters are spatially continuously distributed, the field strengths are continuously differentiable and satisfy the field equations

$$\mathsf{D}_{IP} F_P^{\mathrm{A}} + \partial_t \Phi_I^{\mathrm{A}} = Q_I^{\mathrm{A}}, \tag{8.1}$$

and

$$\mathsf{D}_{PI} F_I^{\mathrm{Z}} + \partial_t \Phi_P^{\mathrm{Z}} = Q_P^{\mathrm{Z}}, \tag{8.2}$$

together with the constitutive relations

$$\Phi_I^{\mathrm{A}}(\boldsymbol{x},t) = X_{IP}^{\mathrm{A}}(\boldsymbol{x},t) \overset{(t)}{*} F_P^{\mathrm{A}}(\boldsymbol{x},t), \quad \text{with } X_{IP}^{\mathrm{A}}(\boldsymbol{x},t) = 0 \text{ for } t < 0, \tag{8.3}$$

and

$$\Phi_P^{\mathrm{Z}}(\boldsymbol{x},t) = X_{PI}^{\mathrm{Z}}(\boldsymbol{x},t) \overset{(t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x},t), \quad \text{with } X_{PI}^{\mathrm{Z}}(\boldsymbol{x},t) = 0 \text{ for } t < 0. \tag{8.4}$$

Furthermore, across interfaces where the constitutive parameters show finite jumps, the interface boundary conditions,

$$N_{IP} F_P^{\mathrm{A}} = \text{continuous across interface} \tag{8.5}$$

and

$$N_{PI} F_I^{\mathrm{Z}} = \text{continuous across interface}, \tag{8.6}$$

hold. The subscripts in equations (8.1)–(8.6) have been adjusted for later convenience.

In the reciprocity theorem of the time-convolution type, the two-dimensional reciprocity *signature array* δ^-_{IM} occurs, a diagonal array with elements $+1$ and -1 according to the scheme given in Appendix A. An important property for reciprocity of the time-convolution type to hold is

$$\delta^-_{IM} \mathsf{D}_{MP} = -\delta^-_{PM} \mathsf{D}_{MI}. \tag{8.7}$$

Equation (8.7) implies that D_{IP} is an array with zero block diagonals, as induced by the signature array δ^-_{MP}, while being symmetric for the rest. The spatial differential operator arrays $\mathcal{D}_{IP}$ that occur in the wave phenomena that we consider do have this property.

The local interaction quantity that occurs in the reciprocity theorem of the time-convolution type is

$$\begin{aligned}\delta^-_{IM}\mathcal{D}_{MP}(F^{\mathrm{A}}_P \overset{(t)}{*} F^{\mathrm{Z}}_I) &= F^{\mathrm{Z}}_I \overset{(t)}{*} \delta^-_{IM}\mathcal{D}_{MP}F^{\mathrm{A}}_P + F^{\mathrm{A}}_P \overset{(t)}{*} \delta^-_{IM}\mathcal{D}_{MP}F^{\mathrm{Z}}_I \\ &= F^{\mathrm{Z}}_I \overset{(t)}{*} \delta^-_{IM}\mathcal{D}_{MP}F^{\mathrm{A}}_P - F^{\mathrm{A}}_P \overset{(t)}{*} \delta^-_{PM}\mathcal{D}_{MI}F^{\mathrm{Z}}_I,\end{aligned}$$

hence

$$\begin{aligned}\delta^-_{IM}\mathcal{D}_{MP}(F^{\mathrm{A}}_P \overset{(t)}{*} F^{\mathrm{Z}}_I) = \delta^-_{IM}F^{\mathrm{Z}}_I \overset{(t)}{*} Q^{\mathrm{A}}_M - \delta^-_{PM}F^{\mathrm{A}}_P \overset{(t)}{*} Q^{\mathrm{Z}}_M \\ - \delta^-_{IM}F^{\mathrm{Z}}_I \overset{(t)}{*} \partial_t\Phi^{\mathrm{A}}_M + \delta^-_{PM}F^{\mathrm{A}}_P \overset{(t)}{*} \partial_t\Phi^{\mathrm{Z}}_M. \qquad (8.8)\end{aligned}$$

Equation (8.8) expresses the *local reciprocity theorem of the time-convolution type.* The first two terms on the right-hand side are representative for the interaction of the two states via their volume source densities; this type of interaction vanishes in a source-free domain. The last two terms can, under the application of the constitutive relations (8.3) and (8.4), be rewritten as

$$-\delta^-_{IM}F^{\mathrm{Z}}_I \overset{(t)}{*} \partial_t\Phi^{\mathrm{A}}_M + \delta^-_{PM}F^{\mathrm{A}}_P \overset{(t)}{*} \partial_t\Phi^{\mathrm{Z}}_M = -\partial_t[F^{\mathrm{Z}}_I \overset{(t)}{*} (\delta^-_{IM}X^{\mathrm{A}}_{MP} - \delta^-_{PM}X^{\mathrm{Z}}_{MI}) \overset{(t)}{*} F^{\mathrm{A}}_P]. \qquad (8.9)$$

Hence, these terms describe the interaction of the two states via their contrast in constitutive properties. This interaction vanishes if

$$\delta^-_{IM}X^{\mathrm{A}}_{MP} = \delta^-_{PM}X^{\mathrm{Z}}_{MI}, \qquad (8.10)$$

i.e. in a (sub)domain where the constitutive properties of the medium in state Z are the *adjoints* of the ones of the medium in state A. In a (sub)domain where the condition (8.10) holds for one and the same medium, the relevant medium is denoted as *self-adjoint* or *reciprocal.*

By integrating equation (8.8) over the subdomains of $\mathcal{D}$ in which the constitutive parameters vary continuously with position, applying Gauss's integral theorem, adding the results, and using the interface boundary conditions (8.5) and (8.6), we obtain

$$\begin{aligned}\int_{\partial\mathcal{D}} \delta^-_{IM}N_{MP}(F^{\mathrm{A}}_P \overset{(t)}{*} F^{\mathrm{Z}}_I)\,\mathrm{d}A \\ = \int_{\mathcal{D}} (\delta^-_{IM}F^{\mathrm{Z}}_I \overset{(t)}{*} Q^{\mathrm{A}}_M - \delta^-_{PM}F^{\mathrm{A}}_P \overset{(t)}{*} Q^{\mathrm{Z}}_M)\,\mathrm{d}V \\ - \int_{\mathcal{D}} (\delta^-_{IM}F^{\mathrm{Z}}_I \overset{(t)}{*} \partial_t\Phi^{\mathrm{A}}_M - \delta^-_{PM}F^{\mathrm{A}}_P \overset{(t)}{*} \partial_t\Phi^{\mathrm{Z}}_M)\,\mathrm{d}V. \qquad (8.11)\end{aligned}$$

Equation (8.11) is the *global reciprocity theorem of the time-convolution type* for the domain $\mathcal{D}$. In this theorem, substituting equation (3.7), the vectorial quantity

$$\{F^{\mathrm{A}}, F^{\mathrm{Z}}\}_k \equiv \delta^-_{IM}\mathcal{D}_{MP}(\boldsymbol{i}_k)(F^{\mathrm{A}}_P \overset{(t)}{*} F^{\mathrm{Z}}_I),$$

appears in the boundary integral; this vector couples the different wave phenomena to one another.

In a number of applications, equation (8.11) will be applied to the entire $\mathbb{R}^3$. This is done by first applying the theorem to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin $\mathcal{O}$ of the chosen reference frame and radius Δ, and then taking the limit $\Delta \to \infty$. For some $\Delta > \Delta_0$, $\mathcal{S}(\mathcal{O},\Delta)$ will be located in the domain $\mathcal{D}^\infty$, where the Green's functions are analytically known and, in particular, their causal behaviour can be established (see § 2). If, now, the volume source distributions in the two states A and Z have bounded supports and the wave fields are causally related to the action of their sources, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes in the limit $\Delta \to \infty$. If, however, the wave field in one of the two states is taken to be anti-causally related and the other causally related to the action of its sources, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ does not vanish, but remains a fixed function of time.

In case state A is chosen equal to state Z, the reciprocity theorem of the time-convolution type reduces to the trivial identity $0 = 0$.

Two applications of the reciprocity theorem of the time-convolution type will be discussed: the transmitter/receiver reciprocity property and the reciprocity property of the Green's function.

(*a*) *Transmitter/receiver reciprocity*

The basis for the transmitter/receiver reciprocity property is furnished by the application of equation (8.11) to the situation where state A is associated with the wave field $\{F_P^{\mathrm{A}}, \Phi_I^{\mathrm{A}}\}$ that is causally related to the action of its source with volume source density Q_I^{A}; and state Z is associated with the wave field $\{F_I^{\mathrm{Z}}, \Phi_P^{\mathrm{Z}}\}$ that is causally related to the action of its source with volume source density Q_P^{Z}. The support of $\{F_P^{\mathrm{A}}, \Phi_I^{\mathrm{A}}\}$ and $\{F_I^{\mathrm{Z}}, \Phi_P^{\mathrm{Z}}\}$ is $\mathbb{R}^3$; Q_I^{A} has the bounded support $\mathcal{D}^{\mathrm{A}}$ and Q_P^{Z} has the bounded support $\mathcal{D}^{\mathrm{Z}}$, with $\mathcal{D}^{\mathrm{A}} \cap \mathcal{D}^{\mathrm{Z}} = \emptyset$. The media in the two states are taken to be each other's adjoints, i.e. $\delta^-_{IM} X^{\mathrm{A}}_{MP} = \delta^-_{PM} X^{\mathrm{Z}}_{MI}$. Application of equation (8.11) to the entire $\mathbb{R}^3$, keeping in mind that in view of the causality of the two wave fields the surface integral in equation (8.11) vanishes, yields

$$\int_{\mathcal{D}^{\mathrm{A}}} \delta^-_{IM} (F_I^{\mathrm{Z}} \overset{(t)}{*} Q_M^{\mathrm{A}})\,\mathrm{d}V = \int_{\mathcal{D}^{\mathrm{Z}}} \delta^-_{PM} (F_P^{\mathrm{A}} \overset{(t)}{*} Q_M^{\mathrm{Z}})\,\mathrm{d}V. \tag{8.12}$$

To establish the transmitter/receiver reciprocity property from this relation, we first consider the case where field F_P^{A}, excited by the source ('transmitter') with volume source density Q_I^{A} and support $\mathcal{D}^{\mathrm{A}}$, is recorded by a receiver with support $\mathcal{D}^{\mathrm{Z}}$ and receiving characteristics R_P^{Z}. The relevant recorded data trace is

$$d^{\mathrm{Z}}(t) = \int_{\mathcal{D}^{\mathrm{Z}}} R_P^{\mathrm{Z}} \overset{(t)}{*} F_P^{\mathrm{A}}\,\mathrm{d}V. \tag{8.13}$$

Secondly, the roles of transmitter and receiver are interchanged and we consider the case where the field F_I^{Z}, excited by the source ('transmitter') with volume source density Q_P^{Z} and support $\mathcal{D}^{\mathrm{Z}}$, is recorded by a receiver with support $\mathcal{D}^{\mathrm{A}}$ and receiving characteristics R_I^{A}. In this case, the recorded data trace is

$$d^{\mathrm{A}}(t) = \int_{\mathcal{D}^{\mathrm{A}}} R_I^{\mathrm{A}} \overset{(t)}{*} F_I^{\mathrm{Z}}\,\mathrm{d}V. \tag{8.14}$$

Upon comparing equations (8.13) and (8.14) with equation (8.12), we conclude that

$$d^{\mathrm{Z}}(t) = d^{\mathrm{A}}(t), \tag{8.15}$$

provided that we interrelate the volume source densities and the receiving characteristics in the two cases via

$$R_P^{\mathrm{Z}} = \delta^-_{PM} Q_M^{\mathrm{Z}} \text{ for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{Z}} \quad \text{and} \quad R_I^{\mathrm{A}} = \delta^-_{IM} Q_M^{\mathrm{A}} \text{ for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{A}}. \tag{8.16}$$

Equations (8.13)–(8.16) express the transmitter/receiver reciprocity property.

(*b*) *Green's function reciprocity*

The Green's function reciprocity property follows from equation (8.11) upon taking the wave field in state A to be the one that is causally related to a point-source excitation at $\{\boldsymbol{x} = \boldsymbol{x}', t = 0\}$ and amplitude a_I^{A}, i.e.

$$Q_I^{\mathrm{A}}(\boldsymbol{x}, t) = a_I^{\mathrm{A}} \delta(\boldsymbol{x} - \boldsymbol{x}', t); \tag{8.17}$$

and taking the wave field in state Z to be the one that is causally related to a point-source excitation at $\{\boldsymbol{x} = \boldsymbol{x}'', t = 0\}$ and amplitude a_P^{Z}, i.e.

$$Q_P^{\mathrm{Z}}(\boldsymbol{x}, t) = a_P^{\mathrm{Z}} \delta(\boldsymbol{x} - \boldsymbol{x}'', t), \tag{8.18}$$

with $\boldsymbol{x}' \neq \boldsymbol{x}''$. The media in the two states are taken to be each other's adjoints, i.e. $\delta^-_{IM} X^{\mathrm{A}}_{MP} = \delta^-_{PM} X^{\mathrm{Z}}_{MI}$, and the support of the wave field is the entire $\mathbb{R}^3$. The two Green's function arrays are introduced via the linear relationships between the field strengths and the point-source amplitudes, namely

$$F_P^{\mathrm{A}}(\boldsymbol{x}, t) = G^{\mathrm{A}}_{PI}(\boldsymbol{x}, \boldsymbol{x}', t) a_I^{\mathrm{A}} \tag{8.19}$$

and

$$F_I^{\mathrm{Z}}(\boldsymbol{x}, t) = G^{\mathrm{Z}}_{IP}(\boldsymbol{x}, \boldsymbol{x}'', t) a_P^{\mathrm{Z}}. \tag{8.20}$$

Application of equation (8.11) to the entire $\mathbb{R}^3$, keeping in mind that in view of the causality of the two wave fields the surface integral in equation (8.11) vanishes, and using the properties of Dirac distributions, yields

$$\delta^-_{IM} a_M^{\mathrm{A}} G^{\mathrm{Z}}_{IP}(\boldsymbol{x}', \boldsymbol{x}'', t) a_P^{\mathrm{Z}} = \delta^-_{PM} a_M^{\mathrm{Z}} G^{\mathrm{A}}_{PI}(\boldsymbol{x}'', \boldsymbol{x}', t) a_I^{\mathrm{A}}. \tag{8.21}$$

This result has to hold for arbitrary values of a_I^{A} and a_P^{Z}. Consequently,

$$\delta^-_{IM} G^{\mathrm{Z}}_{IP}(\boldsymbol{x}', \boldsymbol{x}'', t) = \delta^-_{IP} G^{\mathrm{A}}_{IM}(\boldsymbol{x}'', \boldsymbol{x}', t). \tag{8.22}$$

Equation (8.22) expresses the Green's function reciprocity property. It implies that upon interchanging the roles of observation point (first spatial argument) and source point (second spatial argument), the block diagonal parts of the Green's arrays change into their transposes, while the block off-diagonal parts change into the opposites of their transposes.

In the next section, we will discuss the implications of the reciprocity theorem of the time-convolution type for the remote sensing problem; in particular, we will show how the reciprocity theorem of the time-convolution type underlies the construction of the modelling operators $\boldsymbol{\Omega}^{\mathrm{T}}_{I';m}$ in equation (5.4) and $\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n}$ in equation (6.4).

9. Wave-field extrapolation formula, Huygens's principle, Oseen's extinction theorem

In various branches of remote sensing where properties of a configuration are to be reconstructed from measured data that are collected via wave-field probing techniques, certain wave-field extrapolation formulae form part of the reconstruction process. In general, these wave-field extrapolation formulae express the value of the field strength of the wave field in some bounded subdomain $\mathcal{D}$ of $\mathbb{R}^3$ in terms of the densities of the volume sources present in $\mathcal{D}$ and the values of the field strength at the (closed) boundary $\partial\mathcal{D}$ of $\mathcal{D}$. The wave-field extrapolation formulae arise from the wave-field reciprocity theorem (8.11) by choosing state A to represent the actual physical wave-field state and state Z to correspond to a (causal or anti-causal) point-source excited state in $\mathbb{R}^3$. At the quantities applying to the actual wave-field state, no superscript will be added. The quantities associated with the point-source excitation will be denoted by the superscript 'G' (Green's state).

(*a*) *Wave-field extrapolation formula of the time-convolution type*

The wave-field extrapolation formula of the time-convolution type is obtained by applying equation (8.11) to the domain $\mathcal{D}$. With

$$Q_M^{\mathrm{G}} = a_M \delta(\boldsymbol{x} - \boldsymbol{x}', t) \tag{9.1}$$

and

$$\delta^-_{PM} X^{\mathrm{G}}_{MI} = \delta^-_{IM} X_{MP}, \tag{9.2}$$

we obtain

$$\delta^-_{PM} F_P(\boldsymbol{x}', t) a_M \chi_{\mathcal{D}}(\boldsymbol{x}') = \int_{\mathcal{D}} \delta^-_{IM} F^{\mathrm{G}}_I(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q_M(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}) - \int_{\partial\mathcal{D}} \delta^-_{IM} N_{MP}[F_P(\boldsymbol{x}, t) \overset{(t)}{*} F^{\mathrm{G}}_I(\boldsymbol{x}, \boldsymbol{x}', t)]\,\mathrm{d}A(\boldsymbol{x}), \quad \text{for } \boldsymbol{x}' \in \mathbb{R}^3, \tag{9.3}$$

where $\chi_{\mathcal{D}}(\boldsymbol{x}')$ is the characteristic function of the domain $\mathcal{D}$, namely

$$\chi_{\mathcal{D}}(\boldsymbol{x}') = \{1, \tfrac{1}{2}, 0\}, \quad \text{for } \boldsymbol{x}' \in \{\mathcal{D}, \partial\mathcal{D}, \mathcal{D}'\}, \tag{9.4}$$

in which $\mathcal{D}'$ is the complement of $\mathcal{D} \cup \partial\mathcal{D}$ in $\mathbb{R}^3$. Introducing the Green's function for the medium in state G via

$$F^{\mathrm{G}}_I(\boldsymbol{x}, \boldsymbol{x}', t) = G^{\mathrm{G}}_{IM'}(\boldsymbol{x}, \boldsymbol{x}', t) a_{M'}, \tag{9.5}$$

substituting equation (9.5) in equation (9.3), and accounting for the fact that the result has to hold for arbitrary values of $a_{M'}$, it is found that

$$\delta^-_{PM'} F_P(\boldsymbol{x}', t) \chi_{\mathcal{D}}(\boldsymbol{x}') = \int_{\mathcal{D}} \delta^-_{IM} G^{\mathrm{G}}_{IM'}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q_M(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}) - \int_{\partial\mathcal{D}} \delta^-_{IM} N_{MP}[F_P(\boldsymbol{x}, t) \overset{(t)}{*} G^{\mathrm{G}}_{IM'}(\boldsymbol{x}, \boldsymbol{x}', t)]\,\mathrm{d}A(\boldsymbol{x}), \quad \text{for } \boldsymbol{x}' \in \mathbb{R}^3. \tag{9.6}$$

Using the reciprocity property of the Green's function (cf. equation (8.22))

$$\delta^{-}_{IM} G^{\mathrm{G}}_{IM'}(\boldsymbol{x}, \boldsymbol{x}', t) = \delta^{-}_{IM'} G_{IM}(\boldsymbol{x}', \boldsymbol{x}, t), \tag{9.7}$$

and the fact that each term in the resulting expression is a contraction with the signature array δ^{-}_{IM}, we finally obtain, after renaming the subscripts,

$$F_P(\boldsymbol{x}', t)\chi_{\mathcal{D}}(\boldsymbol{x}') = \int_{\mathcal{D}} G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t) \overset{(t)}{*} Q_I(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}) - \int_{\partial\mathcal{D}} G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t) \overset{(t)}{*} [N_{IP'} F_{P'}(\boldsymbol{x}, t)]\,\mathrm{d}A(\boldsymbol{x}), \quad \text{for } \boldsymbol{x}' \in \mathbb{R}^3. \tag{9.8}$$

For $\boldsymbol{x}' \in \mathcal{D}$, equation (9.8) expresses $F_P(\boldsymbol{x}', t)$ in terms of the volume density $Q_I(\boldsymbol{x}, t)$ of the sources present in $\mathcal{D}$, together with the boundary values of $N_{IP'}F_{P'}(\boldsymbol{x}', t)$ on $\partial\mathcal{D}$ and thus provides the basis for the wave-field extrapolation. The latter boundary values can be regarded as surface source distributions. When the domain $\mathcal{D}'$ is free from sources and $G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t)$ is taken to be the causal Green's function, the boundary integral in equation (9.8) vanishes. When, on the other hand, the domain $\mathcal{D}'$ is free from sources and $G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t)$ is taken to be the anti-causal Green's function, the boundary integral in equation (9.8) is a fixed function of time. For $\boldsymbol{x}' \in \partial\mathcal{D}$, equation (9.8) yields, upon contraction with N_{NP}, an interrelation between the (continuous) components $N_{NP}F_P$ on this surface. This relation plays a role in the boundary-integral equation formulation of direct scattering problems. For $\boldsymbol{x}' \in \mathcal{D}'$, equation (9.8) expresses *Oseen's extinction theorem* (Oseen 1915).

Altogether, equation (9.8) can be regarded as a mathematical formulation of *Huygens's principle* (Huygens 1690). Out of equation (9.8), we extract the linear operator

$$\mathsf{L}_{PI}[Q_I](\boldsymbol{x}', t) = \int_{\mathcal{D}} G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t) \overset{(t)}{*} Q_I(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}). \tag{9.9}$$

When the domain $\mathcal{D}'$ is free from sources, $G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t)$ is taken to be the causal Green's function, and $\boldsymbol{x}' \in \mathcal{D}$, we have

$$F_P = \mathsf{L}_{PI}[Q_I]. \tag{9.10}$$

Upon identifying this equation with equation (4.3), we introduce the operator $\mathsf{L}^{\mathrm{T}}_{PI}$ according to

$$F^{\mathrm{T}}_P = \mathsf{L}^{\mathrm{T}}_{PI'}[Q^{\mathrm{T}}_{I'}]. \tag{9.11}$$

Upon identifying equation (9.10) with equation (4.16), we introduce the operator $\mathsf{L}^{\mathrm{s}}_{PI';n}$ according to

$$F^{\mathrm{s}}_{P;n} = \mathsf{L}^{\mathrm{s}}_{PI';n}[Q^{\mathrm{s}}_{I';n}]. \tag{9.12}$$

(With the same notation, equation (4.6) takes the form

$$F^{\mathrm{i}}_{P;n} = \mathsf{L}^{\mathrm{s}}_{PI';n}[Q^{\mathrm{T}}_{I';n}], \tag{9.13}$$

and with equation (4.8) the field $F^{\mathrm{T}}_{P;n}$ appearing in equation (6.17) is obtained.)

Let us also introduce the detection operator $\mathcal{R}_{P;m}$ (cf. equation (4.4))

$$\mathcal{R}_{P;m}[F_P](t) = \int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} F_P(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}). \tag{9.14}$$

Then equations (4.1) and (5.4) combined lead to the representation

$$\boldsymbol{\Omega}^{\mathrm{T}}_{I';m} = \mathcal{R}_{P;m}\mathcal{L}^{\mathrm{T}}_{PI'}, \tag{9.15}$$

while equations (4.10) and (6.4) combined lead to the representation

$$\boldsymbol{\Omega}^{\mathrm{s}}_{I';m,n} = \mathcal{R}_{P;m}\mathcal{L}^{\mathrm{s}}_{PI';n}. \tag{9.16}$$

10. The wave-field reciprocity theorem of the time-correlation type

The wave-field reciprocity theorem of the time-correlation type is concerned with the interaction of two wave-field states that could be present in one and the same geometrical configuration in space, just as its counterpart of the time-convolution type is. Again, this geometrical configuration consists, for the moment, of the bounded domain $\mathcal{D} \subset \mathbb{R}^3$ with its piecewise smooth closed boundary $\partial\mathcal{D}$. As in § 8, the superscripts A and Z are used to indicate the two states. The wave-field quantities, source quantities, and constitutive parameters are subject to equations (8.1)–(8.6).

In the reciprocity theorem of the time-correlation type, the two-dimensional reciprocity *signature array* δ^{+}_{IM} occurs, the diagonal array with elements +1, i.e. the Kronecker array. An important property for reciprocity of the time-correlation type to hold is

$$\delta^{+}_{IM}\mathcal{D}_{MP} = \delta^{+}_{PM}\mathcal{D}_{MI}. \tag{10.1}$$

Equation (10.1) implies that $\mathcal{D}_{IP}$ is a symmetric array. The spatial differential operator arrays $\mathcal{D}_{IP}$ that occur in the wave phenomena that we consider do have this property (in addition to the one imposed by equation (8.7)). In the formulae to follow, we shall eliminate the signature array δ^{+}_{MP}, since that simplifies the expressions.

The local interaction quantity that occurs in the reciprocity theorem of the time-convolution type is

$$\mathcal{D}_{IP}[F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} F^{\mathrm{Z}}_I(\boldsymbol{x},-t)] = F^{\mathrm{Z}}_I(\boldsymbol{x},-t) \overset{(t)}{*} \mathcal{D}_{IP}F^{\mathrm{A}}_P(\boldsymbol{x},t) + F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} \mathcal{D}_{PI}F^{\mathrm{Z}}_I(\boldsymbol{x},-t),$$

hence

$$\begin{aligned}\mathcal{D}_{IP}&[F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} F^{\mathrm{Z}}_I(\boldsymbol{x},-t)]\\ &= F^{\mathrm{Z}}_I(\boldsymbol{x},-t) \overset{(t)}{*} Q^{\mathrm{A}}_I(\boldsymbol{x},t) + F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} Q^{\mathrm{Z}}_P(\boldsymbol{x},-t)\\ &\quad - F^{\mathrm{Z}}_I(\boldsymbol{x},-t) \overset{(t)}{*} \partial_t\Phi^{\mathrm{A}}_I(\boldsymbol{x},t) - F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} \partial_t\Phi^{\mathrm{Z}}_P(\boldsymbol{x},-t).\end{aligned} \tag{10.2}$$

Equation (10.2) expresses the *local reciprocity theorem of the time-correlation type*. In it, we have written the time correlations as time convolutions. For example, the time correlation of $F^{\mathrm{A}}_P(\boldsymbol{x},t)$ and $Q^{\mathrm{Z}}_P(\boldsymbol{x},t)$ (note the order of the two functions) is

$$\int_{t'=-\infty}^{\infty} F^{\mathrm{A}}_P(\boldsymbol{x},t')Q^{\mathrm{Z}}_P(\boldsymbol{x},t'-t)\,\mathrm{d}t' = F^{\mathrm{A}}_P(\boldsymbol{x},t) \overset{(t)}{*} Q^{\mathrm{Z}}_P(\boldsymbol{x},-t). \tag{10.3}$$

The reason for preferring a notation with convolutions over one with correlations is that convolutions (also compound ones) are invariant under a change in the order of the constituting operands.

The first two terms on the right-hand side of equation (10.2) are representative for the interaction of the two states via their volume source densities; this type of interaction vanishes in a source-free domain. The last two terms can, under the application of the constitutive relations (8.3) and (8.4), be rewritten as

$$\begin{aligned} &-[F_I^{\mathrm{Z}}(\boldsymbol{x},-t) \overset{(t)}{*} \partial_t \Phi_I^{\mathrm{A}}(\boldsymbol{x},t) + F_P^{\mathrm{A}}(\boldsymbol{x},t) \overset{(t)}{*} \partial_t \Phi_P^{\mathrm{Z}}(\boldsymbol{x},-t)] \\ &\qquad = -\partial_t\{F_I^{\mathrm{Z}}(\boldsymbol{x},-t) \overset{(t)}{*} [(X_{IP}^{\mathrm{A}}(\boldsymbol{x},t) - X_{PI}^{\mathrm{Z}}(\boldsymbol{x},-t)] \overset{(t)}{*} F_P^{\mathrm{A}}(\boldsymbol{x},t)\}, \end{aligned} \tag{10.4}$$

where we have used the property

$$\partial_t[X_{PI}^{\mathrm{Z}}(\boldsymbol{x},t) \overset{(-t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x},t)] = -\partial_t[X_{PI}^{\mathrm{Z}}(\boldsymbol{x},-t) \overset{(t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x},-t)]. \tag{10.5}$$

Hence, these terms describe the interaction of the two states via their contrast in constitutive properties upon the time-reversal operation $t \to -t$. This interaction vanishes if

$$X_{IP}^{\mathrm{A}}(\boldsymbol{x},t) = X_{PI}^{\mathrm{Z}}(\boldsymbol{x},-t), \tag{10.6}$$

i.e. in a (sub)domain where the constitutive properties of the medium in state Z are the *time-reverse adjoints* of the ones of the medium in state A. In a (sub)domain where the condition (10.6) holds for one and the same medium, the relevant medium is denoted as *time-reverse self-adjoint*. It is noted that the time-reverse adjoint of a causally reacting medium is an anti-causally reacting medium. If the media for the two states are both to be causal (which is the case if both states apply to a physical situation), they must be *instantaneously reacting*. If one of the two states, or both of them, are computational ones (which will be shown to occur in the realm of the application of error-minimization techniques), there is no objection against anti-causality or no causality at all.

By integrating equation (10.2) over the subdomains of $\mathcal{D}$ in which the constitutive parameters vary continuously with position, applying Gauss's integral theorem, adding the results, and using the interface boundary conditions (8.5) and (8.6), we obtain

$$\begin{aligned} &\int_{\partial\mathcal{D}} N_{IP}[F_P^{\mathrm{A}}(\boldsymbol{x},t) \overset{(t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x},-t)]\,\mathrm{d}A \\ &\quad = \int_{\mathcal{D}} [F_I^{\mathrm{Z}}(\boldsymbol{x},-t) \overset{(t)}{*} Q_I^{\mathrm{A}}(\boldsymbol{x},t) + F_P^{\mathrm{A}}(\boldsymbol{x},t) \overset{(t)}{*} Q_P^{\mathrm{Z}}(\boldsymbol{x},-t)]\,\mathrm{d}V \\ &\qquad - \int_{\mathcal{D}} [F_I^{\mathrm{Z}}(\boldsymbol{x},-t) \overset{(t)}{*} \partial_t \Phi_I^{\mathrm{A}}(\boldsymbol{x},t) + F_P^{\mathrm{A}}(\boldsymbol{x},t) \overset{(t)}{*} \partial_t \Phi_P^{\mathrm{Z}}(\boldsymbol{x},-t)]\,\mathrm{d}V. \end{aligned} \tag{10.7}$$

Equation (10.7) is the *global reciprocity theorem of the time-correlation type* for the domain $\mathcal{D}$.

In a number of applications, equation (8.7) will be applied to the entire $\mathbb{R}^3$. This is done by first applying the theorem to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin $\mathcal{O}$ of the chosen reference frame and radius Δ, and then taking the limit $\Delta \to \infty$. For some $\Delta > \Delta_0$, $\mathcal{S}(\mathcal{O}, \Delta)$ will be located in the domain $\mathcal{D}^\infty$, where the Green's functions are analytically known, and, in particular, their causal (or anti-causal) behaviour can be established (see § 2). If, now, the volume source distributions in the two states A and Z have bounded supports and one of the wave fields is causally related to the action of its sources, while the other wave field is anti-causally related to the action of its sources, the contribution from $\mathcal{S}(\mathcal{O}, \Delta)$ vanishes in the limit $\Delta \to \infty$. If, however, the two wave fields are both causally related or both anti-causally related to the action of their sources, the contribution from $\mathcal{S}(\mathcal{O}, \Delta)$ does not vanish, but remains a fixed function of time.

In case state A is chosen equal to state Z and the result is taken at $t = 0$, the reciprocity theorem of the time-correlation type (10.7) reduces to the balance of energy for the domain $\mathcal{D}$. The left-hand side of equation (10.7) learns that the quantity $N_{IP}[F_P(\boldsymbol{x}, t) \overset{(0)}{*} F_I(\boldsymbol{x}, -t)]$ is the area density of energy flow at $\partial\mathcal{D}$ away from $\mathcal{D}$.

In the next section, we will discuss the implications of the reciprocity theorem of the time-correlation type for the remote sensing problem; in particular, we will show how the reciprocity theorem of the time-correlation type underlies the construction of the adjoint operators $\boldsymbol{\Omega}^{\mathrm{T}*}_{I';m}$ in equation (5.8) and $\boldsymbol{\Omega}^{\mathrm{s}*}_{I';m,n}$ in equation (6.9).

11. Adjoint wave-field extrapolation

To derive a representation for the adjoint extrapolation operator L^*_{PI}, we make the following substitutions in the global reciprocity theorem of the time-correlation type (10.7). The actual state, state A, is causal, and its field is given by equation (9.10), i.e.

$$F_P(\boldsymbol{x}', t) = \mathsf{L}_{PI}[Q_I](\boldsymbol{x}', t) \tag{11.1}$$

(we omit the superscript A). Here,

$$\mathsf{L}_{PI}[Q_I](\boldsymbol{x}', t) = \int_{\mathcal{D}^Q} G_{PI}(\boldsymbol{x}', \boldsymbol{x}, t) \overset{(t)}{*} Q_I(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}), \tag{11.2}$$

with $\mathcal{D}^Q = \operatorname{supp} Q$ (bounded). The state Z is chosen such that its medium is the time-reverse adjoint of the medium of the actual state, hence the term in the reciprocity relation (9.10) containing the fluxes disappears. Let $\mathcal{D}^{\mathrm{R}} = \operatorname{supp} Q^{\mathrm{Z}}$ (bounded) and $\mathcal{D}^Q \cup \mathcal{D}^{\mathrm{R}} \subset \mathcal{D}$; then

$$\begin{aligned}\int_{\mathcal{D}} F_P(\boldsymbol{x}, t) \overset{(t)}{*} Q^{\mathrm{Z}}_P(\boldsymbol{x}, -t)\,\mathrm{d}V(\boldsymbol{x}) \\ = -\int_{\mathcal{D}} F^{\mathrm{Z}}_I(\boldsymbol{x}, -t) \overset{(t)}{*} Q_I(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}) \\ + \int_{\partial\mathcal{D}} N_{IP}(\boldsymbol{x})[F_P(\boldsymbol{x}, t) \overset{(t)}{*} F^{\mathrm{Z}}_I(\boldsymbol{x}, -t)]\,\mathrm{d}A(\boldsymbol{x}).\end{aligned} \tag{11.3}$$

We now substitute equation (11.2) into equation (11.1) and the result into the boundary integral of equation (11.3). Changing the order of integration then yields

$$\begin{aligned}\int_{\mathcal{D}} F_P(\boldsymbol{x}, t) \overset{(t)}{*} Q^{\mathrm{Z}}_P(\boldsymbol{x}, -t)\,\mathrm{d}V(\boldsymbol{x}) = -\int_{\mathcal{D}} F^{\mathrm{Z}}_I(\boldsymbol{x}, -t) \overset{(t)}{*} Q_I(\boldsymbol{x}, t)\,\mathrm{d}V(\boldsymbol{x}) \\ + \int_{\mathcal{D}}\int_{\partial\mathcal{D}} N_{IP}(\boldsymbol{x}')[G_{PI'}(\boldsymbol{x}', \boldsymbol{x}, t) \overset{(t)}{*} Q_{I'}(\boldsymbol{x}, t)] \overset{(t)}{*} F^{\mathrm{Z}}_I(\boldsymbol{x}', -t)\,\mathrm{d}A(\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x}).\end{aligned} \tag{11.4}$$

In the boundary integral, we reorder the time convolutions according to

$$\begin{aligned}\int_{\mathcal{D}}\int_{\partial\mathcal{D}} & N_{IP}(\boldsymbol{x}')[G_{PI'}(\boldsymbol{x}',\boldsymbol{x},t) \overset{(t)}{*} Q_{I'}(\boldsymbol{x},t)] \overset{(t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x}',-t)\,\mathrm{d}A(\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x})\\
&= \int_{\mathcal{D}}\int_{\partial\mathcal{D}} N_{IP}(\boldsymbol{x}')[G_{PI'}(\boldsymbol{x}',\boldsymbol{x},t) \overset{(t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x}',-t)] \overset{(t)}{*} Q_{I'}(\boldsymbol{x},t)\,\mathrm{d}A(\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x})\\
&= \int_{\mathcal{D}}\int_{\partial\mathcal{D}} N_{IP}(\boldsymbol{x}')[G_{PI'}(\boldsymbol{x}',\boldsymbol{x},-t) \overset{(-t)}{*} F_I^{\mathrm{Z}}(\boldsymbol{x}',t)] \overset{(t)}{*} Q_{I'}(\boldsymbol{x},t)\,\mathrm{d}A(\boldsymbol{x}')\mathrm{d}V(\boldsymbol{x})\\
&= \int_{\mathcal{D}}\left\{\int_{\partial\mathcal{D}} G_{PI}(\boldsymbol{x}',\boldsymbol{x},-t) \overset{(-t)}{*} [N_{I'P}(\boldsymbol{x}')F_{I'}^{\mathrm{Z}}(\boldsymbol{x}',t)]\,\mathrm{d}A(\boldsymbol{x}')\right\} \overset{(t)}{*} Q_I(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}),\end{aligned} \tag{11.5}$$

renaming subscripts. Forming an L^2 inner product on $\mathcal{D}\times\mathbb{R}$, the adjoint of L_{PI} is defined through

$$\int_{\mathcal{D}} \mathsf{L}_{PI}[Q_I](\boldsymbol{x},t) \overset{(0)}{*} Q_P^{\mathrm{Z}}(\boldsymbol{x},-t)\,\mathrm{d}V(\boldsymbol{x}) = \int_{\mathcal{D}} Q_I(\boldsymbol{x},t) \overset{(0)}{*} \mathsf{L}^*_{PI}[Q_P^{\mathrm{Z}}](\boldsymbol{x},-t)\,\mathrm{d}V(\boldsymbol{x}). \tag{11.6}$$

Substituting equation (11.5) into equation (11.4) then leads to the identification

$$\mathsf{L}^*_{PI}[Q_P^{\mathrm{Z}}](\boldsymbol{x},t) = -F_I^{\mathrm{Z}}(\boldsymbol{x},t) + \int_{\partial\mathcal{D}} G_{PI}(\boldsymbol{x}',\boldsymbol{x},-t) \overset{(t)}{*} [N_{I'P}(\boldsymbol{x}')F_{I'}^{\mathrm{Z}}(\boldsymbol{x}',t)]\,\mathrm{d}A(\boldsymbol{x}'). \tag{11.7}$$

On the other hand, following the derivation of wave-field extrapolation given by equation (9.8), with the reciprocity theorem of the time-convolution type replaced by the reciprocity theorem of the time-correlation type, leads to the wave-field 'reverse-time' extrapolation

$$\begin{aligned}-F_I^{\mathrm{Z}}(\boldsymbol{x}',t)\chi_{\mathcal{D}}(\boldsymbol{x}') &= \int_{\mathcal{D}} \delta^-_{PM}\delta^-_{NI}G^{\mathrm{Z}}_{NM}(\boldsymbol{x}',\boldsymbol{x},-t) \overset{(t)}{*} Q_P^{\mathrm{Z}}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x})\\
&\quad - \int_{\partial\mathcal{D}} \delta^-_{PM}\delta^-_{NI}G^{\mathrm{Z}}_{NM}(\boldsymbol{x}',\boldsymbol{x},-t) \overset{(t)}{*} [N_{PP'}(\boldsymbol{x})F^{\mathrm{Z}}_{P'}(\boldsymbol{x},t)]\,\mathrm{d}A(\boldsymbol{x}),\\
&\qquad \text{for } \boldsymbol{x}'\in\mathbb{R}^3,\end{aligned} \tag{11.8}$$

where G^{Z}_{NM} is the array of causal Green's functions in the medium of state Z, appearing in a time-reversed fashion. In view of the reciprocity properties (8.22) of the causal Green's functions, we have

$$\delta^-_{PM}\delta^-_{NI}G^{\mathrm{Z}}_{NM}(\boldsymbol{x}',\boldsymbol{x},-t) = G_{PI}(\boldsymbol{x},\boldsymbol{x}',-t). \tag{11.9}$$

Substituting equation (11.9) into equation (11.8) and the result into equation (11.7), we find that the boundary integrals cancel out whence

$$\mathsf{L}^*_{PI}[Q_P^{\mathrm{Z}}](\boldsymbol{x},t) = \int_{\mathcal{D}} G_{PI}(\boldsymbol{x},\boldsymbol{x}',-t) \overset{(t)}{*} Q_P^{\mathrm{Z}}(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}), \tag{11.10}$$

which manifests itself as 'back' propagation.

With the inner product (5.1), the adjoint $\mathsf{R}^{*}_{P;m}$ of the detection operator in (4.1) is defined through

$$\langle \mathsf{R}_{P;m}[F^{\mathrm{T}}_{P}], d^{\mathrm{T}}_{m}\rangle_{\mathrm{d}} = \langle F^{\mathrm{T}}_{P}, \mathsf{R}^{*}_{P;m}[d^{\mathrm{T}}_{m}]\rangle_{\mathrm{d}}. \tag{11.11}$$

Upon substituting equation (4.1) into this equality, we find that

$$\begin{aligned}\mathsf{R}^{*}_{P;m}[d^{\mathrm{T}}_{m}](\boldsymbol{x},t) &= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{T}}_{m,m'} R_{P;m'}(\boldsymbol{x},-t) \overset{(t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{T}}_{m})(t)\\ &= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{T}}_{m,m'} R_{P;m'}(\boldsymbol{x},t) \overset{(-t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{T}}_{m})(-t).\end{aligned} \tag{11.12}$$

With the aid of equation (11.10), we now find the adjoint of equation (9.15) and recover expression (5.8):

$$\boldsymbol{\Omega}^{\mathrm{T}*}_{I';m} = \mathsf{L}^{\mathrm{T}*}_{PI'} \mathsf{R}^{*}_{P;m}. \tag{11.13}$$

With the inner product (6.1), the adjoint $\mathsf{R}^{*}_{P;m}$ of the detection operator in (4.4) follows as

$$\begin{aligned}\mathsf{R}^{*}_{P;m}[d^{\mathrm{s}}_{m,n}](\boldsymbol{x},t) &= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{s}}_{m,m';n} R_{P;m'}(\boldsymbol{x},-t) \overset{(t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{s}}_{m,n})(t)\\ &= \sum_{m,m'=1}^{N^{\mathrm{R}}} w^{\mathrm{s}}_{m,m';n} R_{P;m'}(\boldsymbol{x},t) \overset{(-t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{s}}_{m,n})(-t).\end{aligned} \tag{11.14}$$

With the aid of equation (11.10), we now find the adjoint of equation (9.16) and recover expression (6.9):

$$\boldsymbol{\Omega}^{\mathrm{s}*}_{I';m,n} = \mathsf{L}^{\mathrm{s}*}_{PI';n} \mathsf{R}^{*}_{P;m}. \tag{11.15}$$

12. The first-order low-contrast approximation in inverse scattering

In wave-scattering problems—both direct and inverse ones—the first-order low-contrast approximation or *Rayleigh–Gans–Born approximation* often serves to arrive at a first impression of what features can be expected to show up in the relevant scattering problem. In the first-order low-contrast approximation, the term $F^{\mathrm{s}}_{P;n}$ in the right-hand side of equation (4.14) is neglected with respect to the term $F^{\mathrm{i}}_{P;n}$ as being in the next order of the contrast in constitutive parameters. The resulting approximate expression for $Q^{\mathrm{s}}_{I;n}$ is substituted in equation (4.16), which, upon substitution in equation (4.10), leads to

$$d^{\mathrm{s}}_{m,n} \overset{\mathrm{B}}{=} \boldsymbol{\Omega}^{\mathrm{B}}_{I'P';m,n}[C^{X}_{I'P'}], \tag{12.1}$$

in which $\overset{\mathrm{B}}{=}$ is indicative of the Rayleigh–Gans–Born approximation and the operator

$$\boldsymbol{\Omega}^{\mathrm{B}}_{I'P';m,n} : C^{X}_{I'P'}(\boldsymbol{x}',t) \longmapsto d^{\mathrm{s}}_{m,n}(t) \tag{12.2}$$

is defined through

$$\boldsymbol{\Omega}^{\mathrm{B}}_{I'P';m,n}[C^X_{I'P'}](t) = -\int_{\mathcal{D}^{\mathrm{R}}_m} R_{P;m}(\boldsymbol{x},t) \overset{(t)}{*} \left[\int_{\mathcal{D}^{\mathrm{s}}} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t) \overset{(t)}{*} \partial_t C^X_{I'P'}(\boldsymbol{x}',t) \overset{(t)}{*} F^{\mathrm{i}}_{P';n}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}')\right]\mathrm{d}V(\boldsymbol{x}),$$

$$\text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}. \quad (12.3)$$

For the inner product in the scattered data space, which we denote by $\langle\cdot,\cdot\rangle_{\mathrm{B}}$, we take the expression

$$\langle u^{\mathrm{s}}_{m,n}, v^{\mathrm{s}}_{m';n'}\rangle_{\mathrm{B}} = \int_{\mathcal{T}}\left[\sum_{m,m'=1}^{N^{\mathrm{R}}}\sum_{n,n'=1}^{N^{\mathrm{T}}} w^{\mathrm{B}}_{m,m';n,n'} u^{\mathrm{s}}_{m,n}(t) v^{\mathrm{s}}_{m',n'}(t)\right]\mathrm{d}t, \quad (12.4)$$

where $\mathcal{T}$ is the time window used for the reconstruction procedure, while

$$\{w^{\mathrm{B}}_{m,m';n,n'};\ m,m' = 1,\dots,N^{\mathrm{R}};\ n,n' = 1,\dots,N^{\mathrm{T}}\}$$

is a set of weighting coefficients. This set has the symmetry property

$$w^{\mathrm{B}}_{m,m';n,n'} = w^{\mathrm{B}}_{m',m;n',n},$$

while the expression $\|u^{\mathrm{s}}_{m,n}\|_{\mathrm{B}}$, defined through

$$\|u^{\mathrm{s}}_{m,n}\|^2_{\mathrm{B}} = \langle u^{\mathrm{s}}_{m,n}, u^{\mathrm{s}}_{m',n'}\rangle_{\mathrm{B}}, \quad (12.5)$$

is assumed to have the standard properties of a norm. The weighting coefficients are representative of the algorithm that is used to combine the results from the different irradiating sources and the different receivers to arrive at a value of the contrast in constitutive parameters that is independent of these sources and receivers. From equations (12.3) and (12.4), the inner product in the model space (of contrasts in constitutive properties) that allows for the construction of the operator adjoint to the direct modelling operator follows. Denoting the latter inner product by $\langle\cdot,\cdot\rangle_X$, we obtain

$$\langle U^X_{I'P'}, V^X_{I'P'}\rangle_X = \int_{t=-\infty}^{\infty}\left[\int_{\mathcal{D}^{\mathrm{s}}} U^X_{I'P'}(\boldsymbol{x}',t) V^X_{I'P'}(\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}')\right]\mathrm{d}t. \quad (12.6)$$

This inner product obviously satisfies the requirements of Appendix B. The corresponding norm $\|U^X_{I'P'}\|_X$ follows from

$$\|U^X_{I'P'}\|^2_X = \langle U^X_{I'P'}, U^X_{I'P'}\rangle_X. \quad (12.7)$$

The operator

$$\boldsymbol{\Omega}^{\mathrm{B}*}_{I'P';m,n} : d^{\mathrm{s}}_{m,n}(t) \longmapsto C^X_{I'P'}(\boldsymbol{x}',t), \quad (12.8)$$

adjoint to $\boldsymbol{\Omega}^{\mathrm{B}}_{I'P';m,n}$, is then found to be

$$\boldsymbol{\Omega}^{\mathrm{B}*}_{I'P';m,n}[d^{\mathrm{s}}_{m,n}] = \sum_{m=1}^{N^{\mathrm{R}}}\sum_{n=1}^{N^{\mathrm{T}}} \boldsymbol{\Phi}^{\mathrm{B}}_{I'P';m,n}(\boldsymbol{x}',-t) \overset{(t)}{*} (\chi_{\mathcal{T}} d^{\mathrm{s}}_{m,n})(t), \quad (12.9)$$

in which

$$\boldsymbol{\Phi}^{\mathrm{B}}_{I'P';m,n}(\boldsymbol{x}',t) = -\partial_t \sum_{m'=1}^{N^{\mathrm{R}}} \sum_{n'=1}^{N^{\mathrm{T}}} w^{\mathrm{B}}_{m,m';n,n'} \times \left[\int_{\mathcal{D}^{\mathrm{R}}_{m'}} R_{P;m'}(\boldsymbol{x},t) \overset{(t)}{*} G^{\mathrm{b}}_{PI'}(\boldsymbol{x},\boldsymbol{x}',t)\,\mathrm{d}V(\boldsymbol{x}) \overset{(t)}{*} F^{\mathrm{i}}_{P';n'}(\boldsymbol{x}',t) \right],$$
$$\text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}, \qquad (12.10)$$

and $\chi_{\mathcal{T}}(t)$ is the characteristic function of the set $\mathcal{T}$. Equation (12.9) represents the process of 'imaging'.

With these preliminaries, the iterative solution technique of Appendix B can be used to reconstruct, from the scattered dataset, via the image, the contrast in constitutive parameters (in the first-order low-contrast approximation). In this procedure, the successive data residuals are

$$r^{\mathrm{s};[i]}_{m,n} = d^{\mathrm{s}}_{m,n} - \boldsymbol{\Omega}^{\mathrm{B}}_{I'P';m,n}[C^{X;[i]}_{I'P'}],$$
$$\text{for } m = 1,\dots,N^{\mathrm{R}}, \quad n = 1,\dots,N^{\mathrm{T}}, \quad i = 0,1,2,\dots, \qquad (12.11)$$

with the starting value $C^{X;[0]}_{I'P'} = 0$, while

$$\epsilon^{[i]}_{\mathrm{B}} = \|r^{\mathrm{s};[i]}_{m,n}\|^2_{\mathrm{B}}, \qquad \text{for } i = 0,1,2,\dots, \qquad (12.12)$$

and

$$\bar{\epsilon}^{[i]}_{\mathrm{B}} = \frac{\epsilon^{[i]}_{\mathrm{B}}}{\|d^{\mathrm{s}}_{m,n}\|^2_{\mathrm{B}}}, \qquad \text{for } i = 0,1,2,\dots, \qquad (12.13)$$

are the relevant errors and normalized errors.

This type of inversion, in the high-frequency approximation, leads to the introduction of discrete generalized Radon transforms and their extensions. For a detailed discussion, see De Hoop & Brandsberg-Dahl (2000) and De Hoop & Spencer (1996).

13. Preconditioning methods

In Appendix C, the general aspects of the method of preconditioning applied to a linear operator equation are outlined. The method finds application, and is sometimes an absolute necessity, in all those cases where the *scale* of the scattering configuration is *large*. As an example, we again mention the area of seismic exploration for fossil energy resources or fossil energy reservoir evaluation and monitoring. Due to the geometry of the subsurface of the Earth, here, a method of preconditioning offers itself in a natural fashion, namely the geometrical ray approximation to the wave propagation in question. This approximation has the advantage that it readily gives a geometrical representation of the wave phenomena involved, while, through the generalized Radon transform associated with point sources and point receivers, the inversion via the ray-approximated occurring Green's functions can rapidly be carried out. The generalized Radon transform and its discretization are covered in a number of papers (De Hoop *et al.* 1999; De Hoop & Brandsberg-Dahl 2000; De Hoop & Spencer 1996). The first step of the associated preconditioning method consists of replacing the Green's functions by their geometric ray approximations, using these approximations to construct the 'approximate' operators needed for the inversion, and applying the technique of Appendix C, in which the approximating operator $\tilde{\boldsymbol{\Omega}}$ is taken to be the geometric ray approximation to $\boldsymbol{\Omega}$.

14. Conclusion

A unified space-time domain, multisource/multireceiver approach has been presented to construct models that fit, up to a certain mismatch, the observed data in inverse-source, inverse-scattering and inverse-wave-field-transduction problems whose interrogating agents are acoustic, elastodynamic or electromagnetic wave fields. An operator notation has been introduced that reveals the basic structure of the computational algorithms that are to perform the reconstruction of a 'model' from the 'data', the reconstruction being based on the minimization of an appropriate mismatch in the equality signs in the relevant data and object equations. The media in which the interrogating wave fields travel are of such a type that wave-field reciprocity (both of the time-convolution and the time-correlation types) applies. This feature can be exploited to carry out a sensitivity analysis of a chosen configuration of sources and/or receivers as to their ability to contribute to the reconstruction problem at hand; while the replacement of certain expressions related to a particular domain in space by their reciprocal counterparts related to a domain elsewhere in space can have computational or interpretational advantages, or both. The formulation of the inverse-source problem and the inverse-wave-field-transduction problem leads to linear operator equations. The inverse-scattering problem, although it is, in its ultimate form, nonlinear in the contrast in constitutive parameters to be reconstructed, is formulated as a nested combination of two linear operator equations to be satisfied (two-step approach). For quite a number of practical inversion problems, the computational effort required to carry out the relevant iterations sufficiently often to reach an acceptable level of mismatch is prohibitively big. For this reason, preconditioning methods have, also in the general framework, been discussed. The reconstructed models will, in general, depend on the measuring set-up employed and the choice of the mismatch criterion in the reconstruction algorithms. This, however, should not be the case. At present, we have no way of deciding why models resulting from one measuring set-up and one mismatch criterion should be closer to physical reality than models resulting from a different measuring set-up and/or a different mismatch criterion. As to this aspect, differential semblance analysis (Symes 1992; Kern & Symes 1984; Gockenbach *et al.* 1995) could provide a way out. This is a subject for further research.

Appendix A. Field, flux and constitutive arrays for acoustic, elastodynamic and electromagnetic wave fields

In this appendix, we list the field quantities, flux quantities and constitutive parameters that occur in acoustic, elastodynamic and electromagnetic wave fields in the notation used in the main text. The elements of the different arrays are specified in the subscript notation (with lower-case Latin subscripts) for Cartesian tensors. The components along the x_1-, x_2- and x_3-axes of an orthogonal Cartesian reference frame are denoted by the subscripts 1, 2 and 3, respectively. The notation is close to the one developed by Woodhouse (1974), and refined by De Hoop (1992).

(*a*) *Acoustic wave field*

In acoustic wave theory, the field quantities are the acoustic pressure p and the particle velocity v_r, arranged in F_P according to

$$F = [p, v_1, v_2, v_3]^{\mathrm{T}}; \tag{A 1}$$

the flux quantities are the cubic dilatation θ and the mass flow density Φ_k, arranged in Φ_I according to

$$\Phi = [-\theta, \Phi_1, \Phi_2, \Phi_3]^{\mathrm{T}}. \tag{A 2}$$

From this, the arrangement of the constitutive parameters in X_{IP} (for media with relaxation) or M_{IP} (for lossless media) follows. The diagonal blocks of the array of constitutive parameters reflect the compressibility and inertia properties, respectively; off-diagonal blocks are representative of 'exotic' effects. The corresponding spatial differentiation array follows as

$$D = \begin{bmatrix} 0 & \partial_1 & \partial_2 & \partial_3 \\ \partial_1 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_3 & 0 & 0 & 0 \end{bmatrix}. \tag{A 3}$$

The source quantities are the volume density of injection rate q and the volume density of force f_k; they are arranged in Q_I according to

$$Q = [q, f_1, f_2, f_3]^{\mathrm{T}}. \tag{A 4}$$

The signature array δ^-_{IM} that occurs in the reciprocity theorem of the time-convolution type is

$$\delta^- = \mathrm{diag}[1, -1, -1, -1]. \tag{A 5}$$

For lossless 'conventional' fluids, we have

$$M = \begin{bmatrix} \kappa & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix}, \tag{A 6}$$

where ρ is the volume density of mass, and κ is the compressibility.

(*b*) *Elastodynamic wave field*

In elastodynamic wave theory, the field quantities are the dynamic stress τ_{pq} (with $\tau_{pq} = \tau_{qp}$) and the particle velocity v_r, arranged in F_P according to

$$F = [v_1, v_2, v_3, -\tau_{11}, -\tau_{12}, -\tau_{13}, -\tau_{21}, -\tau_{22}, -\tau_{23}, -\tau_{31}, -\tau_{32}, -\tau_{33}]^{\mathrm{T}}; \tag{A 7}$$

the flux quantities are the deformation e_{ij} (with $e_{ij} = e_{ji}$) and the mass flow density Φ_k, arranged in Φ_I according to

$$\Phi = [\Phi_1, \Phi_2, \Phi_3, -e_{11}, -e_{12}, -e_{13}, -e_{21}, -e_{22}, -e_{23}, -e_{31}, -e_{32}, -e_{33}]^{\mathrm{T}}. \tag{A 8}$$

From this, the arrangement of the constitutive parameters in X_{IP} (for media with relaxation) or M_{IP} (for lossless media) follows. The diagonal blocks of the array of constitutive parameters reflect the compliance and inertia properties, respectively; off-diagonal blocks are representative of 'exotic' effects. The corresponding spatial differentiation array follows as

$$\mathsf{D}_{IP} = \tfrac{1}{2}(\mathsf{D}^{\mathrm{row/col}}_{IP} + \mathsf{D}^{\mathrm{diag}}_{IP}), \tag{A 9}$$

in which

$$\mathsf{D}^{\mathrm{row/col}} = \begin{bmatrix} \mathbf{0} & \begin{matrix} \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 \end{matrix} \\ \begin{matrix} \partial_1 & 0 & 0 \\ \partial_2 & 0 & 0 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & \partial_2 & 0 \\ 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & 0 & \partial_3 \end{matrix} & \mathbf{0} \end{bmatrix} \tag{A 10}$$

and

$$\mathsf{D}^{\mathrm{diag}} = \begin{bmatrix} \mathbf{0} & \begin{matrix} \partial_1 & 0 & 0 & \partial_2 & 0 & 0 & \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 & 0 & \partial_2 & 0 & 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 & \partial_2 & 0 & 0 & \partial_3 \end{matrix} \\ \begin{matrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_3 & 0 \\ 0 & 0 & \partial_3 \end{matrix} & \mathbf{0} \end{bmatrix}. \tag{A 11}$$

The source quantities are the volume density of force f_k and the volume density of induced deformation rate h_{ij}; they are arranged in Q_I according to

$$Q = [f_1, f_2, f_3, h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33}]^{\mathrm{T}}. \tag{A 12}$$

The signature array δ^-_{IM} that occurs in the reciprocity theorem of the time-convolution type is

$$\delta^- = \mathrm{diag}[1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1]. \tag{A 13}$$

For lossless 'conventional' solids, we have

$$M = \begin{bmatrix} -\rho \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\ \boldsymbol{0} & \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{0} & \boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{A 14}$$

where

$$(\boldsymbol{S}_{jl})_{ik} = s_{ijkl} \tag{A 15}$$

relates to the compliance tensor. From the symmetries of the compliance tensor, we obtain

$$\boldsymbol{S}_{kj} = \boldsymbol{S}^{\mathrm{T}}_{jk}. \tag{A 16}$$

(*c*) *Electromagnetic wave field*

In electromagnetic wave theory, the field quantities are the electric field strength E_r and the magnetic field strength H_p, arranged in F_P according to

$$F = [E_1, E_2, E_3, H_1, H_2, H_3]^{\mathrm{T}}; \tag{A 17}$$

the flux quantities are the electric flux density D_k† and the magnetic flux density B_j, arranged in Φ_I according to

$$\Phi = [D_1, D_2, D_3, B_1, B_2, B_3]^{\mathrm{T}}. \tag{A 18}$$

From this, the arrangement of the constitutive parameters in X_{IP} (for media with relaxation) or M_{IP} (for lossless media) follows. The diagonal blocks of the array of constitutive parameters reflect the electric and magnetic properties, respectively; off-diagonal blocks are representative of 'exotic' effects (like the magnetoelectric effect in chiral media). The corresponding spatial differentiation array follows as

$$\mathsf{D} = \left[\begin{array}{ccc|ccc} & & & 0 & \partial_3 & -\partial_2 \\ & \boldsymbol{0} & & \partial_3 & 0 & -\partial_1 \\ & & & -\partial_2 & \partial_1 & 0 \\ 0 & -\partial_3 & \partial_2 & & & \\ -\partial_3 & 0 & \partial_1 & & \boldsymbol{0} & \\ \partial_2 & -\partial_1 & 0 & & & \end{array}\right]. \tag{A 19}$$

† In the Maxwell equations in matter, we have made the assignment $D_k := D_k + H \overset{(t)}{*} j_k$, where j_k denotes the conductive current density and H is the Heaviside function in time.

The source quantities are the volume density of impressed electric current J_k and the volume density of impressed magnetic current K_j; they are arranged in Q_I according to

$$Q = [-J_1, -J_2, -J_3, -K_1, -K_2, -K_3]^{\mathrm{T}}. \tag{A 20}$$

Finally, the signature array δ^-_{IM} that occurs in the reciprocity theorem of the time-convolution type is

$$\delta^- = \operatorname{diag}[1, 1, 1, -1, -1, -1]. \tag{A 21}$$

For lossless 'conventional' matter we have

$$M = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & & & \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & & \mathbf{0} & \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & & & \\ & & & \mu_{11} & \mu_{12} & \mu_{13} \\ & \mathbf{0} & & \mu_{21} & \mu_{22} & \mu_{23} \\ & & & \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}, \tag{A 22}$$

where ε_{ij} denotes the permittivity and μ_{pq} denotes permeability.

Appendix B. Iterative solution to a linear operator equation

In this appendix we construct an iterative solution to a linear operator equation of the type that occurs in inverse-source and inverse-scattering problems. The solution procedure is induced by an appropriate error or mismatch criterion. A rather general notation is used that suffices for the main aspects of the method. Let the operator equation under consideration be given by

$$\boldsymbol{d} = \boldsymbol{\Omega}[\boldsymbol{m}], \tag{B 1}$$

where the *operator*

$$\boldsymbol{\Omega} : \boldsymbol{m} \longmapsto \boldsymbol{d}, \tag{B 2}$$

which maps the *model* $\boldsymbol{m}$ onto the *data* $\boldsymbol{d}$, is assumed to be homogeneous and linear. For our applications, $\boldsymbol{d}$ and $\boldsymbol{m}$ are arrays with discrete or continuous 'subscripts' whose ranges may be different. The data are elements of the *data space*, i.e. a linear space in which an inner product, to be denoted by $\langle \cdot, \cdot \rangle_{\mathrm{d}}$, that has the symmetry property

$$\langle \boldsymbol{d}_1, \boldsymbol{d}_2 \rangle_{\mathrm{d}} = \langle \boldsymbol{d}_2, \boldsymbol{d}_1 \rangle_{\mathrm{d}} \tag{B 3}$$

is defined. Further, we write

$$\|\boldsymbol{d}\|^2_{\mathrm{d}} = \langle \boldsymbol{d}, \boldsymbol{d} \rangle_{\mathrm{d}}, \tag{B 4}$$

where $\|\cdot\|_{\mathrm{d}}$ has the standard properties of a norm. The admissible models are elements of a *model space*, i.e. a linear space in which an inner product, to be denoted by $\langle \cdot, \cdot \rangle_{\mathrm{m}}$, is defined that has the symmetry property

$$\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle_{\mathrm{m}} = \langle \boldsymbol{m}_2, \boldsymbol{m}_1 \rangle_{\mathrm{m}}. \tag{B 5}$$

Further, we write

$$\|\boldsymbol{m}\|_{\mathrm{m}}^2 = \langle \boldsymbol{m}, \boldsymbol{m} \rangle_{\mathrm{m}}, \tag{B 6}$$

where $\|\cdot\|_{\mathrm{m}}$ has the standard properties of a norm.

The iterative procedure to 'solve' equation (B 1) goes as follows. Let $\boldsymbol{m}^{[i]}$ be an 'approximation' to $\boldsymbol{m}$ and define the corresponding *residual* in the operator equation (B 1) as

$$\boldsymbol{r}^{[i]} = \boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}^{[i]}]. \tag{B 7}$$

As the *mismatch* or *error* in the satisfaction of the equality sign in equation (B 1) we introduce the quantity

$$\epsilon^{[i]} = \|\boldsymbol{r}^{[i]}\|_{\mathrm{d}}^2. \tag{B 8}$$

The aim is now to construct an *update* $\boldsymbol{m}^{[i+1]}$ to $\boldsymbol{m}^{[i]}$, such that $\epsilon^{[i+1]} < \epsilon^{[i]}$ (*improvement condition*). To construct such an update, let

$$\boldsymbol{m}^{[i+1]} = \boldsymbol{m}^{[i]} + \boldsymbol{\delta m}^{[i]}. \tag{B 9}$$

Then,

$$\boldsymbol{r}^{[i+1]} = \boldsymbol{r}^{[i]} - \boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}]. \tag{B 10}$$

Substitution of this relation in the expression for $\epsilon^{[i+1]} - \epsilon^{[i]}$ leads to

$$\epsilon^{[i+1]} - \epsilon^{[i]} = -2\langle \boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}], \boldsymbol{r}^{[i]} \rangle_{\mathrm{d}} + \|\boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}]\|_{\mathrm{d}}^2. \tag{B 11}$$

Hence, a necessary and sufficient condition for improvement is

$$-2\langle \boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}], \boldsymbol{r}^{[i]} \rangle_{\mathrm{d}} + \|\boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}]\|_{\mathrm{d}}^2 < 0. \tag{B 12}$$

Now, the last term on the right-hand side is positive for any $\boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}] \neq \boldsymbol{0}$. In order to meet the condition for improvement, the inner product in the first term on the left-hand side must be sufficiently positive for the chosen value of $\boldsymbol{\delta m}^{[i]}$. Now assume that a homogeneous, linear operator $\boldsymbol{\Omega}^*$ *adjoint* to $\boldsymbol{\Omega}$ exists such that, for all $\boldsymbol{\delta m}^{[i]}$ and $\boldsymbol{r}^{[i]}$, the relation

$$\langle \boldsymbol{\Omega}[\boldsymbol{\delta m}^{[i]}], \boldsymbol{r}^{[i]} \rangle_{\mathrm{d}} = \langle \boldsymbol{\delta m}^{[i]}, \boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}] \rangle_{\mathrm{m}} \tag{B 13}$$

holds. Then, a sufficient condition for the requirement expressed by equation (B 8) to be met is

$$\boldsymbol{\delta m}^{[i]} = \alpha^{[i]} \boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}], \tag{B 14}$$

with $\alpha^{[i]} > 0$ suitably chosen. Substitution of equation (B 14) into equation (B 11) yields

$$\epsilon^{[i+1]} - \epsilon^{[i]} = -2\alpha^{[i]} B^{[i]} + \alpha^{[i]2} A^{[i]}, \tag{B 15}$$

where

$$B^{[i]} = \|\boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}]\|_{\mathrm{m}}^2 \tag{B 16}$$

and

$$A^{[i]} = \|\boldsymbol{\Omega}\boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}]\|_{\mathrm{d}}^2. \tag{B 17}$$

The maximum decrease in the error is, from equation (B 15), attained for

$$\alpha^{[i]} = \frac{B^{[i]}}{A^{[i]}}. \tag{B 18}$$

(Note that, indeed, $\alpha^{[i]} > 0$.) This choice leads to the *steepest descent* update

$$\boldsymbol{\delta m}^{[i]} = \frac{B^{[i]}}{A^{[i]}}\boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}], \tag{B 19}$$

and the steepest descent in error

$$\epsilon^{[i+1]} - \epsilon^{[i]} = -\frac{[B^{[i]}]^2}{A^{[i]}}. \tag{B 20}$$

In practice, the computations are carried out with the *normalized error*

$$\bar{\epsilon}^{[i]} = \frac{\epsilon^{[i]}}{\|\boldsymbol{d}\|_{\mathrm{d}}^2}. \tag{B 21}$$

This normalized error has the properties $\bar{\epsilon}^{[i]} = 1$ for $\boldsymbol{r}^{[i]} = \boldsymbol{d}$ (which corresponds to $\boldsymbol{m}^{[i]} = \boldsymbol{0}$) and $\bar{\epsilon}^{[i]} = 0$ for $\boldsymbol{r}^{[i]} = \boldsymbol{0}$ (i.e. for perfect data fit). The steepest descent in the normalized error is

$$\bar{\epsilon}^{[i+1]} - \bar{\epsilon}^{[i]} = -\frac{[B^{[i]}]^2}{A^{[i]}\|\boldsymbol{d}\|_{\mathrm{d}}^2}. \tag{B 22}$$

These results form the basis for an iterative procedure where successive updates of the model lead to successive decreases in the mismatch between the given data and the ones generated by the updates of the model.

As the starting value, we take $\boldsymbol{m}^{[0]} = \boldsymbol{0}$, which entails $\bar{\epsilon}^{[0]} = 1$. With

$$\bar{\epsilon}^{[i+1]} = \bar{\epsilon}^{[i]} - \frac{[B^{[i]}]^2}{A^{[i]}\|\boldsymbol{d}\|_{\mathrm{d}}^2}, \quad \text{for } i = 0, 1, 2, \ldots, \tag{B 23}$$

we generate a sequence $\{\bar{\epsilon}^{[i]};\ i = 0, 1, 2, \ldots\}$. This sequence is positive and decreasing, and, hence, converges to a limit as $i \to \infty$. Denoting this limit by $\bar{\epsilon}^{[\infty]}$, we have

$$\bar{\epsilon}^{[\infty]} = 1 - \sum_{i=0}^{\infty} \frac{[B^{[i]}]^2}{A^{[i]}\|\boldsymbol{d}\|_{\mathrm{d}}^2}. \tag{B 24}$$

Note that, in general, $\bar{\epsilon}^{[\infty]} > 0$.

Another consequence of the convergence of the series on the right-hand side of equation (B 24) is that $B^{[\infty]} = 0$. Therefore, $\boldsymbol{\Omega}^*[\boldsymbol{r}^{[\infty]}] = \boldsymbol{0}$ on account of equation (B 16). Use of equation (B 7) for $i \to \infty$ then leads to

$$\boldsymbol{0} = \boldsymbol{\Omega}^*[\boldsymbol{d}] - \boldsymbol{\Omega}^*\boldsymbol{\Omega}[\boldsymbol{m}^{[\infty]}]. \tag{B 25}$$

In this context, $\boldsymbol{\Omega}^*\boldsymbol{\Omega}$ is the *normal* operator. Therefore, the model reconstructed by the iterative procedure is

$$\boldsymbol{m}^{[\infty]} = (\boldsymbol{\Omega}^*\boldsymbol{\Omega})^{-1}\boldsymbol{\Omega}^*[\boldsymbol{d}]. \tag{B 26}$$

In cases where the inverse $(\boldsymbol{\Omega}^*\boldsymbol{\Omega})^{-1}$ does not exist, $\boldsymbol{m}^{[\infty]}$ is a particular solution to equation (B 25). Since, in general, $\boldsymbol{\Omega}$ in equation (B 1) does not have a unique inverse, $\boldsymbol{m}^{[\infty]}$ will, in general, differ from the model $\boldsymbol{m}$ that has generated the data $\boldsymbol{d}$. The reconstructed model $\boldsymbol{m}^{[\infty]}$ is the *minimum-norm solution* to equation (B 1) and $(\boldsymbol{\Omega}^*\boldsymbol{\Omega})^{-1}\boldsymbol{\Omega}^*$ represents the *pseudo-inverse* of $\boldsymbol{\Omega}$. To determine $\boldsymbol{m}^{[\infty]}$, one has, in most cases, to take recourse to the indicated iterative procedure. However, in some cases, $(\boldsymbol{\Omega}^*\boldsymbol{\Omega})^{-1}$ can be determined from analytical considerations. The quantities $\{\boldsymbol{\Omega}^*[\boldsymbol{r}^{[i]}];\ i = 0, 1, 2, \ldots\}$ are denoted as the (successive) *images* of $\boldsymbol{m}$.

Finally, it is observed that the expression given by equation (B 26) for the reconstructed model also directly follows by considering the expression

$$\begin{aligned} \epsilon &= \|\boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}]\|_{\mathrm{d}}^2 \\ &= \langle \boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}], \boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}]\rangle_{\mathrm{d}}, \end{aligned} \tag{B 27}$$

and defining the *optimum model* $\boldsymbol{m}^{\mathrm{opt}}$ as the one that minimizes ϵ. Upon substituting

$$\boldsymbol{m} = \boldsymbol{m}^{\mathrm{opt}} + \boldsymbol{\delta m}, \tag{B 28}$$

we obtain

$$\begin{aligned} \epsilon &= \epsilon^{\mathrm{opt}} - 2\langle \boldsymbol{\Omega}[\boldsymbol{\delta m}], \boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}^{\mathrm{opt}}]\rangle_{\mathrm{d}} + \|\boldsymbol{\Omega}[\boldsymbol{\delta m}]\|_{\mathrm{d}}^2 \\ &= \epsilon^{\mathrm{opt}} - 2\langle \boldsymbol{\delta m}, \boldsymbol{\Omega}^*[\boldsymbol{d}] - \boldsymbol{\Omega}^*\boldsymbol{\Omega}[\boldsymbol{m}^{\mathrm{opt}}]\rangle_{\mathrm{d}} + \|\boldsymbol{\Omega}[\boldsymbol{\delta m}]\|_{\mathrm{d}}^2, \end{aligned} \tag{B 29}$$

with

$$\epsilon^{\mathrm{opt}} = \|\boldsymbol{d} - \boldsymbol{\Omega}[\boldsymbol{m}^{\mathrm{opt}}]\|_{\mathrm{d}}^2. \tag{B 30}$$

From this, it follows that $\epsilon > \epsilon^{\mathrm{opt}}$ for any $\boldsymbol{\delta m} \neq \boldsymbol{0}$, provided that

$$\boldsymbol{\Omega}^*[\boldsymbol{d}] = \boldsymbol{\Omega}^*\boldsymbol{\Omega}[\boldsymbol{m}^{\mathrm{opt}}], \tag{B 31}$$

which is equation (B 25).

Appendix C. Preconditioning methods

In this appendix we describe a method of preconditioning that is applicable to a linear operator equation of the type

$$\boldsymbol{d} = \boldsymbol{\Omega}[\boldsymbol{m}]. \tag{C 1}$$

An iterative solution procedure for this equation, following from the minimization of a mismatch in the satisfaction of the equality sign, has been presented in Appendix B. The *method of preconditioning* now assumes that a linear operator $\tilde{\boldsymbol{\Omega}}$ can be found that somehow 'approximates' $\boldsymbol{\Omega}$, while, subject to the same mismatch criterion, the 'solution'

$$\tilde{\boldsymbol{m}}^{[\infty]} = (\tilde{\boldsymbol{\Omega}}^*\tilde{\boldsymbol{\Omega}})^{-1}\tilde{\boldsymbol{\Omega}}^*[\boldsymbol{d}], \tag{C 2}$$

to the approximate equation

$$\boldsymbol{d} = \tilde{\boldsymbol{\Omega}}[\tilde{\boldsymbol{m}}], \tag{C 3}$$

is obtainable by either an analytic method or a rapidly converging iterative technique.
With

$$\widehat{\boldsymbol{m}} = \tilde{\boldsymbol{\Omega}}^{*}\tilde{\boldsymbol{\Omega}}[\boldsymbol{m}], \tag{C 4}$$

we precondition equation (C 1) according to

$$\boldsymbol{d} = \boldsymbol{\Upsilon}[\widehat{\boldsymbol{m}}], \tag{C 5}$$

with

$$\boldsymbol{\Upsilon} = \boldsymbol{\Omega}(\tilde{\boldsymbol{\Omega}}^{*}\tilde{\boldsymbol{\Omega}})^{-1}. \tag{C 6}$$

Applying the procedure of Appendix B to this equation, the adjoint

$$\boldsymbol{\Upsilon}^{*} = (\tilde{\boldsymbol{\Omega}}^{*}\tilde{\boldsymbol{\Omega}})^{-1}\boldsymbol{\Omega}^{*} \tag{C 7}$$

now determines the direction of steepest descent. The ‘solution’ to equation (C 5) is obtained as

$$\widehat{\boldsymbol{m}}^{[\infty]} = (\boldsymbol{\Upsilon}^{*}\boldsymbol{\Upsilon})^{-1}\boldsymbol{\Upsilon}^{*}[\boldsymbol{d}], \tag{C 8}$$

and is consistent with the ‘solution’ of equation (C 1) in accordance with the mapping (C 4).

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