A Time-Domain Uniqueness Theorem for Electromagnetic Wavefield Modeling in Dispersive, Anisotropic Media

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Abstract

A uniqueness theorem for the initial-/boundary-value problem arising in the (analytic or computational) modeling of electromagnetic wavefields in arbitrarily dispersive and anisotropic media is presented. It is known that for media where the dispersion takes place via electrically conductive and/or linear magnetic hysteresis losses only, a uniqueness theorem for the initial-/boundary-value problem can be constructed by using direct time-domain arguments in the pertaining energy balance (Poynting's theorem). The case of arbitrary dispersion in the medium's electric and magnetic behavior, however, withstands such an approach. Here, as an intermediate step, the one-to-one correspondence between the causal time-domain field components and material response functions in the constitutive relations on the one hand and their time Laplace transforms for (a set of) real, positive values of the transform parameter on the other hand, seems a necessary tool. It is shown that this approach leads to simple, explicit, sufficiency conditions on the relaxation tensors describing the medium's electric and magnetic behavior, in which the property of causality proves to play an essential role.

1. Introduction

One of the issues one is confronted with in the mathematical modeling – be it with analytical or numerical techniques – of electromagnetic wave phenomena is the question about the uniqueness of the solution to the problem as it is formulated mathematically. Evidently, such a uniqueness should be expected on account of the underlying physics. When investigating wave propagation and scattering problems one expects the pertaining partial differential equations, constitutive relations, boundary conditions at interfaces, excitation conditions at exciting sources, initial values at the time window one considers and the causal relationship that is to exist between the exciting sources and the generated wavefield to play a role. For simple media with instantaneous relations between the intensive quantities (that carry the power flow in the wavefield) and the extensive quantities (that carry the momentum of the wavefield), i.e., for lossless media, the time-domain power balance provides a tool to prove uniqueness. This is also the case when simple loss mechanisms (such as electrically conductive and/or linear magnetic hysteresis losses) are incorporated. A uniqueness proof for the case of arbitrary relaxation effects in the media seems to withstand such a direct time-domain approach. Since for the class of linear, time-invariant, causally reacting media the constitutive relations are expressed via time convolutions, it can be expected that the time Laplace transformation (under which transformation the convolution operation transforms into a simple product of the constituents) might provide a useful tool. This approach is followed in the present paper and applied to the general class of linear, time-invariant, causal, locally reacting, inhomogeneous, anisotropic media. For this class of media, sufficient conditions for the uniqueness of the electromagnetic wavefield problem are specified for the electric and magnetic relaxation tensors in the time Laplace transform domain at real, postitive values of the transform parameter. In the procedure, Lerch's theorem of the one-sided (= causal) Laplace transformation plays an essential role.

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Dedicated to Professor J. Van Bladel on the occasion of his 80th. birthday.

2. Description of the configuration

The configuration for which the uniqueness of the electromagnetic wavefield problem will be proved consists of a linear, time-invariant, locally reacting, inhomogeneous, anisotropic medium with arbitrary electric and magnetic relaxation properties and of bounded support $\mathcal{D} \subset \mathcal{R}^3$. This part of the configuration is embedded in a linear, time-invariant, locally reacting, homogeneous, isotropic, instantaneously reacting medium with permittivity ϵ_∞ and permeability μ_∞. The unbounded domain occupied by the embedding is denoted as $\mathcal{D}^\infty$. The common boundary of $\mathcal{D}$ and $\mathcal{D}^\infty$ is the bounded closed surface $\partial\mathcal{D}$ (Figure 1). The constitutive relaxation functions in $\mathcal{D}$ vary piecewise continuously with position with finite jump discontinuities at a finite number of piecewise smooth, bounded surfaces (interfaces). Position in the configuration is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors is used and the summation convention for repeated subscripts applies. Whenever appropriate, vectors are indicated by boldface symbols, with $\boldsymbol{x}$ as the position vector. The time coordinate is t. Partial differentiation with respect to x_m will be denoted by ∂_m; ∂_t is a reserved symbol indicating partial differentiation with respect to t. Volume source distributions of electric polarization and/or magnetization, with bounded supports, excite a transient electromagnetic field in the configuration. They start to act at the instant $t = 0$. The field that is causally related to the action of these sources then vanishes throughout space for $t < 0$.

3. Formulation of the EM wavefield problem

At any point in the configuration where the electromagnetic field quantities are differentiable they satisfy the Maxwell field equations [1, p. 611]

$$\varepsilon_{k,m,p}\partial_m H_p - \partial_t D_k = 0, \quad (1)$$
$$\varepsilon_{j,n,q}\partial_n E_q + \partial_t B_j = 0, \quad (2)$$

where

E_q = electric field strength (V/m),
H_p = magnetic field strength (A/m),
D_k = electric flux density (C/m^2),
B_j = magnetic flux density (T),

and $\varepsilon_{k,m,p}$ is the completely antisymmetrical unit tensor of rank three: $\varepsilon_{k,m,p} = 1$ for $\{k, m, p\}$ = even permutation of $\{1, 2, 3\}$, $\varepsilon_{k,m,p} = -1$ for $\{k, m, p\}$ = odd permutation of $\{1, 2, 3\}$, $\varepsilon_{k,m,p} = 0$ in all other cases.

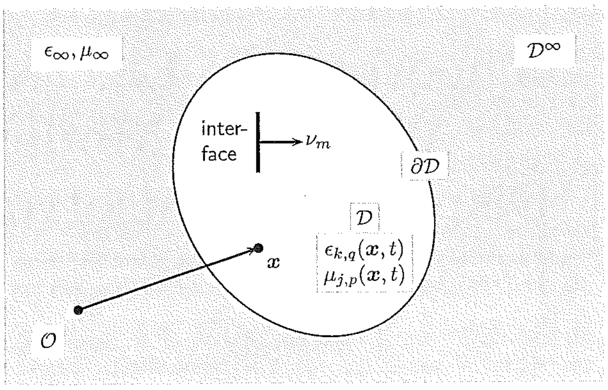

Figure 1: Configuration with inhomogeneous, anisotropic, dispersive medium (with bounded support $\mathcal{D}$) embedded in a homogeneous, isotropic, non-dispersive medium (with unbounded support $\mathcal{D}^\infty$).

The constitutive relations are:

$$D_k(\boldsymbol{x},t) = P_k(\boldsymbol{x},t) + \epsilon_{k,q}(\boldsymbol{x},t) \overset{(t)}{*} E_q(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}, \quad (3)$$

$$B_j(\boldsymbol{x},t) = M_j(\boldsymbol{x},t) + \mu_{j,p}(\boldsymbol{x},t) \overset{(t)}{*} H_p(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}, \quad (4)$$

where $\overset{(t)}{*}$ denotes time convolution, and

$$D_k(\boldsymbol{x},t) = \epsilon_\infty E_k(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}^\infty, \quad (5)$$
$$B_j(\boldsymbol{x},t) = \mu_\infty H_j(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}^\infty. \quad (6)$$

In these relations,

P_k = electric source polarization (C/m^2),
M_j = source magnetization (T),

represent the active parts (source distributions) of the medium in $\mathcal{D}$,

$\epsilon_{k,q}(\boldsymbol{x},t)$ = electric relaxation function (F/m·s),
$\mu_{j,p}(\boldsymbol{x},t)$ = magnetic relaxation function (H/m·s),

of the medium in $\mathcal{D}$, and

ϵ_∞ = electric permittivity (F/m),
μ_∞ = magnetic permeability (H/m),

of the medium in $\mathcal{D}^\infty$. In the representation of the constitutive behavior of the media we have chosen to incorporate the action of field exciting source distributions in them through field-independent excitation terms as is done in the Kirchhoff theory of active networks (Thevenin and Norton representations for the action of source voltages and source electric currents, respectively). Across any interface Σ of jump discontinuity in constitutive properties the boundary conditions of the continuity type

$$\varepsilon_{k,m,p}\nu_m H_p = \text{continuous across } \Sigma, \quad (7)$$
$$\varepsilon_{j,n,q}\nu_n E_q = \text{continuous across } \Sigma, \quad (8)$$

hold, where ν_m is the unit vector along the normal to Σ. This implies that the tangential components of the electric and magnetic fields strengths are continuous across the interface. The constitutive relaxation functions are subject to the causality condition

$$\epsilon_{k,q}(\boldsymbol{x},t) = 0 \quad \text{for } t<0 \text{ and all } \boldsymbol{x}\in\mathcal{D}, \tag{9}$$
$$\mu_{j,p}(\boldsymbol{x},t) = 0 \quad \text{for } t<0 \text{ and all } \boldsymbol{x}\in\mathcal{D}. \tag{10}$$

Further conditions to be laid upon them with regard to the uniqueness of the electromagnetic wavefield problem are investigated further on. On the constitutive coefficients of the embedding we impose the conditions $\epsilon_\infty > 0$ and $\mu_\infty > 0$.

In the embedding the Green's tensors (point-source solutions) can be determined analytically [1, Sections 28.8 and 28.12]. From the corresponding Huygens surface source representations over the surface $\partial\mathcal{D}$ it follows that the outgoing fields in $\mathcal{D}^\infty$ admit the far-field expansions

$$\{E_q,H_p\}(\boldsymbol{x},t) = \frac{\{e_q,h_p\}(\boldsymbol{\theta},t-|\boldsymbol{x}|/c_\infty)}{4\pi|\boldsymbol{x}|}\left[1+O(|\boldsymbol{x}|^{-1})\right] \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{11}$$

where $\boldsymbol{x}$ is the position vector from the chosen far-field reference center to the point of observation, $\boldsymbol{\theta} = \boldsymbol{x}/|\boldsymbol{x}|$ is the unit vector in the direction of observation and $c_\infty = (\epsilon_\infty\mu_\infty)^{-1/2}$ is the electromagnetic wavespeed in $\mathcal{D}^\infty$. The far-field radiation characteristics are mutually related via

$$e_q = -(\mu_\infty/\epsilon_\infty)^{1/2}\varepsilon_{q,m,p}\theta_m h_p, \tag{12}$$
$$h_p = (\epsilon_\infty/\mu_\infty)^{1/2}\varepsilon_{p,n,q}\theta_n e_q. \tag{13}$$

In the following it will be shown that the problem thus formulated has at most one solution, assuming that, for each type of excitation at least one solution exists. The proof puts restrictions on the relaxation functions representing the electric and magnetic properties of the medium in $\mathcal{D}$. For the medium in $\mathcal{D}^\infty$ these simply are $\epsilon_\infty > 0$ and $\mu_\infty > 0$ (as indicated already).

4. The electromagnetic field problem in the time Laplace-transform domain

For the general type of dispersive media considered in the present paper there is, as far as is known, no direct uniqueness proof in the space/time domain based on energy considerations as is the case for media with simple constitutive behavior [2, Section 9.2]. However, because of the causality of both the media's passive field reponse and the field's relation to its activating sources, the time Laplace transformation with real, positive transform parameter offers a tool to specify certain conditions to be imposed on the constitutive relaxation functions in order that the wavefield problem has a unique solution. The relevant transformation is given by

$$\{\hat{E}_q,\hat{H}_p\}(\boldsymbol{x},s) = \int_{t=0}^{\infty}\exp(-st)\{E_q,H_p\}(\boldsymbol{x},t)\mathrm{d}t. \tag{14}$$

For the case of physical interest of excitation functions and relaxation functions that show at most a Dirac delta distribution time behavior, the time Laplace transforms of the field vectors and the relaxation tensors exist for all $\{s\in\mathcal{C};\mathrm{Re}(s)>0\}$, i.e., for all values of the transform parameter in the right half of the complex s-plane. Furthermore, since all time functions involved are real-valued, their Laplace transforms take on real values for real values of s. In relation to our uniqueness proof we now take s to be a *Lerch sequence*: $\{s\in\mathcal{R}; s=s_0+nh, s_0>0, h>0, n=0,1,2,\ldots\}$. Lerch's theorem [3, p. 63] states that if the transformation expressed by Equation (14) is to hold for all s belonging to this sequence, only one (causal) time-domain original corresponds to its related transform. Under the transformation, the time derivative is replaced with a multiplication by s (if zero-value initial conditions apply, as is the case) and the time convolution transforms into the product of the constituents. Using these properties, Equations (1) - (6) lead, upon time Laplace transformation, to

$$\varepsilon_{k,m,p}\partial_m\hat{H}_p - s\hat{\epsilon}_{k,q}\hat{E}_q = s\hat{P}_k \quad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{15}$$
$$\varepsilon_{j,n,q}\partial_n\hat{E}_q + s\hat{\mu}_{j,p}\hat{H}_p = -s\hat{M}_j \quad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{16}$$

and

$$\varepsilon_{k,m,p}\partial_m\hat{H}_p - s\epsilon_\infty\hat{E}_k = 0 \quad \text{for } \boldsymbol{x}\in\mathcal{D}^\infty, \tag{17}$$
$$\varepsilon_{j,n,q}\partial_n\hat{E}_q + s\mu_\infty\hat{H}_j = 0 \quad \text{for } \boldsymbol{x}\in\mathcal{D}^\infty. \tag{18}$$

The interface continuity conditions (7) - (8) are upon Laplace transformation replaced by

$$\varepsilon_{k,m,p}\nu_m\hat{H}_p = \text{continuous across } \Sigma, \tag{19}$$
$$\varepsilon_{j,n,q}\nu_n\hat{E}_q = \text{continuous across } \Sigma, \tag{20}$$

and the far-field expansion (11) by

$$\{\hat{E}_q,\hat{H}_p\}(\boldsymbol{x},s) = \frac{\{\hat{e}_q,\hat{h}_p\}(\boldsymbol{\theta},s)}{4\pi|\boldsymbol{x}|}\exp(-s|\boldsymbol{x}|/c_\infty)\left[1+O(|\boldsymbol{x}|^{-1})\right] \quad \text{as } |\boldsymbol{x}|\to\infty. \tag{21}$$

Upon contracting Equations (15) and (17) with $\hat{E}_k$ and Equations (16) and (18) with $\hat{H}_j$ and combining the results we construct the relations

$$\varepsilon_{m,k,j}\partial_m(\hat{E}_k\hat{H}_j) + s\hat{E}_k\hat{\epsilon}_{k,q}\hat{E}_q + s\hat{H}_j\hat{\mu}_{j,p}\hat{H}_p = -s\hat{E}_k\hat{P}_k - s\hat{H}_j\hat{M}_j \quad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{22}$$

and

$$\varepsilon_{m,k,j}\partial_m(\hat{E}_k\hat{H}_j) + s\hat{E}_k\epsilon_\infty\hat{E}_k + s\hat{H}_j\mu_\infty\hat{H}_j = 0 \quad \text{for } \boldsymbol{x}\in\mathcal{D}^\infty. \tag{23}$$

Integration of Equation (22) over $\mathcal{D}$ and application of Gauss' divergence theorem yields

$$\int_{\partial\mathcal{D}} \varepsilon_{m,k,j}\nu_m \hat{E}_k \hat{H}_j \mathrm{d}A(\boldsymbol{x}) + \int_{\mathcal{D}} (s\hat{E}_k \hat{\epsilon}_{k,q} \hat{E}_q + s\hat{H}_j \hat{\mu}_{j,p} \hat{H}_p)\mathrm{d}V(\boldsymbol{x}) = -\int_{\mathcal{D}} (s\hat{E}_k \hat{P}_k + s\hat{H}_j \hat{M}_j)\mathrm{d}V(\boldsymbol{x}), \qquad (24)$$

where ν_m is the outward unit vector along the normal to $\partial\mathcal{D}$. Next, Equation (23) is integrated over the domain that is bounded internally by $\partial\mathcal{D}$ and externally by the sphere $\mathcal{S}_\Delta$ of radius Δ and center at the far-field reference center, where Δ is chosen so large that $\mathcal{S}_\Delta$ completely surrounds $\partial\mathcal{D}$ (Figure 2). Subsequent application of Gauss' divergence theorem leads to

$$\int_{\mathcal{S}_\Delta} \varepsilon_{m,k,j}\nu_m \hat{E}_k \hat{H}_j \mathrm{d}A(\boldsymbol{x}) - \int_{\partial\mathcal{D}} \varepsilon_{m,k,j}\nu_m \hat{E}_k \hat{H}_j \mathrm{d}A(\boldsymbol{x}) + \int_{\mathcal{D}^\infty \cap \mathcal{D}_\Delta} (s\hat{E}_k \epsilon_\infty \hat{E}_k + s\hat{H}_j \mu_\infty \hat{H}_j)\mathrm{d}V(\boldsymbol{x}) = 0, \qquad (25)$$

where $\mathcal{D}_\Delta$ is the domain interior to $\mathcal{S}_\Delta$. With the use of the far-field representation (21) in the integration over $\mathcal{S}_\Delta$, the limit $\Delta \to \infty$ in Equation (25) leads to (note that the integral over $\mathcal{S}_\Delta$ goes to zero as $\Delta \to \infty$)

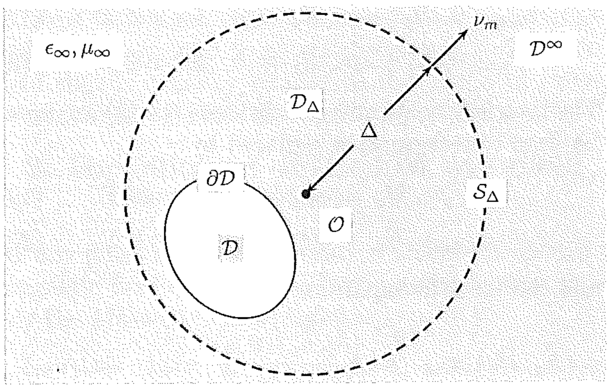

Figure 2: Configuration used in the derivation of the time Laplace-transform domain uniqueness identity. (The limit $\Delta \to \infty$ is taken.)

$$-\int_{\partial\mathcal{D}} \varepsilon_{m,k,j}\nu_m \hat{E}_k \hat{H}_j \mathrm{d}A(\boldsymbol{x}) + \int_{\mathcal{D}^\infty} (s\hat{E}_k \epsilon_\infty \hat{E}_k + s\hat{H}_j \mu_\infty \hat{H}_j)\mathrm{d}V(\boldsymbol{x}) = 0. \qquad (26)$$

Addition of Equations (24) and (26) finally yields

$$\int_{\mathcal{D}} (s\hat{E}_k \hat{\epsilon}_{k,q} \hat{E}_q + s\hat{H}_j \hat{\mu}_{j,p} \hat{H}_p)\mathrm{d}V(\boldsymbol{x}) + \int_{\mathcal{D}^\infty} (s\hat{E}_k \epsilon_\infty \hat{E}_k + s\hat{H}_j \mu_\infty \hat{H}_j)\mathrm{d}V(\boldsymbol{x}) = -\int_{\mathcal{D}} (s\hat{E}_k \hat{P}_k + s\hat{H}_j \hat{M}_j)\mathrm{d}V(\boldsymbol{x}), \qquad (27)$$

where the surface integrals over $\partial\mathcal{D}$ have canceled in view of the continuity of $\varepsilon_{m,k,j}\nu_m \hat{E}_k \hat{H}_j$ across $\partial\mathcal{D}$. Equation (27) will be used in the construction of the uniqueness proof.

5. The uniqueness proof

The uniqueness proof starts by assuming that in the given configuration, for one and the same excitation, there exist at least two non-identical field solutions, which we will distinguish by the superscripts $^{[1]}$ and $^{[2]}$. Obviously, $P_k^{[1]} = P_k^{[2]} = P_k$ and $M_j^{[1]} = M_j^{[2]} = M_j$. Consider the differences in value in the field quantities $\Delta E_q = E_q^{[2]} - E_q^{[1]}$ and $\Delta H_p = H_p^{[2]} - H_p^{[1]}$. Their time Laplace transforms then satisfy the equations (cf. Equations (15) - (18))

$$\varepsilon_{k,m,p}\partial_m \Delta\hat{H}_p - s\hat{\epsilon}_{k,q}\Delta\hat{E}_q = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}, \qquad (28)$$

$$\varepsilon_{j,n,q}\partial_n \Delta\hat{E}_q + s\hat{\mu}_{j,p}\Delta\hat{H}_p = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}, \qquad (29)$$

and

$$\varepsilon_{k,m,p}\partial_m \Delta\hat{H}_p - s\epsilon_\infty\Delta\hat{E}_k = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^\infty, \qquad (30)$$

$$\varepsilon_{j,n,q}\partial_n \Delta\hat{E}_q + s\mu_\infty\Delta\hat{H}_j = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^\infty. \qquad (31)$$

The same operations that have led to Equation (27) now yield

$$\int_{\mathcal{D}} (s\Delta\hat{E}_k \hat{\epsilon}_{k,q} \Delta\hat{E}_q + s\Delta\hat{H}_j \hat{\mu}_{j,p} \Delta\hat{H}_p)\mathrm{d}V(\boldsymbol{x}) + \int_{\mathcal{D}^\infty} (s\Delta\hat{E}_k \epsilon_\infty \Delta\hat{E}_k + s\Delta\hat{H}_j \mu_\infty \Delta\hat{H}_j)\mathrm{d}V(\boldsymbol{x}) = 0. \qquad (32)$$

Evidently, for real, positive values of s the integrand in the integral over $\mathcal{D}^\infty$, and hence the integral itself, is positive for any non identically vanishing $\Delta\hat{E}_k$ and/or any non identically vanishing $\Delta\hat{H}_j$ throughout $\mathcal{D}^\infty$. The integral over $\mathcal{D}$ shares this property if we impose on $\hat{\epsilon}_{k,q}$ and $\hat{\mu}_{j,p}$ the condition that throughout $\mathcal{D}$ they are positive definite tensors of rank two for all real, positive values of s. Under this condition, also the integral over $\mathcal{D}$ is positive for any non identically vanishing $\Delta\hat{E}_k$ and/or any non identically vanishing $\Delta\hat{H}_j$ throughout $\mathcal{D}$. For non identically vanishing $\Delta\hat{E}_k$ and/or non identically vanishing $\Delta\hat{H}_j$ throughout $\mathcal{D} \cup \mathcal{D}^\infty$ Equation (32) leads, in view of the value zero of the right-hand side to a contradiction. Under the given conditions we therefore have $\Delta\hat{E}_k = 0$ and $\Delta\hat{H}_j = 0$ for $\boldsymbol{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$, which implies $\hat{E}_k^{[2]} = \hat{E}_k^{[1]}$ and $\hat{H}_j^{[2]} = \hat{H}_j^{[1]}$ for $\boldsymbol{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$. In view of Lerch's uniqueness theorem of the one-sided Laplace transformation this implies that $E_k^{[2]} = E_k^{[1]}$ and $H_j^{[2]} = H_j^{[1]}$ for $\boldsymbol{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$ and all $t \geq 0$, i.e., there is only one electromagnetic field in the configuration that is causally related to the action of its exciting sources.

It is noted that the conditions imposed on the constitutive relaxation functions are specified through their time Laplace transforms. Strictly speaking the pertaining conditions need only hold on a Lerch sequence. In view of the analyticity of the transforms in $\{s \in \mathcal{C}; \mathrm{Re}(s) > 0\}$, however, they hold for all real, positive values of s. The conditions thus specified are *sufficient ones*, but at present no weaker conditions seem to be in existence. Also, a simple time-domain counterpart does not seem to exist. This, however, is the same situation as in linear, time-invariant, causal system's theory.

6. Examples of relaxation functions

Some examples of relaxation functions that arise in the physics of electric and magnetic materials are given below. They all apply to the simple case of isotropic materials.

Permittivity relaxation function of an isotropic plasma

For an isotropic plasma the (isotropic, scalar) s-domain permittivity relaxation function as based on the Lorentz theory of electrons model is [1, pp. 639-640]

$$\hat{\epsilon} = \epsilon_0 \left[1 + \frac{1}{s} \frac{{\omega_\mathrm{p}}^2}{s + \nu_\mathrm{c}} \right], \qquad (33)$$

where ϵ_0 is the permittivity of vacuum, ω_p the electron angular plasma frequency of the plasma and ν_c the collison frequency. The corresponding time-domain relaxation function is

$$\epsilon = \epsilon_0 \{\delta(t) + ({\omega_\mathrm{p}}^2/\nu_\mathrm{c})[1 - \exp(-\nu_\mathrm{c} t)] H(t)\}, \qquad (34)$$

where $\delta(t)$ is the Dirac delta distribution and $H(t)$ is the Heaviside unit step function. The time-domain magnetic relaxation function is $\mu = \mu_0 \delta(t)$, where μ_0 is the permeability of vacuum.

Lorentzian absorption line of a dielectric material

For the Lorentzian absorption line of a dielectric material the (isotropic, scalar) s-domain permittivity relaxation function is [1, pp. 639-640]

$$\hat{\epsilon} = \epsilon_0 \left[1 + \frac{{\omega_\mathrm{p}}^2}{(s + \Gamma/2)^2 + \Omega^2} \right], \qquad (35)$$

where Γ is a phenomenological damping coefficient, $\Omega = (\omega_0^2 - {\omega_\mathrm{p}}^2/3 - \Gamma^2/4)^{1/2}$ is the natural angular frequency of the oscillations of the movable electric charge, ω_0 is the resonant angular frequency of the (Coulomb force) mechanical model of the atom and ω_p is the angular plasma frequency of the movable electric charge distribution. The corresponding time-domain relaxation function is

$$\epsilon = \epsilon_0 [\delta(t) + ({\omega_\mathrm{p}}^2/\Omega) \exp(-\Gamma t/2) \sin(\Omega t) H(t)]. \qquad (36)$$

The time-domain magnetic relaxation function is $\mu = \mu_0 \delta(t)$, where μ_0 is the magnetic permeability of vacuum.

Linear hysteresis in a magnetic material

Linear hysteresis in an isotropic magnetic material can be modeled via a Debije type of relaxation function

$$\hat{\mu} = \mu_0 \mu_\mathrm{r} (1 + \Gamma/s), \qquad (37)$$

where μ_r is the relative permeability of the material and Γ is a phenomenological (Landau) damping coefficient. The corresponding time-domain relaxation function is

$$\mu = \mu_0 \mu_\mathrm{r} [\delta(t) + \Gamma H(t)]. \qquad (38)$$

The electric properties of the material need further specification.

It is observed that all these relaxation functions satisfy the conditions for uniqueness discussed in Section 5,

7. Conclusion

A time-domain uniqueness theorem for electromagnetic wavefield modeling in arbitrarily dispersive and anisotropic media is presented. Sufficient conditions for the uniqueness to be laid upon the electric and magnetic tensorial relaxation functions are formulated in the (causal) time Laplace-transform domain for real, positive values of the transform parameter. Some simple relaxation functions arising from physical models on an atomic level in plasma and solid-state physics are shown to be in accordance with the criteria developed.

8. References

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