
**Electromagnetics**
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713770615

# The Space-Time Integrated Model of Electromagnetic Field Computation

Adrianus T. De Hoop[a]; Ioan E. Lager[b]; Pieter Jorna[c]
[a] Laboratory of Electromagnetic Research Faculty of Information Technology and Systems Delft University of Technology Delft, The Netherlands.. [b] International Research Centre for Telecommunications - Transmission and Radar Faculty of Information Technology and Systems Delft University of Technology Delft, The Netherlands. [c] Laboratory of Electromagnetic Research Faculty of Information Technology and Systems Delft University of Technology Delft, The Netherlands.

**To cite this Article** De Hoop, Adrianus T. , Lager, Ioan E. and Jorna, Pieter(2002) 'The Space-Time Integrated Model of Electromagnetic Field Computation', Electromagnetics, 22: 5, 371 — 379
**To link to this Article: DOI:** 10.1080/02726340290083978
**URL:** http://dx.doi.org/10.1080/02726340290083978

*Electromagnetics*, 22:371–379, 2002

0272-6343/02 $12.00 + .00
DOI: 10.1080/0272634029008397 8



# The Space-Time Integrated Model of Electromagnetic Field Computation

ADRIANUS T. DE HOOP

Laboratory of Electromagnetic Research
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands

IOAN E. LAGER

International Research Centre for Telecommunications—
Transmission and Radar
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands

PIETER JORNA

Laboratory of Electromagnetic Research
Faculty of Information Technology and Systems
Delft University of Technology
Delft, The Netherlands

*An unconventional model of electromagnetic field computation is presented. It aims at being as close to the physics as possible for the class of strongly heterogeneous and/or anisotropic media. The electric field strength, the magnetic field strength, the electric flux density, and the magnetic flux density are computed separately. Consistently linear edge expansion functions are employed for the discretization of the field strengths and consistently linear face expansion functions are employed for the discretization of the flux densities. The model imposes the exact satisfaction, for each element, of the space-time discretization, of the space-time integrated field equations and compatibility relations, combined with the least-square satisfaction, for each such element, of the space-time integrated constitutive relations.*

**Keywords** electromagnetic field computation, heterogeneous media

Address correspondence to Adrianus T. de Hoop, Laboratory of Electromagnetic Research, Faculty of Information Technology and Systems, Delft University of Technology, Mekelweg 4, Delft 26280D, The Netherlands. E-mail: a.t.dehoop@its.tudelft.nl

## Introduction

Standard computational techniques for evaluating electromagnetic fields in configurations that, due to their degree of complexity, are not amenable to analytical methods involve the finite-element and finite-difference techniques. Both techniques are based on a description of the field behavior in terms of partial differential equations and, hence, can be expected to yield only approximations of a controlled accuracy in subdomains of a configuration where the field quantities are continuously differentiable. The latter property holds wherever the constitutive material parameters vary continuously in space.

However, as soon as discontinuities in material properties occur, additional computational measures have to be taken to circumvent a deterioration of the accuracy of the results in the neighborhood of interfaces. In particular, field values exactly at interfaces (in fact, their limiting values upon approaching either side of an interface) are often inaccurate. An observation of this nature applies more whenever fields are to be evaluated in strongly heterogeneous media (for example, in granular composites or fiber-reinforced materials) where a discontinuous constitutive behavior persists down to the scale where actual measurements are still feasible, a scale that we refer to as the *mesoscopic* scale (in a computational scheme, this scale is taken to correspond to the mesh size of the spatial discretization). Under these circumstances, the standard methods fail to yield a result of uniform accuracy, and other methods, not based on the property of differentiability, but still describing the physics of the problem, have to be called upon. The *domain-integrated field equations* method (see de Hoop & Lager and the references therein) suggests the path to be followed in the quest for the identification of this alternative.



## Prerequisites to Constructing a Model of Electromagnetic Field Computation

The point of departure in our analysis is that all field and source quantities, as well as the constitutive parameters, are to be bounded, piecewise continuous functions in space and time. Across interfaces between adjacent subdomains of continuity, these quantities may show finite jump discontinuities. To enable the handling of radiation from sources in unbounded domains, the (possibly strongly) heterogeneous configuration is assumed to have a bounded support in space, while being embedded in a medium for which the causal space-time Green's field tensors are known in analytic form. The simplest medium in this category is the homogeneous, isotropic one. For the field excited by sources in such a configuration, the existence and the uniqueness of the initial-value problem (which implies a causal interaction between sources and excited fields) can be proven.

Now, it should be noted that on a mesoscopic scale, the property of differentiability cannot be guaranteed anymore (in fact, for a strongly heterogeneous configuration this property is lost throughout its support). To compute electromagnetic fields in a configuration of this kind, we first need a system of field relations that are devoid of derivatives. We denote these relations (and their accompanying "compatibility relations") as the *space-time integrated field equations* and *space-time integrated compatibility relations*. In the interior of each subdomain where differentiability does hold, these relations should meet the condition that they are equivalent to Maxwell's field equations and their accompanying compatibility relations, in a differential form. Note that this is the only set of equations for which, together with the pertaining boundary conditions across interfaces of jump discontinuity in material properties, the existence and the uniqueness of the solution can be proven rigorously.

In view of the wild fluctuations that the field quantities undergo upon crossing an interface where a large jump discontinuity in constitutive parameters manifests itself, the field representations to be used in the computation are the ones that are expressed in terms of their components that remain continuous upon crossing such an interface. These components are:

- the tangential components of the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{H}$ and
- the normal components of the electric flux density $\boldsymbol{D}$ and the magnetic flux density $\boldsymbol{B}$.

On the other hand, along an interface, and in between interfaces intersecting the one under consideration, the field quantities still vary continuously with position, which property allows for a local approximation on a polynomial basis.

The next point of concern is the incorporation of the constitutive behavior. In view of the persistence of heterogeneity on the mesoscopic scale, it makes no sense to attribute a local value to any of the constitutive coefficients at any interior point of an element of the geometrical discretization. Here, we adopt the procedure that an "effective" constitutive coefficient is incorporated in the field evaluations by minimizing the discrepancy that corresponds to the norm of the mismatch in a constitutive relation that would hold in a domain of continuity. This norm is evaluated over the geometrical space-time elements used in the actual computation.

Operating in this manner, the entire physical structure of the field is exploited for carrying out field computations in strongly heterogeneous configurations. Owing to our choice for representing the field quantities exclusively in terms of their components that are continuous across interfaces of jump discontinuity in constitutive parameters, no mesh refinement near such interfaces is required, while accuracy of the results is maintained up to these surfaces. To substantiate this statement, we note that the domain-integrated field equations approach to computing quasi-static magnetic fields (which is the direct counterpart of the method discussed in this analysis in the case of quasi-static magnetic fields) has proven to provide results of superior accuracy, in acceptable computation times, in configurations with relative contrasts in constitutive parameters that go up to 1000 (de Hoop & Lager, 2000), without requiring any mesh refinement in the vicinity of interfaces.

The model, as presented up to this point, accounts for the *spatial* behavior of the field. The *temporal* behavior is still to be accounted for. At this point, we note that, although in the analysis time-varying behaviors of the media (as in parametrically excited media) might be incorporated, we take the media to be time invariant, which implies that only the prescribed source distributions dictate the space-time evolution of the field. Now, in accordance with the spatial behavior of the field and its sources, we assume that the source quantities are bounded functions of time, with possible jump discontinuities at discrete instants. Because of this, differentiation with respect to time should be avoided as well. By assuming a Cartesian space-time decomposition ($\mathbb{R}^3 \times \mathbb{R}$) of the pseudo-Euclidean space-time continuum, an integration with respect to time accomplishes this final step in the setup of our field relations.

## The Space-Time Integrated Model of Electromagnetic Field Computation

The concepts presented above are now used as a basis for constructing a computational model of the electromagnetic field in strongly heterogeneous media. Since all relations pertaining to this model are to be given as integral expressions, we refer to it as the *space-time integrated model of electromagnetic field computation*.

Throughout our analysis, position in the three-dimensional space $\mathbb{R}^3$ is specified by the position vector $\boldsymbol{r}$ with respect to some orthogonal Cartesian reference frame, while the time coordinate is $t \in \mathbb{R}$. The four field quantities describing the physical properties of the electromagnetic field are: electric field strength $\boldsymbol{E}(\boldsymbol{r}, t)$, magnetic field strength $\boldsymbol{H}(\boldsymbol{r}, t)$, electric flux density $\boldsymbol{D}(\boldsymbol{r}, t)$, and magnetic flux density $\boldsymbol{B}(\boldsymbol{r}, t)$. Note in this respect that keeping the four vectorial field quantities $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, and $\boldsymbol{B}$ in the computational scheme (rather than keeping $\boldsymbol{E}$ and $\boldsymbol{H}$ only, which is standard practice in electromagnetic field computation) is also in accordance with the physics of the electromagnetic field, where the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ represents the surface density of electromagnetic power flow and $\boldsymbol{G} = \boldsymbol{D} \times \boldsymbol{B}$ represents the volume density of electromagnetic momentum. Another aspect of our method is that the source distributions that excite the configuration are incorporated in the constitutive relations, which is the standard method in linear, time-invariant systems theory.

### *Field Relations*

Let $\mathcal{D} \subset \mathbb{R}^3$ be any bounded domain with piecewise smooth boundary surface $\partial\mathcal{D}$, the unit vector along the outward normal to $\partial\mathcal{D}$ being denoted as $\boldsymbol{n}$. Let $\mathcal{T} \subset \mathbb{R}$ be any bounded time interval $\mathcal{T} = \{t \in \mathbb{R}, t_1 < t < t_2\}$ with boundary points $t_1$ and $t_2$, and let $\partial\mathcal{T} = \{t \in \mathbb{R}, \{t = t_1\} [ \ \{t = t_2\}\}$. Then, based on our choice for the Cartesian decomposition $\mathbb{R}^3 \times \mathbb{R}$ of the pseudo-Euclidean space-time continuum, the *space-time integrated field relations* are

$$\int_{\partial\mathcal{D}\times\mathcal{T}} \boldsymbol{n} \times \boldsymbol{H}(\boldsymbol{r}, t)\,\mathrm{d}A\mathrm{d}t - \int_{\mathcal{D}} \boldsymbol{D}(\boldsymbol{r}, t)\,\mathrm{d}V\bigg|_{\partial\mathcal{T}} = \boldsymbol{0}, \tag{1}$$

$$\int_{\partial\mathcal{D}\times\mathcal{T}} \boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r}, t)\,\mathrm{d}A\mathrm{d}t + \int_{\mathcal{D}} \boldsymbol{B}(\boldsymbol{r}, t)\,\mathrm{d}V\bigg|_{\partial\mathcal{T}} = \boldsymbol{0}, \tag{2}$$

while the *space-time integrated compatibility relations* are

$$\int_{\partial\mathcal{D}} \boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{r}, t)\,\mathrm{d}A\bigg|_{\partial\mathcal{T}} = 0, \tag{3}$$

$$\int_{\partial\mathcal{D}} \boldsymbol{n} \cdot \boldsymbol{B}(\boldsymbol{r}, t)\,\mathrm{d}A\bigg|_{\partial\mathcal{T}} = 0, \tag{4}$$

where the notation $|_{\partial\mathcal{T}}$ stands for $f(t)|_{\partial\mathcal{T}} = f(t_2) - f(t_1)$.

In any subdomain $\mathcal{D} \times \mathcal{T} \subset \mathbb{R}^3 \times \mathbb{R}$ where the field quantities are continuously differentiable with respect to both space *and* time coordinates, the field equations (1) and (2) are equivalent to the standard electromagnetic field equations in differential form. To show this, an application of Gauss's integral theorem to the first terms on the lefthand sides and an application of the theorem of one-dimensional calculus to the second terms leads to

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) - \partial_t \boldsymbol{D} = \boldsymbol{0}, \tag{5}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) + \partial_t \boldsymbol{B} = \boldsymbol{0}, \tag{6}$$

which are the desired expressions. Analogously, from Equations (3) and (4), it follows that

$$\partial_t \nabla \cdot \boldsymbol{D}(\boldsymbol{r}, t) = 0, \tag{7}$$

$$\partial_t \nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0, \tag{8}$$

which are the standard compatibility relations in differential form. While the carrying out of the procedure indicated above requires the continuous differentiability of the field quantities throughout $\mathcal{D} \times \mathcal{T}$, the space-time integrated form of the relations requires only piecewise continuity:

- of the *tangential* components of $\boldsymbol{H}$ and $\boldsymbol{E}$ on $\partial\mathcal{D} \times \mathcal{T}$,
- of the *normal* components of $\boldsymbol{D}$ and $\boldsymbol{B}$ on $\partial\mathcal{D}$,
- and of $\boldsymbol{D}$ and $\boldsymbol{B}$ on $\mathcal{D}$.



### *Continuity Conditions in Space and Time*

By applying Equations (1) and (2) to a space-time "pillbox" (of vanishing height in space and vanishing duration in time) at an interface between two media with different constitutive parameters, and using the boundedness of the field quantities, these interface boundary conditions follow:

$$\boldsymbol{\nu} \times \boldsymbol{H}(\boldsymbol{r}, t) = \text{continuous across interface}, \tag{9}$$

$$\boldsymbol{\nu} \times \boldsymbol{E}(\boldsymbol{r}, t) = \text{continuous across interface}, \tag{10}$$

with $\boldsymbol{\nu}$ denoting the unit vector along the normal to the interface. Application of the same procedure to Equations (3) and (4) leads to

$$\boldsymbol{\nu} \cdot \boldsymbol{D}(\boldsymbol{r}, t) = \text{continuous across an interface}, \tag{11}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = \text{continuous across an interface}. \tag{12}$$

By applying Equations (1) and (2) to a time interval of vanishing duration, using the boundedness of the first terms in the righthand side over the interval $\mathcal{T}$, and bearing in mind that the resulting relation has to hold for any spatial domain $\mathcal{D}$, it follows that

$$\boldsymbol{D}(\boldsymbol{r}, t) = \text{continuous across a jump discontinuity in time}, \tag{13}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \text{continuous across a jump discontinuity in time}. \tag{14}$$

From the interface boundary conditions in space, Equations (9)–(12), we learn that, as far as their behavior in space is concerned, the field representations to be used in the computational scheme are the ones whose expansion coefficients should be directly related to the values of the *tangential* components of $\boldsymbol{H}$ and $\boldsymbol{E}$ and the *normal* components of $\boldsymbol{D}$ and $\boldsymbol{B}$ on the boundary surfaces of the elementary cells in the discretized geometry. To exploit such representations in Equations (1) and (2), the analytical continuation of $\boldsymbol{D}$ and $\boldsymbol{B}$ into the interior of the elementary cells is also needed. Additionally, accounting for the constitutive relation (see below) requires the analytical continuation of $\boldsymbol{H}$ and $\boldsymbol{E}$ into the interior of the elementary cells as well. As will be shown below, spatially linear expansions on a simplicial mesh (tetrahedra in $\mathbb{R}^3$) provide a consistent framework in this respect.

As far as the time behavior is concerned, we start from the continuity conditions of Equations (13) and (14) taken at the possible jump discontinuities in time and analytically continue the relevant representations into the interior of the interval in between these instances, via a linear interpolation. This still leaves $\boldsymbol{H}$ and $\boldsymbol{E}$ to be represented in time. Here, we observe that the configuration is excited via impressed volume source densities (see below), whose time behavior is bound to be such as to lead to a continuous time behavior of the field quantities everywhere in space. By considering the class of time-invariant configurations, $\boldsymbol{H}$ and $\boldsymbol{E}$ are then also to be continuous in time. Therefore, as far as time is concerned, the same type of expansion as for $\boldsymbol{D}$ and $\boldsymbol{B}$ applies.



### *Constitutive Relations*

In the standard constitutive relations, the quantities $\boldsymbol{D}$ and $\boldsymbol{B}$ are taken to be partly related to the quantities $\boldsymbol{E}$ and $\boldsymbol{H}$ (passive response), and partly to direct excitation by external sources (active response). The media in our configuration are taken to be linear, time invariant, locally reacting, and without magneto-electric or chiral properties. Then, in any subdomain $\mathcal{D} \times \mathcal{T} \subset \mathbb{R}^3 \times \mathbb{R}$ where the material parameters are continuous functions of position, the *exact* field quantities (i.e., those that satisfy Equations (5)–(8)) are taken to satisfy the following constitutive relations:

$$\boldsymbol{D}(\boldsymbol{r},t) = [\boldsymbol{\epsilon}(\boldsymbol{r},t) \overset{(t)}{*} \boldsymbol{E}(\boldsymbol{r},t)] + \boldsymbol{P}(\boldsymbol{r},t), \tag{15}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = [\boldsymbol{\mu}(\boldsymbol{r},t) \overset{(t)}{*} \boldsymbol{H}(\boldsymbol{r},t)] + \mu_0 \boldsymbol{M}(\boldsymbol{r},t), \tag{16}$$

where $\boldsymbol{\epsilon}(\boldsymbol{r},t)$ and $\boldsymbol{\mu}(\boldsymbol{r},t)$ denote the permittivity and the permeability relaxation tensors, respectively, $\overset{(t)}{*}$ denotes time convolution, $\boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{M}(\boldsymbol{r},t)$ denote the impressed electric polarization and magnetization, respectively, and $\mu_0$ denotes the permeability of the vacuum. The use of the factor $\mu_0$ is dictated by SI. Note that the manner of introducing external volume source densities in Equations (15) and (16) deviates from the one employed in de Hoop (1995, 610–613) (the change affecting the values of $\boldsymbol{D}$ and $\boldsymbol{B}$ inside subdomains occupied by external sources only).

As far as the possible discontinuous behavior in time is concerned, we observe that, from a physical point of view, on intervals of differentiability, the quantities $\partial_t \boldsymbol{P}$ and $\mu_0 \partial_t \boldsymbol{M}$ (which are the volume source densities of external electric and magnetic current, respectively (de Hoop, 1995, 610)) are to be piecewise continuous in time (as well as in space). This property implies that $\boldsymbol{P}$ and $\boldsymbol{M}$ themselves are continuous in time (with possible finite jumps in the time derivative), which, in turn, based on the continuity in time of $\boldsymbol{D}$ and $\boldsymbol{B}$ and on the assumed time-invariance of the configurations, implies that $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous in time as well.

In the case of a computational model that employs an expansion of the field quantities on $\partial\mathcal{D}$ and $\partial\mathcal{T}$, together with an analytical continuation of these values into $\mathcal{D}$ and $\mathcal{T}$ (as indicated above), a pointwise satisfaction of the constitutive relations of Equations (15) and (16) no longer makes sense. In our computational model, we replace these pointwise constitutive relations by the space-time integrated constitutive relations that are taken as

$$\left\{\int_{\mathcal{D}\times\mathcal{T}} \left| \boldsymbol{D}(\boldsymbol{r},t) - \left[\tilde{\boldsymbol{\epsilon}}(\boldsymbol{r},t) \overset{(t)}{*} \boldsymbol{E}(\boldsymbol{r},t)\right] - \boldsymbol{P}(\boldsymbol{r},t)\right|^2 \,\mathrm{d}V\,\mathrm{d}t\right\}^{1/2} = \text{minimum}, \quad (17)$$

$$\left\{\int_{\mathcal{D}\times\mathcal{T}} \left| \boldsymbol{B}(\boldsymbol{r},t) - \left[\tilde{\mu}(\boldsymbol{r},t) \overset{(t)}{*} \boldsymbol{H}(\boldsymbol{r},t)\right] - \mu_0\boldsymbol{M}(\boldsymbol{r},t)\right|^2 \,\mathrm{d}V\,\mathrm{d}t\right\}^{1/2} = \text{minimum}, \quad (18)$$

where $\tilde{\boldsymbol{\epsilon}}(\boldsymbol{r},t)$ and $\tilde{\mu}(\boldsymbol{r},t)$ denote some "effective" permittivity and permeability relaxation functions, respectively, that are representative for the properties of the medium on a mesoscopic scale. The volume integral in the lefthand side of Equation (17) defines a norm over the function space of electric flux densities (hereafter denoted as $\|\cdot\|_{\boldsymbol{D}}$), while the volume integral in the lefthand side of Equation (18) defines a norm over the function space of magnetic flux densities (hereafter denoted as $\|\cdot\|_{\boldsymbol{B}}$). For carrying out the minimization procedure in Equations (17) and (18) we keep fixed the expansions of $\boldsymbol{D}$ and $\boldsymbol{B}$ as well as the source terms $\boldsymbol{P}$ and $\boldsymbol{M}$ (that have prescribed values), and vary the expansions of $\boldsymbol{H}$ and $\boldsymbol{E}$.

Note that the substitution of the exact field quantities in a subdomain of continuity (those that satisfy the pointwise constitutive relations in Equations (15) and (16)) in Equations (17) and (18) yields the (absolute) minimum of zero for those expressions.

### *Causality Condition*

The construction of the space-time integrated model of electromagnetic field computation is concluded by stating that the causality condition requires the field quantities to vanish throughout space prior to the onset of the action of the field source distributions.

## Discretized Computational Model

The relations that have been derived above for arbitrary space-time domains $\mathcal{D}\times\mathcal{T}\subset\mathbb{R}^3\times\mathbb{R}$ will now be applied to a specific, discretized computational model. For reasons of algebraic topology, we adopt a spatial decomposition of the domain of computation into simplices, i.e., tetrahedra in $\mathbb{R}^3$ (triangles in $\mathbb{R}^2$). The temporal discretization is achieved by means of a decomposition of the time interval $\mathcal{T}$ into simplices as well, i.e., segments in $\mathbb{R}$ (hereafter referred to as *time divisions*).

The spatial and temporal discretization aspects are discussed separately below.

### *Spatial Discretization*

The (mesoscopic) domains $\mathcal{D}$ in Equations (1)–(4), (17), and (18) are now identified with each of the simplicial cells $\mathcal{S}_m$ ($m = 1,\ldots,M$) of the simplicial decomposition of the domain of computation.

To identify the appropriate set of expansion functions, we first note that at any given vertex $i$ ($i = 0,1,2,3$) of an arbitrary tetrahedron $\mathcal{S}$ with vertices at $\boldsymbol{r}_i$ ($i = 0,1,2,3$), the oriented edges emerging from that vertex $\boldsymbol{e}_{i,j}$ ($j = 0,1,2,3$, $j \neq i$) and the outwardly oriented areas of the faces meeting at that vertex $\boldsymbol{A}_j$ ($j = 0, 1, 2, 3,\ j \neq i$, with $j$ being the index of the vertex opposite to the relevant face) satisfy the relation

$$\boldsymbol{e}_{i,j_1} \cdot \boldsymbol{A}_{j_2} = -3V_{\mathcal{S}}\delta_{j_1,j_2}, \quad \text{for } i = 0, 1, 2, 3,\ j_1, j_2 = 0, 1, 2, 3,\ j_1 \neq i,\ j_2 \neq i, \tag{19}$$

where $V_{\mathcal{S}}$ denotes the volume of the relevant tetrahedron and $\delta_{j_1,j_2}$ denotes the Kronecker symbol. Equation (19) implies that the sets $\{\boldsymbol{e}_{i,j}\}$ and $\{\boldsymbol{A}_j\}$ ($j = 0, 1, 2, 3,\ j \neq i$) form at the vertex $i$ ($i = 0, 1, 2, 3$) a set of reciprocal base vectors in $\mathbb{R}^3$ (with $-3V_{\mathcal{S}}$ as their interrelation coefficient). It then follows that an arbitrary vector $\boldsymbol{X}(\boldsymbol{r}, t) \in \mathbb{R}^3 \times \mathbb{R}$ can be represented at any vertex $i$ of the tetrahedron $\mathcal{S}$ and at any instant $t$:

- as an *edge expansion*

$$\boldsymbol{X}(\boldsymbol{r}_i, t) = -\sum_{\substack{j=0\\ j\neq i}}^{3} \alpha^{\boldsymbol{X}}_{i,j}(t)\frac{\boldsymbol{A}_j}{3V_{\mathcal{S}}} \quad \text{for } i = 0, 1, 2, 3, \tag{20}$$

  the expansion coefficients $\alpha^{\boldsymbol{X}}_{i,j}(t)$ being

$$\alpha^{\boldsymbol{X}}_{i,j}(t) = \boldsymbol{X}(\boldsymbol{r}_i, t) \cdot \boldsymbol{e}_{i,j} \quad \text{for } i = 0, 1, 2, 3,\ j = 0, 1, 2, 3,\ j \neq i, \tag{21}$$

- or as a *face expansion*

$$\boldsymbol{X}(\boldsymbol{r}_i, t) = -\sum_{\substack{j=0\\ j\neq i}}^{3} \beta^{\boldsymbol{X}}_{i,j}(t)\frac{\boldsymbol{e}_{i,j}}{3V_{\mathcal{S}}} \quad \text{for } i = 0, 1, 2, 3, \tag{22}$$

  the expansion coefficients $\beta^{\boldsymbol{X}}_{i,j}(t)$ being

$$\beta^{\boldsymbol{X}}_{i,j}(t) = \boldsymbol{X}(\boldsymbol{r}_i, t) \cdot \boldsymbol{A}_j \quad \text{for } i = 0, 1, 2, 3,\ j = 0, 1, 2, 3,\ j \neq i. \tag{23}$$

(In practice, the basis functions are scaled down such that they become dimensionless.)

Once the type of expansion has been chosen at each vertex, the values of the quantity to be discretized are first extrapolated along edges and across faces and, subsequently, analytically continued in the interior of the tetrahedron. Since, presumably, the mesh size is chosen such that spatial polynomial expansions of degree one suffice for the approximation of the values throughout the tetrahedra $\mathcal{S}_m$, the representation along edges and across faces and in the interior of the tetrahedron is constructed out of the vertex values by means of a spatial linear interpolation. Note that the simplex is the only geometrical object for which such a consistently linear interpolation can be carried out (Naber, 1980).

We now observe that a violation of the continuity conditions of Equations (9)–(12) across interfaces would manifest itself as an action of spurious surface sources at such interfaces, with the consequence of a deterioration of the field values, certainly in the neighborhood of the interface and possibly, through error propagation, also away from them. For this reason, we invoke these interface continuity conditions (in machine precision) across *all* faces of adjacent tetrahedra. To achieve this, edge expansion is to be used for $\boldsymbol{E}$ and $\boldsymbol{H}$ and face expansion for $\boldsymbol{D}$ and $\boldsymbol{B}$.

***Temporal Discretization***

Similar arguments for the discretization in time lead to a representation of linearly interpolated values in between the values of the field quantities at the boundary points $t_n$ and $t_{n+1}$ $(n = 1, \dots N - 1)$ of the time division with the index $n$ in the partition of $\partial \mathcal{T}$, i.e.,

$$\{\alpha^X_{i,j}(t), \beta^X_{i,j}(t)\} = \{\alpha^X_{i,j}(t_n), \beta^X_{i,j}(t_n)\}\frac{t_{n+1} - t}{t_{n+1} - t_n} + \{\alpha^X_{i,j}(t_{n+1}), \beta^X_{i,j}(t_{n+1})\}\frac{t - t_n}{t_{n+1} - t_n}$$

$$\text{for } i = 0, 1, 2, 3,\ j = 0, 1, 2, 3,\ j \neq i,\ t_n < t < t_{n+1}. \tag{24}$$

***Construction of the System of Equations in the Expansion Coefficients***

The field expansions thus constructed are substituted in the space-time integrated field and compatibility relations of Equations (1)–(4) and in the space-time integrated constitutive relations of Equations (17) and (18). As for the minimization procedure in Equations (17) and (18), we carry out

- the minimization in space and time of the $\|\cdot\|_{\boldsymbol{D}}$-norm in Equation (17) by varying the expansion coefficients of $\boldsymbol{E}$, while keeping those of $\boldsymbol{D}$ fixed

and

- the minimization in space and time of the $\|\cdot\|_{\boldsymbol{B}}$-norm in Equation (18) by varying the expansion coefficients of $\boldsymbol{H}$, while keeping those of $\boldsymbol{B}$ fixed.

This procedure yields a system of linear, algebraic equations that can be shown to be overdetermined for all cases of practical interest. A "best" solution to this system is constructed via the minimization of the mismatch between the "known" and the "unknown" quantities, according to a suitable mismatch criterion.

## Conclusions

The computational scheme presented offers a range of perspectives in all those cases where heterogeneous media, in particular those with high contrasts in constitutive properties, are involved. The exclusive use of expansions of continuously varying field components and the satisfaction of the integrated field equations over the elementary subdomains of the discretized geometry, rather than a pointwise satisfaction of the field equations in their differential form, has the main advantage of making mesh refinement superfluous.

So, the method can, in its theoretical aspects, serve as a well-defined mathematical basis for any physical "effective-medium" theory of strongly heterogeneous components. For engineering applications, it yields numerical results of a desired accuracy in acceptable computation times, as experience with the computation of quasi-static magnetic fields in strongly heterogeneous configurations has shown (de Hoop & Lager, 2000).

## References

de Hoop, A. T. 1995. *Handbook of radiation and scattering of waves*. London: Academic Press.

de Hoop, A. T., & I. E. Lager. Domain-integrated field equations approach to static magnetic field computation—Application to some two-dimensional configurations. *IEEE Trans. Magnetics* Vol. 36, No. 4, pp. 654–658, 2000.

Naber, G. L. 1980. *Topological methods in Euclidean spaces*. Cambridge, UK: Cambridge University Press.