A uniqueness theorem for the time-domain elastic-wave scattering in inhomogeneous, anisotropic solids with relaxation

Adrianus T. de Hoop[a)]
Laboratory of Electromagnetic Research, Faculty of Information Technology and Systems, Delft University of Technology, 4 Mekelweg, 2628 CD Delft, the Netherlands

(Received 16 March 2003; revised 11 June 2003; accepted 23 February 2004)

A uniqueness theorem for the (analytic or computational) time-domain modeling of the elastic wave motion in a scattering configuration that consists of inhomogeneous, anisotropic solids with arbitrary relaxation properties, occupying a bounded subdomain in an unbounded homogeneous, isotropic, perfectly elastic embedding, is presented. No direct time-domain uniqueness proof seems to exist for this kind of configuration. As an intermediate step, the one-to-one correspondence between the causal time-domain wavefield components and the constitutive material response functions on the one hand, and their time Laplace-transform counterparts for (a sequence of) real, positive values of the transform parameter on the other hand, seems a necessary tool. It is shown that such an approach leads to simple, explicit, sufficiency conditions on the inertial loss and compliance relaxation tensors describing the solid's constitutive behavior for uniqueness to hold. In it, the property of causality plays an essential role. In Christensen [*Theory of Viscoelasticity—An Introduction* (Academic, New York, 1971)] a similar approach is applied to the problem of uniqueness of the elastodynamic initial-/boundary-value problem associated with a viscoelastic object of bounded extent, the surface of which is subject to an admissible set of explicit boundary values. In the scattering configuration of unbounded extent, no explicit boundary values occur and the far-field compressional and shear wave radiation characteristics at "infinity" in the embedding play a key role in the proof. [DOI: 10.1121/1.1710876]

PACS numbers: 43.20.Bi, 43.20.Px [JJM] Pages: 2711–2715

I. INTRODUCTION

One of the issues one is confronted with in the mathematical modeling—be it with analytical or numerical techniques—of elastodynamic wave phenomena is the question about the uniqueness of the solution of the problem as it is formulated mathematically. Such a uniqueness should be expected on account of the underlying physics. When investigating wave propagation and scattering problems, the pertaining partial differential equations, constitutive relations, boundary conditions at interfaces, excitation conditions at exciting sources, initial values at the time window one considers, and the causal relationship that is to exist between the exciting sources and the generated wavefield are expected to play a role. For simple solids with instantaneous constitutive relations, i.e., for lossless solids, the elastodynamic time-domain power balance provides a tool to prove uniqueness. This is also the case when simple loss mechanisms (such as the Kelvin–Voigt relaxation model and frictional force losses) are incorporated. A uniqueness proof for the case of arbitrary relaxation effects in the solids seems to withstand such a direct time-domain approach. Since for the class of linear, time-invariant, causally reacting solids the constitutive relations are expressed via time convolutions (Christensen, 1971, Chap. I; Achenbach, 1973, pp. 399–402), the time Laplace transformation (under which transformation the convolution operation transforms into a simple product of the relevant constituents) can be expected to provide a useful tool. In Christensen (1971) such an approach is applied to the problem of uniqueness of the elastodynamic initial-/boundary-value problem associated with a viscoelastic object of bounded extent, the surface of which is subject to an admissible set of explicit boundary values. In the present paper the method is applied to a scattering configuration that consists of inhomogeneous, anisotropic solids with arbitrary relaxation properties, occupying a bounded subdomain in an unbounded homogeneous, isotropic, perfectly elastic embedding. In such a configuration no explicit boundary values occur and the far-field compressional and shear wave radiation characteristics at "infinity" in the embedding play a key role in the proof. Simple, explicit, sufficiency conditions on the inertial loss and compliance relaxation tensors describing the solid's constitutive behavior in the inhomogeneous subdomain of the scattering configuration are derived. In view of Lerch's theorem of the one-sided (=causal) Laplace transformation (Widder, 1946), they are to hold at a sequence of equidistant, real, positive values of the time Laplace transform parameter.

II. DESCRIPTION OF THE CONFIGURATION

The configuration for which the uniqueness of the elastodynamic wavefield problem is proved consists of a linear, time-invariant, locally reacting, inhomogeneous, anisotropic solid with arbitrary inertia and compliance relaxation properties and of bounded support $\mathcal{D}\subset\mathcal{R}^3$. This part of the configuration is embedded in a linear, time-invariant, locally reacting, homogeneous, isotropic, instantaneously reacting solid with volume density of mass ρ^∞ and Lamé stiffness

[a)]Electronic mail: a.t.dehoop@its.tudelft.nl

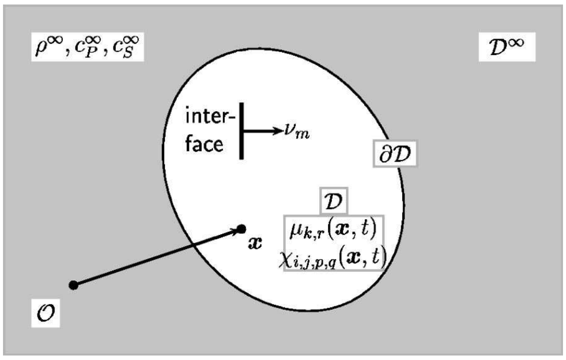

FIG. 1. Scattering configuration with inhomogeneous, anisotropic solid with relaxation (with bounded support $\mathcal{D}$) embedded in a homogeneous, isotropic, lossless solid (with unbounded support $\mathcal{D}^\infty$).

coefficients λ^∞ and μ^∞. These constitutive coefficients satisfy the conditions $\rho^\infty > 0$, $\lambda^\infty > -2\mu^\infty/3$ and $\mu^\infty > 0$. The corresponding compressional or P-wave speed is $c_P^\infty = [(\lambda^\infty + 2\mu^\infty)/\rho^\infty]^{1/2}$, and the corresponding shear or S-wave speed is $c_S^\infty = (\mu^\infty/\rho^\infty)^{1/2}$. The unbounded domain occupied by the embedding is denoted as $\mathcal{D}^\infty$. The common boundary of $\mathcal{D}$ and $\mathcal{D}^\infty$ is the bounded closed surface $\partial\mathcal{D}$ (Fig. 1). The configuration thus defined is typical for the modeling of elastodynamic wave scattering problems. The constitutive relaxation functions in $\mathcal{D}$ vary piecewise continuously with position, with finite jump discontinuities at a finite number of piecewise smooth, bounded surfaces (interfaces). Position in the configuration is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to an orthogonal Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors is used and the summation convention for repeated subscripts applies. Whenever appropriate, vectors are indicated by boldface symbols, with $\mathbf{x}$ as the position vector. The time coordinate is t. Partial differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol indicating partial differentiation with respect to t. Volume source distributions of force and of deformation rate, with bounded supports, excite a transient elastodynamic wavefield in the configuration. Without loss of generality we locate the sources in $\mathcal{D}$. They start to act at the instant $t=0$. The field that is causally related to the action of these sources, then vanishes throughout the configuration for $t<0$.

III. FORMULATION OF THE ELASTODYNAMIC WAVEFIELD PROBLEM

At any point in the configuration where the elastodynamic wavefield quantities are differentiable they satisfy the linearized, coupled, first-order elastic wave equations (De Hoop, 1995, pp. 311, 314, and 320)

$$\Delta^+_{k,m,p,q}\partial_m \tau_{p,q} - \partial_t(\mu_{k,r} \overset{(t)}{*} v_r) = -f_k, \qquad (1)$$

$$\Delta^+_{i,j,n,r}\partial_n v_r - \partial_t(\chi_{i,j,p,q} \overset{(t)}{*} \tau_{p,q}) = h_{i,j}, \qquad (2)$$

where $\tau_{p,q}$=dynamic stress (Pa), v_r=particle velocity (m/s), f_k=volume source density of force (N/m^3), $h_{i,j}$=volume source density of deformation rate (s^{-1}), $\mu_{k,r}$=inertia relaxation tensor (kg/m$^3\cdot$s), and $\chi_{i,j,p,q}$=compliance relaxation tensor (Pa/s).

The symbol $\overset{(t)}{*}$ denotes time convolution and $\Delta^+_{k,m,p,q}$ is the symmetrical unit tensor of rank four: $\Delta^+_{k,m,p,q} = (\delta_{k,p}\delta_{m,q} + \delta_{k,q}\delta_{m,p})/2$, with $\delta_{k,p} = \{1,0\}$ for $\{k=p,\ k\neq p\}$ as the symmetrical unit tensor of rank two (Kronecker tensor). The symmetrical unit tensor of rank four extracts out of any tensor of rank two with which it has contracted its symmetrical part. So, $\Delta^+_{k,m,p,q}\tau_{p,q} = (\frac{1}{2})(\tau_{k,m} + \tau_{m,k})$ and $\Delta^+_{i,j,n,r}\partial_n v_r = (\frac{1}{2})(\partial_i v_j + \partial_j v_i)$. The constitutive relaxation tensors are piecewise continuous functions of position in $\mathcal{D}$, while in the embedding (De Hoop, 1995, pp. 320–321)

$$\mu_{k,r} = \rho^\infty \delta_{k,r}\delta(t) \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty, \qquad (3)$$

$$\chi_{i,j,p,q} = [\Lambda^\infty \delta_{i,j}\delta_{p,q} + M^\infty(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})]\delta(t) \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty, \qquad (4)$$

in which $\Lambda^\infty = -\lambda^\infty/(3\lambda^\infty + 2\mu^\infty)2\mu^\infty$ and $M^\infty = 1/4\mu^\infty$.

Across any interface Σ of jump discontinuity in constitutive properties the boundary conditions of the continuity type (De Hoop, 1995, pp. 322–323)

$$\Delta^+_{k,m,p,q}\nu_m \tau_{p,q} = \text{continuous across } \Sigma, \qquad (5)$$

$$v_r = \text{continuous across } \Sigma, \qquad (6)$$

hold, where ν_m is the unit vector along the normal to Σ. This implies that the dynamic traction (=the normal component of the dynamic stress) and all components of the particle velocity are continuous across the interface. The constitutive relaxation functions are subject to the causality condition

$$\mu_{k,r}(\mathbf{x},t) = 0 \quad \text{for } t<0 \quad \text{and all } \mathbf{x} \in \mathcal{D}, \qquad (7)$$

$$\chi_{i,j,p,q}(\mathbf{x},t) = 0 \quad \text{for } t<0 \quad \text{and all } \mathbf{x} \in \mathcal{D}. \qquad (8)$$

Further conditions to be laid upon them with regard to the uniqueness of the elastodynamic wavefield problem are investigated further on.

In the embedding, the Green's tensors (dynamic stress and particle velocity due to point-sources of force and of deformation rate) can be determined analytically (De Hoop, 1995, Secs. 15.8 and 15.12). From the corresponding surface source representations over $\partial\mathcal{D}$ it follows that the outgoing fields in $\mathcal{D}^\infty$ admit the far-field expansion

$$\{\tau_{p,q}, v_r\}(\mathbf{x},t) = \left[\frac{\{T^P_{p,q}, V^P_r\}(\boldsymbol{\theta}, t - |\mathbf{x}|/c_P^\infty)}{4\pi|\mathbf{x}|} + \frac{\{T^S_{p,q}, V^S_r\}(\boldsymbol{\theta}, t - |\mathbf{x}|/c_S^\infty)}{4\pi|\mathbf{x}|}\right] \times [1 + O(|\mathbf{x}|^{-1})] \quad \text{as } |\mathbf{x}| \to \infty, \qquad (9)$$

where $\mathbf{x}$ is the position vector from the chosen far-field reference center to the point of observation and $\boldsymbol{\theta} = \mathbf{x}/|\mathbf{x}|$ is the unit vector in the direction of observation. The far-field radiation characteristics $T^{P,S}_{p,q}$ for the dynamic stress and $V^{P,S}_r$ for the particle velocity are mutually related via

$$V_k^P = -(\rho^\infty c_P^\infty)^{-1}\Delta^+_{k,m,p,q}\theta_m T^P_{p,q}, \quad (10)$$

$$V_k^S = -(\rho^\infty c_S^\infty)^{-1}\Delta^+_{k,m,p,q}\theta_m T^S_{p,q}, \quad (11)$$

while

$$V_k^P = (V_r^P\theta_r)\theta_k, \quad (12)$$

$$V_k^S = V_k^S - (V_r^S\theta_r)\theta_k, \quad (13)$$

implying that the *P*-wave far-field particle velocity is longitudinal with respect to its radial direction of propagation and the *S*-wave far-field particle velocity is transverse with respect to its radial direction of propagation.

In the following it is shown that the problem thus formulated has at most one solution, assuming that, for each type of excitation, at least one solution exists. The proof puts restrictions on the relaxation functions representing the inertia and complicance properties of the solid in $\mathcal{D}$. For the medium in $\mathcal{D}^\infty$ the conditions simply are $\rho^\infty > 0$, $\lambda^\infty > -2\mu^\infty/3$ and $\mu^\infty > 0$ (as indicated already).

IV. THE ELASTODYNAMIC WAVEFIELD PROBLEM IN THE TIME LAPLACE-TRANSFORM DOMAIN

The general type of solids with relaxation properties as considered in the present paper withstands, as far as is known, a direct uniqueness proof in the space/time domain based on energy considerations as is the case for media with simple constitutive behavior (Achenbach, 1973, pp. 80–82). However, taking into account the causality of both the solid's passive constitutive response and the wavefield's relation to its activating sources, the time Laplace transformation with real, positive transform parameter yields a tool to specify certain conditions to be imposed on the constitutive relaxation functions in order that the wavefield problem has a unique solution. The relevant transformation is given by

$$\{\hat{\tau}_{p,q}, \hat{v}_r\}(\mathbf{x},s) = \int_{t=0}^{\infty} \exp(-st)\{\tau_{p,q}, v_r\}(\mathbf{x},t)\,\mathrm{d}t. \quad (14)$$

For the case of physical interest of excitation functions and relaxation functions that are bounded in space and at most show a Dirac delta distribution time behavior, the time Laplace transforms of the wavefield quantities and the relaxation tensors exist for all $\{s \in \mathcal{C}; \mathrm{Re}(s) > 0\}$, i.e., for all values of the transform parameter in the right half of the complex *s*-plane. Furthermore, since all time functions involved are real-valued, their Laplace transforms take on real values for real values of *s*. In relation to our uniqueness proof we now take *s* to be a *Lerch sequence*: $\{s \in \mathcal{R};\ s = s_0 + nh,\ s_0 > 0,\ h > 0,\ n = 0,1,2,...\}$. Lerch's theorem (Widder, 1946, p. 63) states that if the transformation expressed by Eq. (14) is to hold for all *s* belonging to such a sequence, only one (causal) time-domain original corresponds to its related transform. Recalling that under the transformation the time derivative is replaced with a multiplication by *s* (if zero-value initial conditions apply, as is the case) and that the time convolution transforms into the product of the constituents, Eqs. (1)–(4) lead, upon time Laplace transformation, to

$$\Delta^+_{k,m,p,q}\partial_m\hat{\tau}_{p,q} - s\hat{\mu}_{k,r}\hat{v}_r = -\hat{f}_k \quad \text{for } \mathbf{x} \in \mathcal{D}, \quad (15)$$

$$\Delta^+_{i,j,n,r}\partial_n\hat{v}_r - s\hat{\chi}_{i,j,p,q}\hat{\tau}_{p,q} = \hat{h}_{i,j} \quad \text{for } \mathbf{x} \in \mathcal{D}, \quad (16)$$

and

$$\Delta^+_{k,m,p,q}\partial_m\hat{\tau}_{p,q} - s\rho^\infty\hat{v}_k = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty, \quad (17)$$

$$\Delta^+_{i,j,n,r}\partial_n\hat{v}_r - s[\Lambda^\infty\delta_{i,j}\hat{\tau}_{p,p} + M^\infty(\hat{\tau}_{i,j} + \hat{\tau}_{j,i})] = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty. \quad (18)$$

The interface continuity conditions (5) and (6) are upon Laplace transformation replaced by

$$\Delta^+_{k,m,p,q}\nu_m\hat{\tau}_{p,q} = \text{continuous across } \Sigma, \quad (19)$$

$$\hat{v}_r = \text{continuous across } \Sigma, \quad (20)$$

and the far-field expansion (9) by

$$\{\hat{\tau}_{p,q}, \hat{v}_r\}(\mathbf{x},t) = \left[\{\hat{T}^P_{p,q}, \hat{V}^P_r\}(\boldsymbol{\theta},s)\frac{\exp(-s|\mathbf{x}|/c_P^\infty)}{4\pi|\mathbf{x}|} + \{\hat{T}^S_{p,q}, \hat{V}^S_r\}(\boldsymbol{\theta},s)\frac{\exp(-s|\mathbf{x}|/c_S^\infty)}{4\pi|\mathbf{x}|}\right] \times [1 + O(|\mathbf{x}|^{-1})] \quad \text{as } |\mathbf{x}| \to \infty. \quad (21)$$

Upon contracting Eqs. (15) and (17) with $\hat{v}_k$ and Eqs. (16) and (18) with $\hat{\tau}_{i,j}$ and combining the results, the relations

$$-\partial_m\Delta^+_{m,r,p,q}(\hat{\tau}_{p,q}\hat{v}_r) + s\hat{v}_k\hat{\mu}_{k,r}\hat{v}_r + s\hat{\tau}_{i,j}\hat{\chi}_{i,j,p,q}\hat{\tau}_{p,q} = \hat{v}_k\hat{f}_k - \hat{\tau}_{i,j}\hat{h}_{i,j} \quad \text{for } \mathbf{x} \in \mathcal{D}, \quad (22)$$

and

$$-\partial_m\Delta^+_{m,r,p,q}(\hat{\tau}_{p,q}\hat{v}_r) + s\hat{v}_k\rho^\infty\hat{v}_k + s\hat{\tau}_{i,j}[\Lambda^\infty\delta_{i,j}\hat{\tau}_{p,p} + M^\infty(\hat{\tau}_{i,j} + \hat{\tau}_{j,i})] = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty \quad (23)$$

are constructed. Integration of Eq. (22) over $\mathcal{D}$ and application of Gauss' divergence theorem yields

$$-\int_{\partial\mathcal{D}} \nu_m\Delta^+_{m,r,p,q}\hat{\tau}_{p,q}\hat{v}_r\,\mathrm{d}A(\mathbf{x}) + \int_{\mathcal{D}}(s\hat{v}_k\hat{\mu}_{k,r}\hat{v}_r + s\hat{\tau}_{i,j}\hat{\chi}_{i,j,p,q}\hat{\tau}_{p,q})\mathrm{d}V(\mathbf{x}) = \int_{\mathcal{D}}(\hat{v}_k\hat{f}_k - \hat{\tau}_{i,j}\hat{h}_{i,j})\mathrm{d}V(\mathbf{x}), \quad (24)$$

where ν_m is the outward unit vector along the normal to $\partial\mathcal{D}$. Next, Eq. (23) is integrated over the domain that is bounded internally by $\partial\mathcal{D}$ and externally by the sphere $\mathcal{S}_\Delta$ of radius Δ and center at the far-field reference center, where Δ is chosen so large that $\mathcal{S}_\Delta$ completely surrounds $\partial\mathcal{D}$ (Fig. 2). Subsequent application of Gauss' divergence theorem leads to

$$-\int_{\mathcal{S}_\Delta} \nu_m\Delta^+_{m,r,p,q}\hat{\tau}_{p,q}\hat{v}_r\,\mathrm{d}A(\mathbf{x}) + \int_{\partial\mathcal{D}} \nu_m\Delta^+_{m,r,p,q}\hat{\tau}_{p,q}\hat{v}_r\,\mathrm{d}A(\mathbf{x}) + \int_{\mathcal{D}^\infty\cap\mathcal{D}_\Delta}\{s\hat{v}_k\rho^\infty\hat{v}_k + s\hat{\tau}_{i,j}[\Lambda^\infty\delta_{i,j}\hat{\tau}_{p,p} + M^\infty(\hat{\tau}_{i,j} + \hat{\tau}_{j,i})]\}\mathrm{d}V(\mathbf{x}) = 0, \quad (25)$$

where $\mathcal{D}_\Delta$ is the domain interior to $\mathcal{S}_\Delta$. Using the far-field representation (21) in the integration over $\mathcal{S}_\Delta$ and taking the

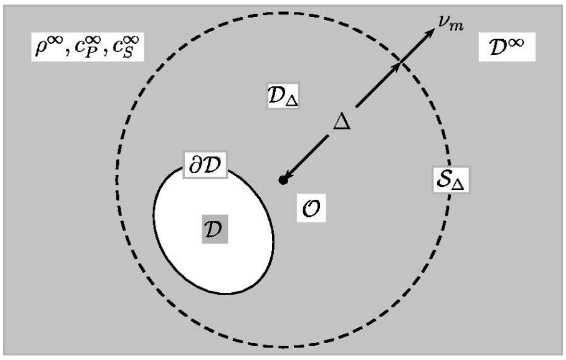

FIG. 2. Configuration used in the derivation of the time Laplace-transform domain uniqueness criterion. (The limit $\Delta \to \infty$ is taken.)

limit $\Delta \to \infty$, Eq. (25) leads to (note that the integral over $\mathcal{S}_\Delta$ goes to zero as $\Delta \to \infty$)

$$\int_{\partial\mathcal{D}} \nu_m \Delta^+_{m,r,p,q} \hat{\tau}_{p,q} \hat{v}_r \, \mathrm{d}A(\mathbf{x}) + \int_{\mathcal{D}^\infty} \{ s \hat{v}_k \rho^\infty \hat{v}_k + s \hat{\tau}_{i,j} [\Lambda^\infty \delta_{i,j} \hat{\tau}_{p,p} + M^\infty (\hat{\tau}_{i,j} + \hat{\tau}_{j,i})] \} \mathrm{d}V(\mathbf{x}) = 0. \quad (26)$$

Addition of Eqs. (24) and (26) finally yields

$$\int_{\mathcal{D}} (s \hat{v}_k \hat{\mu}_{k,r} \hat{v}_r + s \hat{\tau}_{i,j} \hat{\chi}_{i,j,p,q} \hat{\tau}_{p,q}) \mathrm{d}V(\mathbf{x}) + \int_{\mathcal{D}^\infty} \{ s \hat{v}_k \rho^\infty \hat{v}_k + s \hat{\tau}_{i,j} [\Lambda^\infty \delta_{i,j} \hat{\tau}_{p,p} + M^\infty (\hat{\tau}_{i,j} + \hat{\tau}_{j,i})] \} \mathrm{d}V(\mathbf{x}) = \int_{\mathcal{D}} (\hat{v}_k \hat{f}_k - \hat{\tau}_{i,j} \hat{h}_{i,j}) \mathrm{d}V(\mathbf{x}), \quad (27)$$

where the surface integrals over $\partial\mathcal{D}$ have canceled in view of the continuity of $\nu_m \Delta^+_{m,r,p,q} \hat{\tau}_{p,q} \hat{v}_r$ across $\partial\mathcal{D}$. Equation (27) is the basis for the construction of the uniqueness proof.

V. THE UNIQUENESS PROOF

The uniqueness proof follows the standard procedure (see, for example, Christensen, 1971, Sec. 5.1). It starts by assuming that in the given configuration, for one and the same source excitation, there exist at least two nonidentical wavefield solutions, which are distinguished by the superscripts $^{[1]}$ and $^{[2]}$. Then, $f_k^{[1]} = f_k^{[2]} = f_k$ and $h_{i,j}^{[1]} = h_{i,j}^{[2]} = h_{i,j}$. Consider the differences in value in the field quantities $\Delta \tau_{p,q} = \tau_{p,q}^{[2]} - \tau_{p,q}^{[1]}$ and $\Delta v_r = v_r^{[2]} - v_r^{[1]}$. Their time Laplace transforms then satisfy the equations [cf. Eqs. (15)–(18)]

$$\Delta^+_{k,m,p,q} \partial_m \Delta \hat{\tau}_{p,q} - s \hat{\mu}_{k,r} \Delta \hat{v}_r = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}, \quad (28)$$

$$\Delta^+_{i,j,n,r} \partial_n \Delta \hat{v}_r - s \hat{\chi}_{i,j,p,q} \Delta \hat{\tau}_{p,q} = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}, \quad (29)$$

and

$$\Delta^+_{k,m,p,q} \partial_m \Delta \hat{\tau}_{p,q} - s \rho^\infty \Delta \hat{v}_k = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty, \quad (30)$$

$$\Delta^+_{i,j,n,r} \partial_n \Delta \hat{v}_r - s [\Lambda^\infty \delta_{i,j} \Delta \hat{\tau}_{p,p} + M^\infty (\Delta \hat{\tau}_{i,j} + \Delta \hat{\tau}_{j,i})] = 0 \quad \text{for } \mathbf{x} \in \mathcal{D}^\infty. \quad (31)$$

The same operations that have led to Eq. (27) now lead to

$$\int_{\mathcal{D}} (s \Delta \hat{v}_k \hat{\mu}_{k,r} \Delta \hat{v}_r + s \Delta \hat{\tau}_{i,j} \hat{\chi}_{i,j,p,q} \Delta \hat{\tau}_{p,q}) \mathrm{d}V(\mathbf{x}) + \int_{\mathcal{D}^\infty} \{ s \Delta \hat{v}_k \rho^\infty \Delta \hat{v}_k + s \Delta \hat{\tau}_{i,j} [\Lambda^\infty \delta_{i,j} \Delta \hat{\tau}_{p,p} + M^\infty (\Delta \hat{\tau}_{i,j} + \Delta \hat{\tau}_{j,i})] \} \mathrm{d}V(\mathbf{x}) = 0. \quad (32)$$

By observing that

$$\Delta \hat{\tau}_{i,j} [\Lambda^\infty \delta_{i,j} \Delta \hat{\tau}_{p,p} + M^\infty (\Delta \hat{\tau}_{i,j} + \Delta \hat{\tau}_{j,i})] = (\Lambda^\infty + 2M^\infty/3) \Delta \hat{\tau}_{i,i} \Delta \hat{\tau}_{p,p} + M^\infty (\Delta \hat{\tau}_{i,j} - \Delta \hat{\tau}_{p,p} \delta_{i,j}/3) \times (\Delta \hat{\tau}_{i,j} - \Delta \hat{\tau}_{q,q} \delta_{i,j}/3), \quad (33)$$

and taking into account that $\Lambda^\infty + 2M^\infty/3 > 0$ and $M^\infty > 0$ in view of the conditions laid upon λ^∞ and μ^∞, it follows that for real, positive values of s the integrand in the integral over $\mathcal{D}^\infty$, and hence the integral itself, is positive for any nonidentically vanishing $\Delta \hat{\tau}_{p,q}$ and/or any nonidentically vanishing $\Delta \hat{v}_r$ throughout $\mathcal{D}^\infty$. The integral over $\mathcal{D}$ shares this property if we impose on $\hat{\mu}_{k,r}$ and $\hat{\chi}_{i,j,p,q}$ the condition that throughout $\mathcal{D}$ they are positive definite tensors of ranks two and four, respectively, for all real, positive values of s. Under this condition, also the integral over $\mathcal{D}$ is positive for any nonidentically vanishing $\Delta \hat{\tau}_{p,q}$ and/or any nonidentically vanishing $\Delta \hat{v}_r$ throughout $\mathcal{D}$. For nonidentically vanishing $\Delta \hat{\tau}_{p,q}$ and/or nonidentically vanishing $\Delta \hat{v}_r$ throughout $\mathcal{D} \cup \mathcal{D}^\infty$ Eq. (32) leads, in view of the value zero of the right-hand side, to a contradiction. Under the given conditions we therefore have $\Delta \hat{\tau}_{p,q} = 0$ and $\Delta \hat{v}_r = 0$ for $\mathbf{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$, which implies $\hat{\tau}_{p,q}^{[2]} = \hat{\tau}_{p,q}^{[1]}$ and $\hat{v}_r^{[2]} = \hat{v}_r^{[1]}$ for $\mathbf{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$. In view of Lerch's uniqueness theorem of the one-sided Laplace transformation this implies that $\tau_{p,q}^{[2]} = \tau_{p,q}^{[1]}$ and $v_r^{[2]} = v_r^{[1]}$ for $\mathbf{x} \in \{\mathcal{D} \cup \mathcal{D}^\infty\}$ and all $t \geq 0$, i.e., there is only one elastodynamic wavefield in the scattering configuration that is causally related to the action of its exciting sources.

It is noted that the conditions imposed on the constitutive relaxation functions are specified through their time Laplace transforms. Strictly speaking the pertaining conditions need only hold on a Lerch sequence. In view of the analyticity of the transforms in $\{s \in \mathcal{C}; \mathrm{Re}(s) > 0\}$, however, they hold for all real, positive values of s. The conditions thus specified are *sufficient ones*, but at present no weaker conditions seem to be in existence. Also, a simple time-domain counterpart does not seem to exist. This, however, is the same situation as in general linear, time-invariant, causal, passive system's theory.

VI. EXAMPLES OF RELAXATION FUNCTIONS

Some examples of relaxation functions that are in use to model elastic wave propagation in dissipative solids are given below. They all apply to the simple case of isotropic solids. Their dependence on $\mathbf{x}$ is not indicated explicitly.

A. Frictional-force and Maxwell-type viscosity

For an isotropic solid with frictional-force and Maxwell-type viscosity loss terms the constitutive coefficients are of the form (Kolsky, 1964, p. 107)

$$\hat{\mu}_{k,r} = \rho (1 + 1/s\tau_\mathrm{f}) \delta_{k,r}, \quad (34)$$

$$\hat{\chi}_{i,j,p,q}=[\Lambda\,\delta_{i,j}\delta_{p,q}+M(\delta_{i,p}\delta_{j,q}+\delta_{i,q}\delta_{j,p})]\times(1+1/s\tau_{\mathrm{M}}), \tag{35}$$

in which ρ is the volume density of mass of the solid, $\Lambda=-\lambda/(3\lambda+2\mu)\mu$, and $M=1/4\mu$, with λ and μ the Lamé stiffness coefficients of the solid, τ_{f} the frictional-force relaxation time and τ_{M} the Maxwell viscosity relaxation time. The coefficients satisfy the conditions $\rho>0$, $\lambda>-2\mu/3$, $\mu>0$, $\tau_{\mathrm{f}}>0$ and $\tau_{\mathrm{M}}>0$. The corresponding time-domain relaxation functions are

$$\mu_{k,r}=\rho[\delta(t)+\tau_{\mathrm{f}}^{-1}H(t)]\delta_{k,r}, \tag{36}$$

$$\chi_{i,j,p,q}=[\Lambda\,\delta_{i,j}\delta_{p,q}+M(\delta_{i,p}\delta_{j,q}+\delta_{i,q}\delta_{j,p})]\times[\delta(t)+\tau_{\mathrm{M}}^{-1}H(t)], \tag{37}$$

where $\delta(t)$ is the Dirac delta distribution and $H(t)$ is the Heaviside unit step function.

B. The standard linear solid with creep/relaxation loss mechanism

For the standard linear solid with creep/relaxation loss mechanism, or Zener solid, the constitutive coefficients are of the form (Carcione, 2001; Mainardi 2002)

$$\hat{\mu}_{k,r}=\rho\,\delta_{k,r}, \tag{38}$$

$$\hat{\chi}_{i,j,p,q}=[\Lambda\,\delta_{i,j}\delta_{p,q}+M(\delta_{i,p}\delta_{j,q}+\delta_{i,q}\delta_{j,p})]\times\left(1+\frac{1/\tau_\sigma-1/\tau_\epsilon}{1/\tau_\epsilon+s}\right), \tag{39}$$

in which $\tau_\sigma>0$ is the stress relaxation time and $\tau_\epsilon>0$ is the strain relaxation time. On account of the physical condition of creep yield, the condition

$$\tau_\sigma<\tau_\epsilon \tag{40}$$

holds. The corresponding time-domain relaxation functions are

$$\mu_{k,r}=\rho\,\delta(t)\,\delta_{k,r}, \tag{41}$$

$$\chi_{i,j,p,q}=[\Lambda\,\delta_{i,j}\delta_{p,q}+M(\delta_{i,p}\delta_{j,q}+\delta_{i,q}\delta_{j,p})]\times[\delta(t)+(1/\tau_\sigma-1/\tau_\epsilon)\exp(-t/\tau_\epsilon)H(t)]. \tag{42}$$

It is observed that the relaxation functions shown here do satisfy the conditions for uniqueness discussed in Sec. V.

VII. CONCLUSION

A time-domain uniqueness theorem for elastodynamic wavefield scattering in a configuration consisting of inhomogeneous, anisotropic solids with arbitrary relaxation properties, occupying a bounded subdomain in an unbounded homogeneous, isotropic, perfectly elastic embedding, is presented. Sufficient conditions for the uniqueness to be laid upon the tensorial relaxation functions are formulated in the (causal) time Laplace-transform domain for real, positive values of the transform parameter. Some simple relaxation functions that are in use in the modeling of wave phenomena in nonperfectly elastic solids are shown to be in accordance with the criteria developed.

ACKNOWLEDGMENT

The author wishes to express his sincere thanks to an (anonymous) reviewer for bringing to his attention the most valuable reference to Christensen (1971). For some inexplicable reason, this reference had hitherto escaped the author's attention.

Achenbach, J. D. (**1973**). *Wave Propagation in Elastic Solids* (North-Holland (Elsevier), Amsterdam).

Carcione, J. M. (**2001**). *Waves in Real Media: Wave Propagation in Anisotropic, Anelastic and Porous Media* (Pergamon, Oxford).

Christensen, R. M. (**1971**). *Theory of Viscoelasticity—An Introduction* (Academic, New York); also (Dover, New York, 2003).

De Hoop, A. T. (**1995**). *Handbook of Radiation and Scattering of Waves* (Academic, London).

Kolsky, H. (**1964**). *Stress Waves in Solids* (Dover, New York).

Mainardi, F. (**2002**). "Linear Viscoelasticiy," Chap. 4 in *Acoustic Interactions with Submerged Elastic Structures*, edited by A. Guran, A. Boström, O. Leroy, and G. Maze (World Scientific, Singapore).

Widder, D. V. (**1946**). *The Laplace Transform* (Princeton U.P., Princeton).