ELECTROMAGNETIC WAVEFIELD COMPUTATION - A STRUCTURED APPROACH BASED ON RECIPROCITY

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The **objective** is to construct a **computational discretization** of the (forward) electromagnetic wave problem that is

- as **coherent** and **internally consistent** as possible
- as **close** as possible to the **physics of the problem**

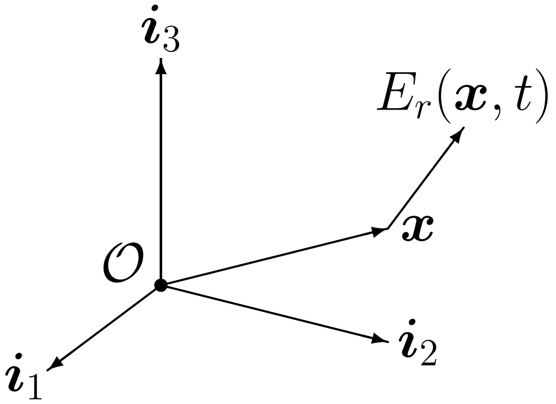
Type of **configuration**:

- piecewise continuous distribution of constitutive parameters
- piecewise continuous volume source distributions

Guideline:

- avoid spatial delta distributions!

Reference frame in $\mathcal{R}^3$:



Position vector:

- $\boldsymbol{x} \;=\; x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$

Subscript notation, summation convention:

- $\epsilon_{k,r}E_r = \Sigma_{r=1}^{3}\,\epsilon_{k,r}E_r$ for $k = 1, 2, 3$
- $\partial_m =$ derivative with respect to x_m

Cartesian reference frame, subscript notation

- **Electromagnetic wave quantities**:

$$E_r = \text{electric field strength (V/m)}$$
$$H_p = \text{magnetic field strength (A/m)}$$
$$D_k = \text{electric flux density (C/m}^2\text{)}$$
$$B_j = \text{magnetic flux density (T)}$$

- **Electromagnetic source quantities**:

$$J_k = \text{volume source density of electric current (A/m}^2\text{)}$$
$$K_j = \text{volume source density of magnetic current (V/m}^2\text{)}$$

Electromagnetic wave and source quantities

- **Electromagnetic medium parameters for an inhomogeneous, anisotropic medium with relaxation**:

$$\kappa^{\mathrm{e}}_{k,r} = \text{electric relaxation function (F/m·s)}$$

$$\kappa^{\mathrm{m}}_{j,p} = \text{magnetic relaxation function (H/m·s)}$$

- **Electromagnetic constitutive relations**:

$$D_k = \kappa^{\mathrm{e}}_{k,r} \overset{(t)}{*} E_r$$

$$B_j = \kappa^{\mathrm{m}}_{j,p} \overset{(t)}{*} H_p$$

where

$$\overset{(t)}{*} = \text{time convolution}$$

- **Electromagnetic constitutive relations for a simple inhomogeneous, anisotropic medium**:

$$\partial_t D_k = \sigma_{k,r} E_r + \epsilon_{k,r} \partial_t E_r$$
$$\partial_t B_j = \chi_{j,p} H_p + \mu_{j,p} \partial_t H_p$$

where

$\sigma_{k,r}$ = conductivity (S/m)

$\epsilon_{k,r}$ = permittivity (F/m)

$\chi_{j,p}$ = linear magnetic hysteresis loss coefficient (H•s/m)

$\mu_{j,p}$ = permeability (H/m)

Electromagnetic medium properties (continued)

- **Electromagnetic field equations**:

$$-\boldsymbol{\epsilon}_{k,m,p}\partial_m H_p + \partial_t D_k = -J_k$$
$$\boldsymbol{\epsilon}_{j,n,r}\partial_n E_r + \partial_t B_j = -K_j$$

- **Electromagnetic compatibility relations**:

$$\int_{\mathcal{S}} \nu_k \partial_t D_k \mathrm{d}A = -\int_{\mathcal{S}} \nu_k J_k \mathrm{d}A$$
$$\int_{\mathcal{S}} \nu_j \partial_t B_j \mathrm{d}A = -\int_{\mathcal{S}} \nu_j K_j \mathrm{d}A$$

for any closed surface $\mathcal{S}$ ($\nu_m =$ unit vector along outward normal to $\mathcal{S}$)

- **Levi-Civita tensor**:

$\boldsymbol{\epsilon}_{k,m,p} = \{+1, -1, 0\}$ if $\{k, m, p\}$ is {even, odd, no} permutation of $\{1, 2, 3\}$

Electromagnetic field equations, compatibility relations

- **Interface boundary conditions**: across any surface of discontinuity in medium properties

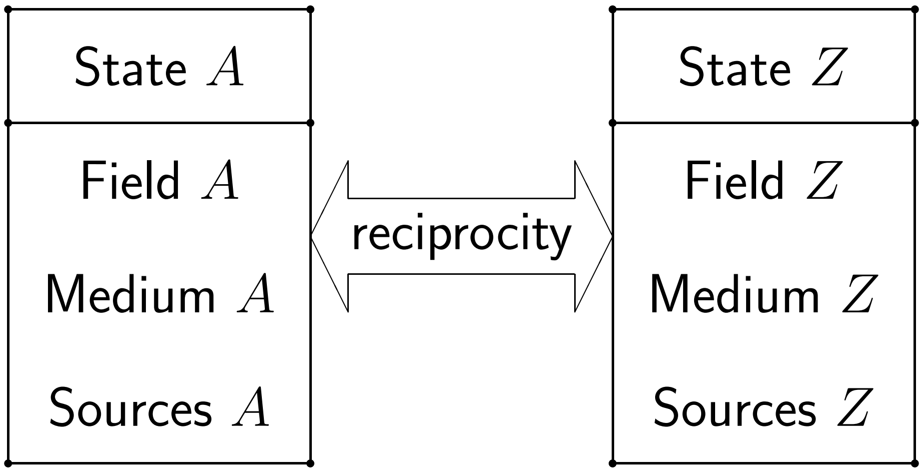
$$\begin{aligned}\boldsymbol{\epsilon}_{k,m,p}\nu_m H_p &= \text{continuous}\\ \boldsymbol{\epsilon}_{j,n,r}\nu_n E_r &= \text{continuous}\end{aligned}$$

$\nu_m =$ unit vector along the normal to the interface

- **Compatibility boundary conditions**: across any surface of discontinuity in medium properties and/or volume source distributions

$$\begin{aligned}\nu_k(\partial_t D_k + J_k) &= \text{continuous}\\ \nu_j(\partial_t B_j + K_j) &= \text{continuous}\end{aligned}$$

$\nu_k =$ unit vector along the normal to the surface of discontinuity



The two admissible states
in the reciprocity theorem.

- **Electromagnetic reciprocity theorem of the time-convolution type**:

$$\epsilon_{m,r,p} \int_{\partial\mathcal{D}} \nu_m [E_r^A \overset{(t)}{*} H_p^Z - E_r^Z \overset{(t)}{*} H_p^A] \mathrm{d}A$$

$$= \int_{\mathcal{D}} [\partial_t D_k^A \overset{(t)}{*} E_k^Z - \partial_t D_r^Z \overset{(t)}{*} E_r^A$$

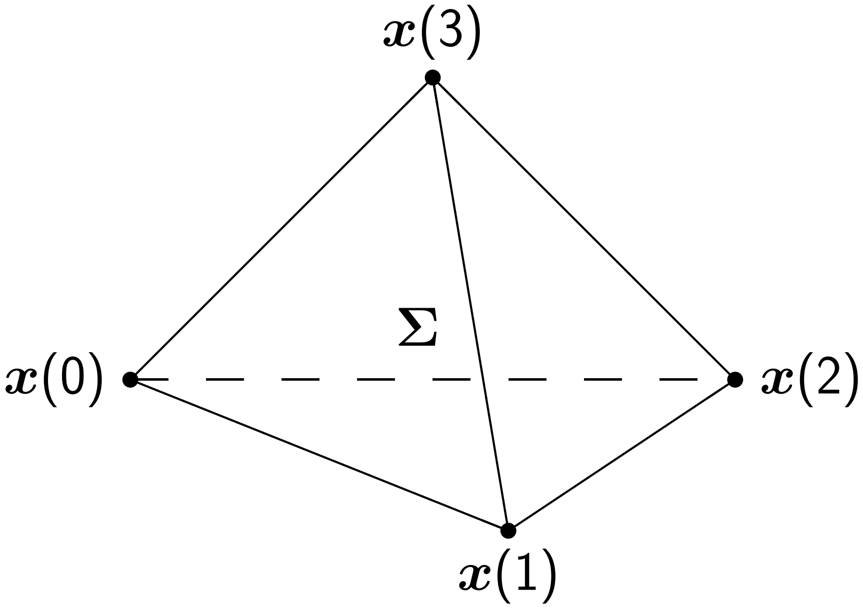
$$-\partial_t B_j^A \overset{(t)}{*} H_j^Z + \partial_t B_p^Z \overset{(t)}{*} H_p^A] \mathrm{d}V$$

$$+ \int_{\mathcal{D}} [J_k^A \overset{(t)}{*} E_k^Z - K_j^A \overset{(t)}{*} H_j^Z$$

$$-J_r^Z \overset{(t)}{*} E_r^A + K_p^Z \overset{(t)}{*} H_p^A] \mathrm{d}V$$

- **The reciprocity relation is a "weak" formulation of the field problem: for the theorem to hold for arbitrary States Z, State A must satisfy the field equations**

Reciprocity theorem of the time-convolution type



Oriented **simplex** $\boldsymbol{\Sigma}$ in $\mathcal{R}^3$ with

ordered set of vertices

$$\{\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \boldsymbol{x}(3)\}$$

Simplex in $\mathcal{R}^3$ (tetrahedron)

- **Vectorial edges** leaving the vertex $\boldsymbol{x}(I)$:

$$\{\boldsymbol{x}(J) - \boldsymbol{x}(I)\} \text{ for } I = 0, 1, 2, 3; J = 0, 1, 2, 3; \\ I \neq J$$

- Outwardly oriented **vectorial faces** meeting at the vertex $\boldsymbol{x}(I)$:

$$\{\boldsymbol{A}(J)\} \text{ for } I = 0, 1, 2, 3; J = 0, 1, 2, 3; I \neq J$$

where

- $\boldsymbol{A}(I) =$ outwardly oriented **face opposite the vertex** $\boldsymbol{x}(I)$

Vectorial edges and faces of a simplex in $\mathcal{R}^3$

- **Property:**

$$[x_m(J) - x_m(I)]A_m(K)$$
$$-3V_{\boldsymbol{\Sigma}}[\delta(J,K) - \delta(I,K)]$$
$$\text{for } I = 0,1,2,3;\ \ J = 0,1,2,3;\ \ K = 0,1,2,3$$

where

- $V_{\boldsymbol{\Sigma}}$ is the **volume** of $\boldsymbol{\Sigma}$

$\Longrightarrow$ **At the vertex $\boldsymbol{x}(I)$, the set of base vectors $\{\boldsymbol{x}(J) - \boldsymbol{x}(I); I \neq J\}$ is reciprocal to the set of base vectors $\{-(3V_{\boldsymbol{\Sigma}})^{-1}\boldsymbol{A}(K); K \neq I\}$**

Base vectors at the vertex of a simplex $\boldsymbol{\Sigma}$

All field quantities, medium parameters and source quantities are locally (i.e. in each tetrahedron Σ) represented as

• **spatially linear interpolations**

of their values at the vertices of Σ

Vertex-based local representation

• A **vertex-based** representation is a local, spatially linear representation of a scalar, vector or tensor quantity whose vertex values are **decomposed along the axes of the background Cartesian reference frame**

Local, spatially linear (vertex-based) representation

Edge-based local vector representation

- An **edge-based** representation is a local, spatially linear representation of a vector quantity whose vertex "components" are **the projections of that vector on the vectorial edges leaving that vertex**

Face-based local vector representation

- A **face-based** representation is a local, spatially linear representation of a vector quantity whose vertex "components" are **the projections of that vector on the vectorial faces meeting at that vertex**

Edge- and face-based local vector representation

The **number of local expansion coefficients of a vector quantity** in a tetrahedron is:

- *Vertex-based representation*:

  4 (vertices) $\times$ 3 (components)

- *Edge-based representation*:

  6 (edges) $\times$ 2 (projections)

- *Face-based representation*:

  4 (faces) $\times$ 3 (projections)

Number of local expansion coefficients (vector quantity)

The **global representations of the field quantities** $\{E_r, H_p, D_k, B_j\}$ on the discretized geometry are constructed out of their

- **local representations**

plus the application of the

- **interface boundary conditions** to the **edge-based** expansion coefficients of $\{E_r, H_p\}$
- **compatibility boundary conditions** to the **face-based** expansion coefficients of $\{D_k, B_j\}$

The **global representations** of the

- **medium parameters** $\{\kappa^{\mathrm{e}}_{k,r}, \kappa^{\mathrm{m}}_{j,p}\}$

and the

- **source quantities** $\{J_k, K_j\}$

on the discretized geometry are constructed out of their

- **local vertex-based representations**

To accomodate **radiation problems** without explicit boundary conditions on the boundary of the domain of computation, an **embedding procedure** is applied; the **embedding** has:

- $\mathcal{R}^3$ as support
- **medium parameters** $\{\kappa^{\mathrm{e;b}}_{k,r}, \kappa^{\mathrm{m;b}}_{j,p}\}$, such that the **Green's functions (point-source solutions)** are **analytically known**

$\Longrightarrow$ Field computation problem can be reformulated as a **scattering problem** with **contrast source distributions** that have the contrasting domain of computation as their support

In the global reciprocity theorem of the time-convolution type, applied to the domain of computation, we substitute

- **State** A = constructed **global expansions of field and source quantities**
- **State** Z = sequence of suitable **"computational" States** C

Depending on the **choice of State** C, the

- **finite-element method**
- **integral-equation method**
- **domain integration method**

result

The **finite-element method** is characterized by the **computational state**:

- $\{\kappa^{\mathrm{e};C}_{r,k}, \kappa^{\mathrm{m};C}_{p,j}\} = \{0, 0\}$
  $\Longrightarrow \bullet \ \{D^C_j, B^C_p\} = \{0, 0\}$,

together with **either**

- $E^C_k \in \delta(t)\{$**edge-based** global expansion function$\}$, $H^C_j = 0$
  $\Longrightarrow \bullet \ \{J^C_r = 0, K^C_p = -\epsilon_{p,n,k}\partial_n E^C_k\}$

or

- $E^C_k = 0, H^C_j \in \delta(t)\{$**edge-based** global expansion function$\}$
  $\Longrightarrow \bullet \ \{J^C_r = \epsilon_{r,m,j}\partial_m H^C_j, K^C_p = 0\}$

Finite-element method (computational state)

Finite-element method:

• Substitute expanded (actual) field state and chosen computational state in **reciprocity theorem** of the time-convolution type, applied to the **domain of computation**

• Invoke **constitutive relations** at all vertices

• Relate **boundary values on the boundary of the domain of computation** to **contrast source densities** via source-type field representation in embedding

$\Longrightarrow$

• **"square" system** of equations in the time evolution of the **expansion coefficients**

(continued ...)

Finite-element method (procedure)

- Substitute expanded (actual) field state in **compatibility relations** and **compatibility boundary conditions**

$\Longrightarrow$

- **additional equations** in the same set of expansion coefficients

An **overdetermined system** of equations in the time evolution of the expansion coefficients results

- **Solution method**: **iterative procedure** with **guaranteed decrease** in the **norm of the residual**

Finite-element method (procedure, continued)

The **integral-equation method** is characterized by the **computational state**:

- $\{\kappa^{\mathrm{e};C}_{r,k}, \kappa^{\mathrm{m};C}_{p,j}\} = \{\kappa^{\mathrm{e;b}}_{k,r}, \kappa^{\mathrm{m;b}}_{j,p}\}$ together with

either

- $J^C_r \in \delta(t)\{$**vertex-based** expansion function$\}$, $K^C_p = 0$
  $\Longrightarrow$ • $\{E^C_k, H^C_j\}$ via **source-type integral representations**

or

- $J^C_r = 0, K^C_p \in \delta(t)\{$**vertex-based** expansion function
  $\Longrightarrow$ • $\{E^C_r, H^C_p\}$ via **source-type integral representations**

Integral-equation method (computational state)

Integral-equation method:

• Substitute expanded (actual) field state and chosen computational state in the **reciprocity theorem** of the time-convolution type, applied to **the entire** $\mathcal{R}^3$

• Invoke **reproduction of volume densities of contrast sources** at all vertices

$\Longrightarrow$

• **"square" system** of equations in the time evolution of the **expansion coefficients**

(continued ...)

- Substitute expanded (actual) field state in **compatibility relations** and **compatibility boundary conditions**

$\Longrightarrow$

- **additional equations** in the same set of expansion coefficients

An **overdetermined system** of equations in the time evolution of the expansion coefficients results

- **Solution method**: **iterative procedure** with **guaranteed decrease** in the **norm of the residual**

Integral-equation method (procedure, continued)

The **domain integration method** is characterized by the **computational state**:

- $\{\kappa^{\mathrm{e};C}_{r,k}, \kappa^{\mathrm{m};C}_{p,j}\} = \{0,0\}$
  $\Longrightarrow$ • $\{D^C_r, B^C_p\} = \{0,0\}$,

together with **either**

- $\{E^C_k \in \delta(t)\{$**global constant** with the domain of computation as support$\}$, $H^C_j = 0\}$
  $\Longrightarrow$ • $\{J^C_r, K^C_p\} = \{0,0\}$

or

- $\{E^C_k = 0, H^C_j \in \delta(t)\{$**global constant** with the domain of computation as support$\}$
  $\Longrightarrow$ • $\{J^C_r, K^C_p\} = \{0,0\}$

Domain integration method (computational state)

Domain integration method:

• Substitute expanded (actual) field state and chosen computational state in **reciprocity theorem** of the time-convolution type, applied to the **domain of computation**

• Invoke **constitutive relations** at all vertices

• Relate **boundary values on the boundary of the domain of computation** to contrast source densities via source-type field representation in embedding

$\Longrightarrow$

• **"square" system** of equations in the time evolution of the **expansion coefficients**

(continued . . .)

Domain integration method (procedure)

• Substitute expanded (actual) field state in **compatibility relations** and **compatibility boundary conditions**

$\Longrightarrow$

• **additional equations** in the same set of expansion coefficients

An **overdetermined system** of equations in the time evolution of the expansion coefficients results

• **Solution method**: **iterative procedure** with **guaranteed decrease** in the **norm of the residual**

Domain integration method (procedure, continued)

Note:

In the substitution of the computational states in the reciprocity theorem of the time-convolution type, the value of the chosen constant drops out and the resulting equations are the same as when the field equations are integrated over the domain of computation and Gauss' integral theorem is applied. The latter **domain integration method** can therefore also be approached directly.

Operator equation to be "solved":

- $\mathsf{L}u \simeq q$

Inner product:

- $< v, u >$ with
- $< v, u > = < u, v >^*$
- $< v, \mathsf{L}u > = < \mathsf{L}^{\mathsf{H}} v, u >$

  ($\mathsf{L}^{\mathsf{H}} =$ **Hermitean conjugate** of L)

Associated norm:

- $||u|| = < u, u >^{1/2}$

Residual:

- $r = q - \mathsf{L}u$

Operator equation, inner product, norm, residual

Iterative solution procedure:

- $u^{[n+1]} = u^{[n]} + \delta u^{[n]}$ for $n = 0, 1, 2, \ldots$

Residual after n steps:

- $r^{[n]} = q - \mathsf{L}u^{[n]}$ for $n = 0, 1, 2, \ldots$

$\Longrightarrow r^{[n+1]} = r^{[n]} - \mathsf{L}\delta u^{[n]}$

"Improvement condition":

- $||r^{[n]}||^2 > ||r^{[n+1]}||^2$ for $n = 0, 1, 2, \ldots$

Sufficient condition for improvement:

- $< r^{[n+1]}, \mathsf{L}\delta u^{[n]} > + < \mathsf{L}\delta u^{[n]}, r^{[n+1]} >= 0$

which is satisfied by*

- $\delta u^{[n]} = \alpha^{[n]}\mathsf{L}^{\mathsf{H}}r^{[n]}$ with • $\alpha^{[n]} = \frac{||\mathsf{L}^{\mathsf{H}}r^{[n]}||^2}{||\mathsf{L}\mathsf{L}^{\mathsf{H}}r^{[n]}||^2}$

*More sophisticated choices for $\delta u^{[n]}$ exist

Iterative solution procedure, "improvement condition"

Mapping of **causal time functions** to the **complex frequency domain** takes place via the **Laplace transformation**:

- $\hat{u}(\boldsymbol{x}, s) = \int_{t_0}^{\infty} \exp(-st) u(\boldsymbol{x}, t) \mathrm{d}t$
  for $\{s \in \mathcal{C}, \mathrm{Re}(s) > s_0\}$

Properties:

- $\partial_t \to s$
- $u(\boldsymbol{x}, t) \overset{(t)}{*} v(\boldsymbol{x}, t) \to \hat{u}(\boldsymbol{x}, s) \hat{v}(\boldsymbol{x}, s)$
- **Real frequences:** $s = \mathrm{j}\omega$ with $\omega \in \mathcal{R}$

- The spatial discretization parts run parallel to the ones for the time-domain analysis