I. INTRODUCTION

In a variety of applications in acoustics (for example, in outdoor sound propagation, traffic noise analysis, jet-engine sound absorption in aircraft engineering and architectural acoustics), the analysis of the point-source excited reflection of sound waves by a boundary surface with certain absorptive and dispersive properties is of interest. In all these cases, the absorptive and dispersive properties of the boundary need characterization by a judiciously chosen set of parameters. Following the pioneering paper by Ingard (1951), such a characterization goes via a local acoustic admittance, i.e., a linear, time-invariant, causal, passive behavior. A parametrization of the admittance function is put forward that has the property of showing up explicitly, and in a relatively simple manner, in the expression for the reflected acoustic pressure. The partial fraction representation of the complex frequency domain admittance is shown to have such a property. The result opens the possibility of constructing inversion algorithms that enable the extraction of the relevant parameters from the measured time traces of the acoustic pressure at different offsets, parallel as well as normal to the boundary, between source and receiver. Illustrative theoretical numerical examples are presented. © 2005 Acoustical Society of America. [DOI: 10.1121/1.1954567]

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Parametrization of acoustic boundary absorption and dispersion properties in time-domain source/receiver reflection measurement

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Closed-form analytic time-domain expressions are obtained for the acoustic pressure associated with the reflection of a monopole point-source excited impulsive acoustic wave by a planar boundary with absorptive and dispersive properties. The acoustic properties of the boundary are modeled as a local admittance transfer function between the normal component of the particle velocity and the acoustic pressure. The transfer function is to meet the conditions for linear, time-invariant, causal, passive behavior. A parametrization of the admittance function is put forward that has the property of showing up explicitly, and in a relatively simple manner, in the expression for the reflected acoustic pressure. The partial fraction representation of the complex frequency domain admittance is shown to have such a property. The result opens the possibility of constructing inversion algorithms that enable the extraction of the relevant parameters from the measured time traces of the acoustic pressure at different offsets, parallel as well as normal to the boundary, between source and receiver. Illustrative theoretical numerical examples are presented. © 2005 Acoustical Society of America. [DOI: 10.1121/1.1954567]

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I. INTRODUCTION

In a variety of applications in acoustics (for example, in outdoor sound propagation, traffic noise analysis, jet-engine sound absorption in aircraft engineering and architectural acoustics), the analysis of the point-source excited reflection of sound waves by a boundary surface with certain absorptive and dispersive properties is of interest. In all these cases, the absorptive and dispersive properties of the boundary need characterization by a judiciously chosen set of parameters. Following the pioneering paper by Ingard (1951), such a characterization goes via a local acoustic admittance, i.e., via a linear, time-invariant, causal, passive transfer function that links the normal component of the particle velocity on the boundary to the local acoustic pressure. For the canonical configuration consisting of a planar boundary, a monopole acoustic (volume injection) source and a monopole acoustic (pressure) point receiver, we derive closed-form time-domain expressions for the received signal. For the same configuration and along similar lines, a recent paper (Lam et al., 2004) discusses some ad-hoc cases, where the boundary’s properties are expressed via a complex-frequency domain Padé representation, the coefficients in which are matched to experimental data available in the literature. The approach via the Padé representation appears, however, to be limited to at most the Padé (2,2) one. In the present paper, a more systematic approach is followed where the complex-frequency domain characterization of the boundary admittance goes via

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transforms are related such that the Laplace transform of the latter arises out of the Laplace transform of the former upon replacing the transform parameter $s$ with a certain function $\phi(s)$, where $\phi(s)$ belongs to the class of functions for which a causal time function corresponding to $\exp(-\phi(s)\tau)$, with $\tau > 0$, exists.

The analysis can be carried out for an arbitrary number of terms in the partial-fraction characterization of the boundary’s acoustic admittance, each of them provided with its associated two parameters. This implies that a rather accurate tuning of the parameters to match the measured values of the admittance (a procedure that is usually carried out in the frequency domain) can be achieved by incorporating a sufficient number of terms.

Some theoretical numerical examples illustrate how some physical phenomena can be attributed to certain ranges of the values of the parameters involved.

II. FORMULATION OF THE PROBLEM

Position in the configuration is specified by the coordinates $\{x,y,z\}$ with respect to an orthogonal, Cartesian reference frame with the origin $O$ and the three mutually perpendicular base vectors $\{i_x,i_y,i_z\}$ of unit length each; they form, in the indicated order, a right-handed system. The position vector is $r = xi_x + yi_y + zi_z$. The vectorial spatial differentiation operator is $\nabla = i_x \partial_x + i_y \partial_y + i_z \partial_z$. The time coordinate is $t$; differentiation with respect to time is denoted by $\partial_t$.

The acoustic wave motion is studied in the half-space $D=[-\infty < x < \infty, -\infty < y < \infty, 0 < z < \infty]$, which is filled with a fluid with volume density of mass $\rho_0$ and compressibility $\kappa_0$. The speed of sound waves in it is $c_0 = (\rho_0\kappa_0)^{-1/2}$. The acoustic wave motion is excited by an acoustic monopole point source with volume injection rate $Q_0(t)$ and located at $r_0 = (0,0,h)$, with $h \geq 0$. We assume that $Q_0(t) = 0$ for $t < 0$. The acoustic pressure $p(r,t)$ and the particle velocity $v(r,t)$ then satisfy the first-order acoustic wave equations (De Hoop, 1995, p. 44)

\[ \nabla p + \rho_0 \partial_t v = 0, \]

\[ \nabla \cdot v + \kappa_0 \partial_t p = Q_0(t) \delta(r - r_0). \]

Causality entails that $p(r,t) = 0$ and $v(r,t) = 0$ for $t < 0$ and all $r \in D$. The acoustic properties of the planar boundary are modeled via the local, linear, time-invariant, causal, passive acoustic admittance relation

\[ v_i(x,y,0,t) = - (\rho_0 c_0)^{-1} Y(t) * p(x,y,0,t), \]

where $*$ denotes time convolution and $Y(t)$ is the boundary’s acoustic time-domain admittance transfer function, normalized with respect to the acoustic plane-wave admittance $(\rho_0 c_0)^{-1}$ of the fluid. Figure 1 shows the configuration.

The acoustic wave field in the fluid is written as the superposition of the incident wave field to be denoted by the superscript $i$ and the reflected wave field to be denoted by the superscript $r$. The incident wave field is the wave field that is generated by the source and would be the total wave field in the absence of the boundary. Its acoustic pressure satisfies the scalar wave equation

\[ \nabla^2 p^i - c_0^2 \partial_t^2 p^i = -\rho_0 \partial_t Q_0(t) \delta(x,y,z-h). \]  (4)

From this equation we obtain (see, for example, De Hoop, 1995, pp. 93–97)

\[ p^i(r,t) = \rho_0 c_0^2 Q_0(t) * G^i(r,t), \]  (5)

in which the incident-wave Green’s function is

\[ G^i(r,t) = \frac{H(t-T_0)}{4\pi D_0} \]  for $D_0 > 0$,  (6)

with

\[ D_0 = [x^2 + y^2 + (z-h)^2]^{1/2} \geq 0 \]  (7)

as the distance from the source to the receiver,

\[ T_0 = D_0 / c_0 \]  (8)

as the travel time from source to receiver and $H(t)$ as the Heaviside unit step function.

III. THE COMPLEX SLOWNESS REPRESENTATION FOR THE ACOUSTIC WAVE FIELDS

The time invariance of the configuration and the causality of the sound waves are taken into account by the use of the unilateral Laplace transform:

\[ \{\hat{p},\hat{v}\}(r,s) = \int_{t=0}^{\infty} \exp(-st)\{p,v\}(r,t)dt. \]  (9)

The Laplace transform parameter $s$ is taken positive and real. Then, according to Lerch’s theorem (Widder, 1946) a one-to-one mapping exists between $\{p,v\}(r,t)$ and their time-Laplace transformed counterparts $\{\hat{p},\hat{v}\}(r,s)$. The fluid is initially at rest, which has the consequence that the transformation property $\partial_r \rightarrow s$ holds. Next, the complex slowness representations for $\{\hat{p},\hat{v}\}(r,s)$ are introduced as
\begin{equation}
\{\hat{\varphi}, \vec{\mathbf{B}}\}(x, y, z, s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\hat{\varphi}, \vec{\mathbf{B}}\}(\alpha, \beta, z, s) \times \exp[-i\delta(\alpha x + \beta y)] d\beta, \tag{10}
\end{equation}

where \(\alpha\) and \(\beta\) are the wave slownesses in the \(x\) and \(y\) directions, respectively. This representation entails the properties \(\varphi_{x} \rightarrow -is\alpha\), \(\varphi_{y} \rightarrow -is\beta\). Use of the transforms in Eqs. (11)-(4) yields for the incident wave

\begin{equation}
\vec{\mathbf{P}}(\alpha, \beta, z, s) = \frac{p_0 Q_0(s)}{2\gamma_0} \exp(-s\gamma_0[z - h]), \tag{11}
\end{equation}

while for the reflected wave we write

\begin{equation}
\vec{\mathbf{P}}'(\alpha, \beta, z, s) = \frac{p_0 Q_0(s)}{2\gamma_0} \hat{R} \exp(-s\gamma_0[z + h]), \tag{12}
\end{equation}

in which

\begin{equation}
\gamma_0(\alpha, \beta) = (c_0^{-2} + \alpha^2 + \beta^2)^{1/2} \quad \text{with} \quad Re(\gamma_0) \geq 0 \tag{13}
\end{equation}

is the wave slowness normal to the boundary and \(\hat{R}\) denotes the slowness-domain reflection coefficient. Use of Eqs. (11) and (12) in the complex slowness domain counterpart of the admittance boundary condition (3), together with the property [cf. Eq. (1)]

\begin{equation}
\vec{\mathbf{v}}_1 = -(s\gamma_0)^{-1} \partial_s \vec{\mathbf{p}}, \tag{14}
\end{equation}

Eqs. (11), (12), and (14) lead to

\begin{equation}
(s\gamma_0 p_0)(1 + \hat{R}) = (s\gamma_0 c_0)^{-1} Y(\gamma)(1 + \hat{R}), \tag{15}
\end{equation}

from which it follows that

\begin{equation}
\hat{R} = \frac{c_0 \gamma_0 - \hat{Y}(\gamma)}{c_0 \gamma_0 + \hat{Y}(\gamma)} = 1 - \frac{2\hat{Y}(\gamma)}{c_0 \gamma_0 + \hat{Y}(\gamma)}. \tag{16}
\end{equation}

IV. SPACE-TIME EXPRESSIONS FOR THE ACOUSTIC WAVE FIELD CONSTITUENTS

The expressions for the time Laplace transformed reflected wave field quantities are written as

\begin{equation}
\vec{\mathbf{P}}'(r, s) = p_0 s^2 \hat{Q}_0(s) \hat{G}'(r, s), \tag{17}
\end{equation}

\begin{equation}
\vec{\mathbf{v}}'(r, s) = -s \hat{Q}_0(s) \nabla \hat{G}'(r, s), \tag{18}
\end{equation}

in which

\begin{equation}
\hat{G}'(r, s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{R} \frac{1}{2\gamma_0} \exp(-sl[\alpha x + \beta y]) d\beta \tag{19}
\end{equation}

is the time Laplace transformed reflected-wave Green’s function. The time-domain counterparts of Eqs. (17)-(19) are determined with an aid of an extension (De Hoop, 2002) of the standard modified Cagniard method (De Hoop, 1960; De Hoop and Van der Hijken, 1984). First, upon writing \(s = r \cos(\theta), \quad y = r \sin(\theta)\), the transformation

\(\alpha = ip \cos(\theta) - q \sin(\theta), \tag{20}\)

is carried out, which for the slowness normal to the boundary leads to \(\tilde{\gamma}(q, p) = [\Omega(q^2 - p^2)]^{1/2}\), with \(\Omega(q) = (c_0^{-2} + q^2)^{1/2}\). Next, the integrand in the integration with respect to \(p\) is continued analytically into the complex \(p\) plane, away from the imaginary axis and, under the application of Cauchy’s theorem and Jordan’s lemma, the integration along the imaginary \(p\) axis is replaced by one along the hyperbolic path (modified Cagniard path) consisting of \(pr + \tilde{\gamma}(q, p)(z + h) = \tau\), together with its complex conjugate, for \(T_1(q) < \tau < \infty\), where \(T_1(q) = \Omega(q)D_1 \) and \(D_1 = [x^2 + y^2 + (z + h)^2]^{1/2} > 0\) is the distance from the image of the source to the receiver, while \(\tau\) is introduced as the variable of integration. In the relevant Jacobian, the relation \(dp/\delta \tau = i\tilde{\gamma}(q)/[\tau^2 - T_1^2(q)]^{1/2}\) is used. Next, Schwarz’s reflection principle of complex function theory is used to combine the integrations in the upper and lower halves of the complex \(p\) plane, the orders of integration with respect to \(\tau\) and \(q\) are interchanged, and in the resulting integration with respect to \(q\), that extends over the interval \(0 < q < (\tau^2 / D_1^2 - c_0^{-2})^{1/2}\), the variable of integration \(q\) is replaced with \(\psi\) defined through \(q = (\tau^2 / D_1^2 - c_0^{-2})^{1/2} \sin(\psi)\), with \(0 \leq \psi < \pi / 2\). This procedure leads to

\begin{equation}
\hat{G}'(r, s) = \frac{1}{4\pi D_1} \int_{T_1}^{\infty} \exp(-s\tau) \hat{K}'(r, \tau, s) d\tau, \tag{21}
\end{equation}

in which

\begin{equation}
\hat{K}'(r, \tau, s) = \frac{2}{\pi} \int_{\psi=0}^{\pi/2} \frac{re d\psi}{\rho_0 \tilde{\gamma}_0(Y/s) + \hat{Y}(s)} \tag{22}
\end{equation}

with

\begin{equation}
c_0 \tilde{\gamma}_0 = \Gamma_1(r, \tau) - i\Gamma_2(r, \tau) \cos(\psi), \tag{23}
\end{equation}

\begin{equation}\Gamma_1(r, \tau) = c_0 \tau(z + h)/D_1^2, \tag{24}\end{equation}

\begin{equation}\Gamma_2(r, \tau) = c_0 (\tau^2 - T_1^2)^{1/2} / D_1^2, \tag{25}\end{equation}

is the reflected-wave kernel function and

\begin{equation}
T_1 = T_1(0) = D_1/c_0 \tag{26}\end{equation}

is the travel time from the image of the source to the receiver. Evaluation of the integral in the right-hand side of Eq. (22) yields (see the Appendix)

\begin{equation}
\hat{K}'(r, \tau, s) = 1 - \frac{2\hat{Y}(\gamma)}{[\Gamma_1(r, \tau) + \hat{Y}(\gamma)]^2 + \Gamma_2^2(r, \tau)} \tag{27}\end{equation}

Since the right-hand side of Eq. (27) is an analytic function of \(s\) in the right half \([Re(s) > 0]\) of the complex \(s\) plane, it has a causal time-domain counterpart \(K'(r, \tau, t)\) that vanishes for \(t < 0\). In terms of the latter, Eq. (21) leads to the time-domain expression

\begin{equation}
K'(r, t) = \frac{1}{4\pi D_1} \int_{0}^{\infty} K'(r, \tau, t - \tau) d\tau H(t - T_1). \tag{28}\end{equation}

To further separate in the second term on the right-hand side of Eq. (27) the influence of the configurational parameters of the measurement setup from the influence of the parameters de Hoop et al.: Parametrization of an acoustic admittance-boundary

associated with the boundary’s acoustic admittance on the reflected field acoustic pressure, we make use of the Schouten–Van der Pol theorem of the unilateral Laplace transformation (Schouten, 1934, 1961; Van der Pol, 1934; Van der Pol and Bremmer, 1950) and employ the Laplace-transform integral formula (29.3.55) from Abramowitz and Stegun (1968, p. 1024), together with some elementary rules of the Laplace transformation to obtain

\[
\hat{Y}(s) = \left( [\Gamma_1(r, \tau) + \hat{Y}(s)]^2 + \Gamma_2^2(r, \tau) \right)^{1/2}
\]

\[
= 1 - \int_{w=0}^{\infty} K_{\hat{y}}(r, \tau, w) \hat{K}_y(w, s) dw,
\]

in which

\[
\hat{K}_y(w, s) = \exp(-\hat{Y}(w)H(w))
\]

and

\[
K_{\hat{y}}(r, \tau, w) = \exp[-\Gamma_1(r, \tau)w][\Gamma_1(r, \tau)J_0[\Gamma_2(r, \tau)w] + \Gamma_2(r, \tau)J_1[\Gamma_2(r, \tau)w]]H(w),
\]

where \(J_0\) and \(J_1\) are the Bessel functions of the first kind and orders zero and one, respectively. Use of this result in Eq. (27) yields

\[
\hat{K}'(r, \tau, s) = -1 + 2 \int_{w=0}^{\infty} K_{\hat{y}}(r, \tau, w) \hat{K}_y(w, s) dw.
\]

In terms of the (causal) time-domain counterpart \(K_y(w, t)\) of \(\hat{K}_y(w, s)\) we end up with

\[
K'(r, \tau, t) = -\delta(t) + 2 \left[ \int_{w=0}^{\infty} K_{\hat{y}}(r, \tau, w)K_y(w, t) dw \right] H(t).
\]

Note that in this expression the space-time configurational parameters of the fluid only occur in the kernel function \(K_{\hat{y}}(r, \tau, w)\), while the parameters of \(Y(t)\) only occur in the kernel function \(K_y(w, t)\). The space-time expressions for the reflected acoustic wave field quantities are from Eqs. (17) and (18) finally obtained as

\[
p^{(i)}(r, t) = \rho_0 \partial^2_0 Q_0(t) \ast G'(r, t),
\]

\[
v^{(i)}(r, t) = -\partial_0 Q_0(t) \ast \nabla G'(r, t).
\]

V. PARTIAL-FRACTION PARAMETRIZATION OF THE COMPLEX FREQUENCY DOMAIN ACOUSTIC ADMITTANCE AND ITS COROLLARIES

In this section an expression for the kernel function \(K_{\hat{y}}(w, t)\), introduced via Eq. (30), is constructed for the case where \(\hat{Y}(s)\) is parametrized through a partial fraction representation. Let

\[
\hat{Y}(s) = \sum_{n=0}^{N} \hat{Y}^{(n)}(s),
\]

with

\[
\hat{Y}^{(0)}(s) = Y^{\infty},
\]

\[
\hat{Y}^{(n)}(s) = \frac{A_n}{s + \alpha_n} \quad \text{for } n = 1, \ldots, N.
\]

Since the underlying assumption of such a representation is that \(\hat{Y}(s)\) arises as the causal response from a rational time differentiation operator with real-valued coefficients and a finite number of degrees of freedom, a number of properties hold (Kwakernaak and Sivan, 1991). First, \(\hat{Y}(s)\) has to be real and positive for \(s\) real and positive, which entails that \(Y^{\infty}\) is real and \(\geq 0\). Furthermore, \(\hat{Y}(s)\) has, in general, simple poles at \(s=-\alpha_n (n=1, \ldots, N)\) that should be located in the left half of the complex \(s\)-plane. As to the terms \(\hat{Y}^{(n)}(n=1, \ldots, N)\) two possibilities arise: either \(\alpha_n (n=1, \ldots, N)\) is real and \(\geq 0\) and the residues \(A_n (n=1, \ldots, N)\) at the poles \(s=-\alpha_n (n=1, \ldots, N)\) are real, or pairs of \(\alpha_n (n=1, \ldots, N)\) are complex conjugate with positive real parts and the residues \(A_n (n=1, \ldots, N)\) at such pair of poles \(s=-\alpha_n (n=1, \ldots, N)\) are each other’s complex conjugate. (The case of higher-order poles is most easily handled by a limiting confluence procedure.) Equation (36) entails a representation of \(\hat{K}_{\hat{y}}(w, s)\) of the form

\[
\hat{K}_{\hat{y}}(w, s) = \prod_{n=0}^{N} \hat{K}_{\hat{y}}^{(n)}(w, s),
\]

with

\[
\hat{K}_{\hat{y}}^{(0)}(w, s) = \exp(-Y^{\infty}w)H(w),
\]

\[
\hat{K}_{\hat{y}}^{(n)}(w, s) = \exp[-Y^{(n)}(s)w]H(w) \text{ for } n = 1, \ldots, N.
\]

The time-domain counterpart of Eq. (40) is

\[
K_{\hat{y}}^{(0)}(w, t) = \exp(-Y^{\infty}w)H(w) \delta(t).
\]

To construct the time-domain counterpart of Eq. (41) we again use the Schouten–Van der Pol theorem and employ formula (29.3.75) of Abramowitz and Stegun (1968, p. 1026), together with some elementary rules of the time Laplace transformation to obtain

\[
K_{\hat{y}}^{(n)}(w, t) = H(w) \delta(t) - \exp(-\alpha_n t) \times (A_n w/t)^{1/2} J_{1/2}[2(A_n w/t)^{1/2}]H(w)H(t)
\]

for \(n = 1, \ldots, N\).

In terms of Eq. (43) (that also holds for complex values of \(w\))
the parameters), the time-domain counterpart of Eq. (39) follows as

$$K_f(w,t) = K_1^{(0)}(w,t) \ast K_1^{(1)}(w,t) \ast \cdots \ast K_1^{(N)}(w,t).$$

(44)

In this expression each of the factors contains only two parameters, a property that can facilitate the parameter sensitivity analysis of the reflection measurement setup.

VI. PLANE-WAVE ADMITTANCE PARAMETRIZATION OF THE COMPLEX FREQUENCY DOMAIN ACOUSTIC ADMITTANCE AND ITS COROLLARIES

In this section an expression for the kernel function $K_f(w,t)$, introduced via Eq. (30), is constructed for the case where $\hat{Y}(s)$ is parametrized through a plane-wave admittance expression, applying to a fluid with volume density of mass $\rho_1$, compressibility $\kappa_1$, normalized inertia relaxation function $\hat{\alpha}_i(s)$, and normalized compressibility relaxation function $\hat{\beta}_i(s)$. Accordingly, we write (De Hoop, 1995, p. 42)

$$\hat{Y}_w(s) = Y_\infty^w[\hat{X}(s)]^{1/2},$$

(45)
in which

$$Y_1 = \rho_0 c_0 (\rho_1 c_1)^{1/2} = \rho_0 c_0 \rho_1 / c_1,$$

(46)
with $c_1 = (\rho_1 \kappa_1)^{1/2}$ as the corresponding wave speed, is representative for the instantaneous response and

$$\hat{X}(s) = \frac{s + \hat{\alpha}_i(s)}{s + \hat{\beta}_i(s)}$$

(47)
is representative for the absorptive and dispersive properties. To construct the time-domain counterpart $K_w(w,t)$ of the corresponding kernel function

$$\hat{K}_w(w,s) = \exp[-\hat{Y}_w(s)w]$$

(48)
we again use the Schouten–Van der Pol theorem and employ Formula (29.3.82) of Abramowitz and Stegun (1968, p. 1026) to obtain:

$$\hat{K}_w(w,s) = \int_{w=0}^{\infty} \exp[-\hat{X}(s)u]Y(w,u)du,$$

(49)
where

$$Y(w,u) = \frac{Y_\infty^w}{(4\pi u)^{1/2}} \exp \left[ - \frac{(Y_\infty^w)^2}{4u} \right] H(w)H(u).$$

(50)
Since $\hat{\alpha}_i(s)$ and $\hat{\beta}_i(s)$ are system’s response functions of the linear, time-invariant, causal, passive type, $\hat{X}(s)$ admits a partial-fraction parametrization of the type (36)–(38) and the time-domain counterpart of $\exp[-\hat{X}(s)u]$ follows from Eq. (44).

VII. SOME ILLUSTRATIVE NUMERICAL EXAMPLES

In the following, some illustrative numerical examples are presented. The source is placed at the boundary ($h=0$). Two receiver positions are considered, viz. one at the boundary ($r>0$, $z=0$), i.e., the propagation takes place parallel to the boundary, and one at the normal to the boundary through the source ($r=0$, $z>0$), i.e., the propagation takes place normal to the boundary. With regard to the boundary’s acoustic admittance, two examples are discussed: (A) the zero-order (single-term) admittance and (B) the first-order (two-terms) admittance. Figure 2 shows the normalized incident-wave Green’s function as a function of normalized time [cf. Eq. (6)].

A. Zero-order boundary admittance

For the zero-order boundary admittance we have

$$\hat{Y}(s) = Y_\infty^w,$$

(51)
which corresponds to the time-domain acoustic admittance

$$Y(t) = Y_\infty^w \delta(t)$$

(52)
and the time-domain boundary condition [cf. Eq. (3)]

$$v_c(x,y,0,t) = -Y_\infty^w p(x,y,0,t).$$

(53)
This section mainly serves to illustrate the influence of $Y_\infty^w$ on the reflection problem. Figure 3 shows the normalized reflected-wave Green’s function as a function of normalized time [cf. Eqs. (27) and (28)] at (a) $r=10$ m, $z=0$ (propagation parallel to the boundary) and (b) $r=0$, $z=1$ m (propagation normal to the boundary), for three different values of $Y_\infty^w$. Note that for propagation parallel to the boundary the normalized Green’s function always starts at the value $-1$, irrespective of the value of $Y_\infty^w$, while for propagation normal to the boundary the starting value is positive for $Y_\infty^w > 1$, zero for $Y_\infty^w = 1$ (admittance matched to the plane-wave value at normal incidence), and negative for $Y_\infty^w < 1$.

B. First-order boundary admittance

For the first-order boundary admittance we have [cf. Eqs. (36)–(38)]

$$\hat{Y}(s) = Y_\infty^w + \frac{A_1}{s + \alpha_1},$$

(54)
which we rewrite as

FIG. 2. Normalized incident-wave Green’s function as a function of normalized time $t/T_0$. 

FIG. 3. Normalized reflected-wave Green’s function 4πD_0/G as a function of normalized time t/T_1. Zero-order acoustic boundary admittance Y = Y’. Source at boundary (h=0); c_0 = 330 m/s. (a) Propagation parallel to boundary (r=10 m, z=0), (b) propagation normal to boundary (r=0, z=1 m). Curves: (---) Y’=2.0, (-- ...) Y’=1.0 (matched to plane-wave value at normal incidence), (--- ...) Y’=0.5.

\[ \hat{Y}(s) = \frac{Y'' s + z_1}{s + p_1}, \]  
\[ (t_1 + 1/\tau_v)u(x,y,0,t) = - (\rho_0 c_0) Y'' (t_1 + 1/\tau_p) p(x,y,0,t), \]  
where \( -p_1 = -\alpha_1 \) is the pole of \( \hat{Y}(s) \) and \( z_1 \) is the zero of \( \hat{Y}(s) \), both located in the left half of the complex s plane, and

is the residue at the pole. Equations (54) and (55) correspond to the time-domain acoustic admittance

\[ Y(t) = Y'' \delta(t) + A_1 \exp(-\alpha_1 t) H(t) \]  
and the time-domain boundary condition [cf. Eq.(3)]

where \( \tau_v = 1/p_1 \) is the velocity relaxation time and \( \tau_p = 1/z_1 \) is the pressure relaxation time (Christensen, 2003, pp. 17–19; Meinardi, 2002, p. 105). This section mainly serves to illustrate the influence of \( \tau_v \) and \( \tau_p \) on the reflection problem. Therefore, we take \( Y'' = 1 \), which implies matching to the plane-wave admittance at normal incidence.

Figure 4 shows the normalized reflected-wave Green’s function as a function of normalized time [cf. Eqs. (28) and (33)] at (a) \( r=10 \text{ m, } z=0 \) (propagation parallel to the boundary), for four different values of \( \tau_v \), with \( \tau_p \) fixed. As Fig. 4(a) shows, strong oscillations occur at propagation parallel to the boundary, which phenomenon has been referred to in Sec. I. No such oscillations show up in the propagation normal to the boundary, as Fig. 4(b) shows. It can be argued that this behavior can be inferred from Eq. (31), where \( \Gamma_1 \) is related to the offset normal to the boundary and occurs in the damping exponential function, while \( \Gamma_2 \) is related to the offset parallel to the boundary and occurs in the oscillating Bessel functions. Apparently, such an easy interpretation does not apply to Eq. (43), where for \( A_0 > 0 \) the Bessel functions are oscillatory, while for \( A_0 < 0 \) they change into modified Bessel functions of the first kind that show a monotonic behavior.

FIG. 4. Normalized reflected-wave Green’s function 4πD_0/G as a function of normalized time t/T_1. First-order acoustic boundary admittance: \( (t_1 + 1/\tau_v)u = (\rho_0 c_0)^{-1} Y'' (t_1 + 1/\tau_p) p \) at boundary. Source at boundary (h=0); Y’=1.0 (matched to plane-wave value at normal incidence), c_0 = 330 m/s. (a) Propagation parallel to boundary (r=10 m, z=0), (b) propagation normal to boundary (r=0, z=1 m). Curves: (--- ...) \( \tau_v = 1.0 \times 10^{-3} \text{ s, } \tau_p = 5.0 \times 10^{-2} \text{ s, } A_1 = -4.8 \times 10^2 \text{ s}^{-1}; (--- ... \) \( \tau_v = 5.0 \times 10^{-3} \text{ s, } \tau_p = 5.0 \times 10^{-2} \text{ s, } A_1 = -1.8 \times 10^3 \text{ s}^{-1}; (--- ... \) \( \tau_v = 1.0 \times 10^{-2} \text{ s, } \tau_p = 5.0 \times 10^{-2} \text{ s, } A_1 = 1.0 \times 10^3 \text{ s}^{-1}. \)
persive boundary has been expressed as a multiple sequence of operations acting on the source signature. Each of the kernel functions in the expression contains separately the configurational parameters of the measurement setup (location of source, receiver and boundary, and propagation through the fluid) and the parameters by which the absorptive and dispersive properties of the boundary can be characterized. Two parametrizations of the boundary’s complex frequency domain acoustic admittance have been discussed in detail: the partial-fraction parametrization and the plane-wave admittance parametrization. The explicit attribution of a sequence of parameters to their corresponding kernel functions is conjectured to play an illuminating role in the use of the reflection measurement setup to characterize (via an appropriate inversion algorithm applied to the measured values of the acoustic pressure) the absorption and dispersion properties of the boundary, while the obtained expression itself is directly amenable to carry out the relevant parameter sensitivity analysis. It is noted that the multiple time convolutions that occur in the final expression for the acoustic pressure can numerically most profitably be evaluated through the use of the FFT algorithm.

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**APPENDIX: EVALUATION OF THE INTEGRAL IN EQ. (22)**

In this Appendix the integral occurring in Eq. (22)

$$ I = \frac{2}{\pi} \text{Re} \left[ \int_{\phi=0}^{\pi/2} \frac{1}{c_0 \gamma_0} \frac{dy}{y} \right] $$

$$ = \frac{2}{\pi} \text{Re} \left[ \int_{\phi=0}^{\pi/2} \frac{1}{\Gamma_1 - i\Gamma_2 \cos(\phi) + \hat{Y}(s)} \right] $$

$$ = \frac{2}{\pi} \int_{\phi=0}^{\pi/2} \frac{\Gamma_1 + \hat{Y}(s)}{(\Gamma_1 + \hat{Y}(s))^2 + \Gamma_2^2 \cos^2(\phi)} d\phi, \quad (A1) $$

with [cf. Eqs. (24) and (25)]

$$ \Gamma_1 = c_0 \gamma(z + h)/D_1^2, \quad (A2) $$

$$ \Gamma_2 = c_0 (\tau^2 - T_{ij}^2)^{1/2}/D_1, \quad (A3) $$

is evaluated. Using the standard integral

$$ \frac{2}{\pi} \int_{\phi=0}^{\pi/2} \frac{A}{A^2 + B^2 \cos^2(\phi)} d\phi = \frac{1}{(A^2 + B^2)^{1/2}}, \quad (A4) $$

we obtain

$$ I = \frac{1}{{[\Gamma_1 + \hat{Y}(s)]^2 + \Gamma_2^2}^{1/2}} \quad (A5) $$

This result is used in the main text.


