WAVEFIELD RECIPROCITY - FUNDAMENTALS AND APPLICATIONS

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We shall dicuss **RECIPROCITY** for the following **WAVE PHENOMENA**:

- Acoustic waves in fluids
  (i.e., gases and liquids)
- Elastic waves in solids (condensed matter)
- Electromagnetic waves
  (in vacuum and all matter)

RECIPROCITY finds its origins in the following papers:

- 1828 **G. Green**: Static electric and magnetic fields
- 1894 **Lord Rayleigh**: Acoustic waves in fluids
- 1871-1873 **A. Betti** and
  1873 **Lord Rayleigh**: Elastic waves in solids
- 1896 **H. A. Lorentz**: Electromagnetic waves

A **wavefield** is physically characterized by

- the **sources** that generate it (**excitation**)
- the **medium** in which it is present (**propagation**)
- the **wavefield state quantities** through which it is accessible to measurement (**reception**)

Wavefield (characterization)

Acoustic waves in fluids:

- **Sources**: the human voice, musical intruments, loudspeakers, ultrasonic transmitting transducers, airgun in marine seismics
- **Media**: air, water
- **Receivers**: the human ear, microphones, ultrasonic receiving transducers

Elastic waves in solids:

- **Sources**: earthquake, mechanical vibrator (VIBROSEIS$^{\mathrm{TM}}$), ultrasonic transmitting transducer
- **Media**: the Earth, mechanical structures
- **Receivers**: geophone, seismometer, ultrasonic receiving transducer

Elastic waves in solids (excitation, propagation, reception)

Electromagnetic waves:

- **Sources**: transmitting antenna for radio and television broadcast, radar, satellite communication, lightning discharge, ElectroStatic Discharge
- **Media**: vacuum & all matter
- **Receivers**: receiving antenna for radio and television broadcast, radar, satellite communication, wire loops for detection in vehicle traffic control, any loop in a digital integrated circuit

- **Acoustic source quantities**:

$$q = \text{volume source density of injection rate}$$
$$f_k = \text{volume source density of force}$$

- **Acoustic medium properties**:

$$\rho_{k,r} = \text{volume density of (inertia) mass}$$
$$\kappa = \text{compressibility}$$

- **Acoustic wavefield quantities**:

$$p = \text{acoustic pressure}$$
$$v_r = \text{particle velocity}$$
$$\Phi_k = \text{mass flow density}$$
$$\theta = \text{cubic dilatation}$$

- **Elastic source quantities**:

$$f_k = \text{volume source density of force}$$
$$h_{i,j} = \text{volume source density of deformation rate}$$

- **Elastic medium properties**:

$$\rho_{k,r} = \text{volume density of (inertial) mass}$$
$$S_{i,j,p,q} = \text{compliance}$$

- **Elastic wavefield quantities**:

$$\tau_{p,q} = \text{dynamic stress}$$
$$v_r = \text{particle velocity}$$
$$\Phi_k = \text{mass flow density}$$
$$e_{i,j} = \text{deformation}$$

- **Electromagnetic source quantities**:

$$J_k = \text{volume source density of electric current}$$
$$K_j = \text{volume source density of magnetic current}$$

- **Electromagnetic medium properties**:

$$\epsilon_{k,r} = \text{(electrical) permittivity}$$
$$\mu_{i,j} = \text{(magnetic) permeability}$$

- **Electromagnetic wavefield quantities**:

$$E_r = \text{electric field strength}$$
$$K_p = \text{magnetic field strength}$$
$$D_k = \text{electric flux density}$$
$$B_j = \text{magnetic flux density}$$

Wave phenomena are phenomena in

- **(three-dimensional) space** ($\boldsymbol{x} \in \boldsymbol{\mathcal{R}}^3$)

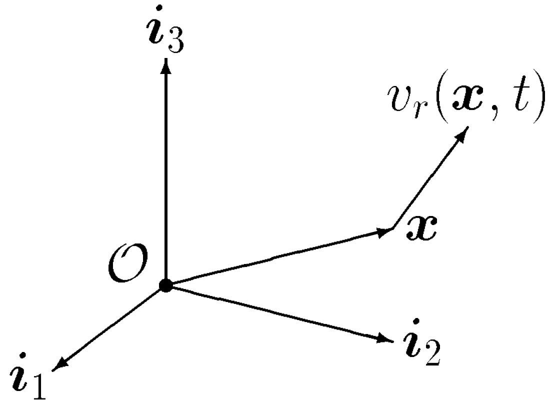
and

- **time** ($t \in \mathcal{R}$)

They

- are **causally related to the action of their sources**
- **transfer energy from source to receiver**
- **SHARE** the property of **RECIPROCITY**

Reference frame in $\mathcal{R}^3$:



Position vector:

- $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$

Subscript notation, summation convention:

- $\rho_{k,r}v_r = \Sigma_{r=1}^{3}\,\rho_{k,r}v_r$ for $k = 1, 2, 3$
- $\partial_m =$ derivative with respect to x_m

Cartesian reference frame, subscript notation

Wave phenomena are **mathematically** characterized by a

- **hyperbolic system** of **first-order partial differential equations**

that

- **couple** the **relative changes of the wavefield in space** ($x \in \mathcal{R}^3$) to the **relative changes of the wavefield in time**($t \in \mathcal{R}$)

in such a manner that

- **RECIPROCITY holds**

Wave phenomena (mathematical characterization)

- **Acoustic wave quantities**:

$$\begin{aligned} p &= \text{acoustic pressure (Pa)} \\ v_r &= \text{particle velocity (m/s)} \\ \Phi_k &= \text{mass flow density (kg/m}^2\cdot\text{s)} \\ \theta &= \text{cubic dilatation} \end{aligned}$$

- **Acoustic source quantities**:

$$\begin{aligned} f_k &= \text{volume source density of force (N/m}^3) \\ q &= \text{volume source density of injection rate (s}^{-1}) \end{aligned}$$

Acoustic wave and source quantities

- **Acoustic medium parameters for an inhomogeneous, anisotropic medium with relaxation**:

$\mu_{k,r}$ = inertia relaxation function (kg/m^3·s)

χ = compliance relaxation function (s^{-1})

- **Acoustic constitutive relations**:

$$\Phi_k = \mu_{k,r} \overset{(t)}{*} v_r$$
$$\theta = -\chi \overset{(t)}{*} p$$

where

$$\overset{(t)}{*} = \text{time convolution}$$

- **Acoustic constitutive relations for an inhomogeneous, anisotropic fluid with simple loss mechanisms**:

$$\partial_t \Phi_k = K_{k,r} v_r + \rho_{k,r} \partial_t v_r$$
$$-\partial_t \theta = \Gamma p + \kappa \partial_t p$$

where

$K_{k,r}$ = coefficient of frictional force (kg/m^3·s)

$\rho_{k,r}$ = volume density of (inertial) mass (kg/m^3)

Γ = coefficient of bulk inviscidness (Pa^{-1}·s^{-1})

κ = compressibility (Pa^{-1})

- **Acoustic wavefield equations**:

$$\partial_k p + \partial_t \Phi_k = f_k$$
$$\partial_r v_r - \partial_t \theta = q$$

- **Acoustic compatibility relation**:

$$\boldsymbol{\epsilon}_{p,m,k} \int_{\mathcal{S}} \nu_m \partial_t \Phi_k \mathrm{d}A = \boldsymbol{\epsilon}_{p,m,k} \int_{\mathcal{S}} \nu_m f_k \mathrm{d}A$$

for any closed surface $\mathcal{S}$ ($\nu_n =$ unit vector along outward normal to $\mathcal{S}$)

- **Levi-Civita tensor**:

$\boldsymbol{\epsilon}_{p,m,k} = \{+1, -1, 0\}$ if $\{p, m, k\}$ is {even, odd, no} permutation of $\{1, 2, 3\}$

Acoustic wavefield equations, compatibility relations

- **Interface boundary conditions**:
  across any surface of discontinuity in medium properties

$$p = \text{continuous}$$
$$\nu_r v_r = \text{continuous}$$

ν_m = unit vector along normal to interface

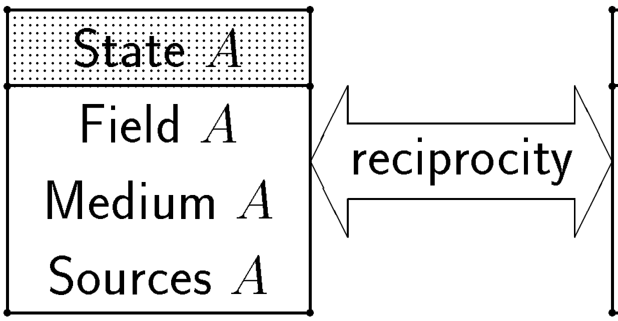
- **Compatibility boundary conditions**:
  across any surface of discontinuity in medium properties and/or volume source distributions

$$\boldsymbol{\epsilon}_{p,m,k}\nu_m(\partial_t\Phi_k - f_k) = \text{continuous}$$

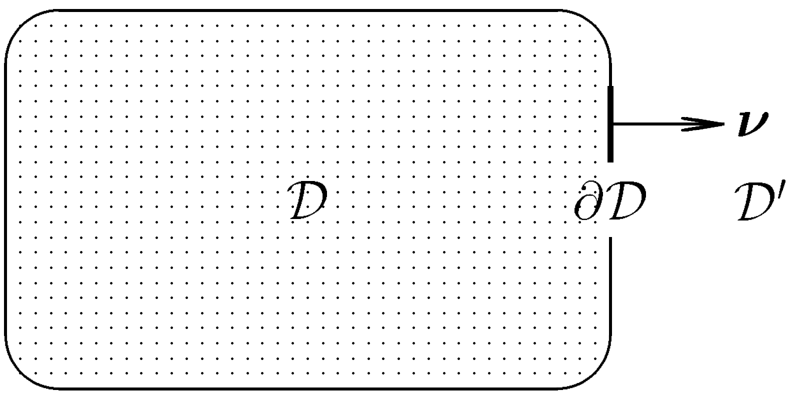
ν_m = unit vector along normal to surface of discontinuity

Acoustic boundary conditions

- **The two States A and Z:**



- **Domain of application:**



- **Acoustic reciprocity theorem of the time-convolution type**:

$$\int_{\partial\mathcal{D}} \nu_r [p^A \overset{(t)}{*} v_r^Z - p^Z \overset{(t)}{*} v_r^A] \mathrm{d}A$$

$$= \int_{\mathcal{D}} [-\partial_t \Phi_k^A \overset{(t)}{*} v_k^Z + \partial_t \Phi_r^Z \overset{(t)}{*} v_r^A$$

$$-\partial_t \theta^A \overset{(t)}{*} p^Z + \partial_t \theta^Z \overset{(t)}{*} p^A] \mathrm{d}V$$

$$+ \int_{\mathcal{D}} [f_k^A \overset{(t)}{*} v_k^Z - q^A \overset{(t)}{*} p^Z$$

$$-f_r^Z \overset{(t)}{*} v_r^A + q^Z \overset{(t)}{*} p^A] \mathrm{d}V$$

- **The reciprocity relation is a "weak" formulation of the field problem: for the theorem to hold for arbitrary States Z, State A must satisfy the field equations**

Reciprocity theorem of the time-convolution type

- **Elastic wave quantities**:

$$\begin{aligned} \tau_{p,q} &= \text{dynamic stress (Pa)} \\ v_r &= \text{particle velocity (m/s)} \\ \Phi_k &= \text{mass flow density (kg/m}^2\text{·s)} \\ e_{i,j} &= \text{deformation} \end{aligned}$$

- **Elastic wave source quantities**:

$$\begin{aligned} f_k &= \text{volume source density of force (N/m}^3\text{)} \\ h_{i,j} &= \text{volume source density of deformation rate (s}^{-1}\text{)} \end{aligned}$$

Elastic wave and source quantities

- **Elastic wave medium parameters for an inhomogeneous, anisotropic medium with relaxation**:

$\mu_{k,r}$ = inertia relaxation function (kg/m^3·s)

$\chi_{i,j,p,q}$ = compliance relaxation function (s^{-1})

- **Elastic wave constitutive relations**:

$$\Phi_k = \mu_{k,r} \overset{(t)}{*} v_r$$

$$e_{i,j} = -\chi_{i,j,p,q} \overset{(t)}{*} \tau_{p,q}$$

where

$$\overset{(t)}{*} = \text{time convolution}$$

- **Elastic wave constitutive relations for an inhomogeneous, anisotropic fluid with simple loss mechanisms**:

$$\partial_t \Phi_k = K_{k,r} v_r + \rho_{k,r} \partial_t v_r$$
$$-\partial_t e_{i,j} = \Gamma_{i,j,p,q} \tau_{p,q} + S_{i,j,p,q} \partial_t \tau_{p,q}$$

where

$K_{k,r}$ = coefficient of frictional force (kg/m^3·s)

$\rho_{k,r}$ = volume density of (inertial) mass (kg/m^3)

$\Gamma_{i,j,p,q}$ = coefficient of bulk inviscidness (Pa^{-1}·s^{-1})

$S_{i,j,p.q}$ = compliance (Pa^{-1})

Elastic wave medium properties (continued)

- **Elastic wavefield equations**:

$$
\begin{aligned}
-\Delta^{+}_{k,m,p,q}\partial_m \tau_{p,q} + \partial_t \Phi_k &= f_k \\
\Delta^{+}_{i,j,n,r}\partial_n v_r - \partial_t e_{i,j} &= h_{i,j}
\end{aligned}
$$

- **Elastic wave compatibility relation**:

Elastic wavefield equations, compatibility relations

- **Interface boundary conditions**:

  across any surface of discontinuity in medium properties

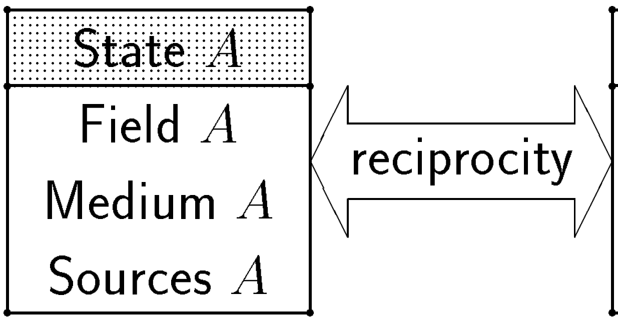
$$\Delta^{+}_{k,m,p,q} \nu_m \tau_{p,q} = \text{continuous}$$
$$v_r = \text{continuous}$$

ν_m = unit vector along normal to interface

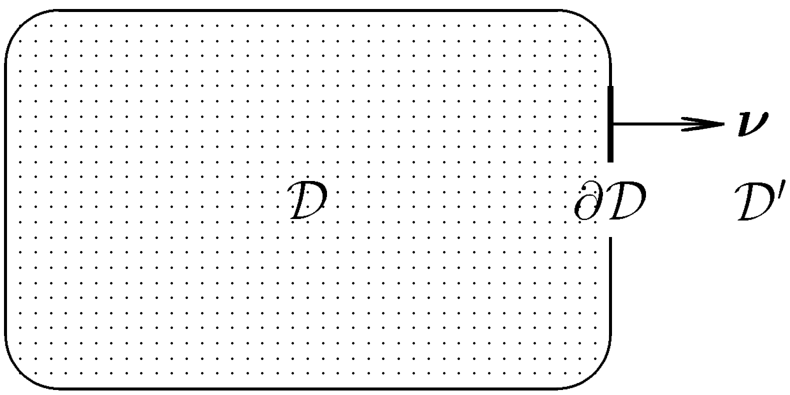
- **Compatibility boundary conditions**:

Elastic wave boundary conditions

- **The two States A and Z:**



- **Domain of application:**



- **Elastic wave reciprocity theorem of the time-convolution type**:

$$\Delta^{+}_{m,r,p,q} \int_{\partial\mathcal{D}} \nu_m[-\tau^A_{p,q} \overset{(t)}{*} v^Z_r + \tau^Z_{p,q} \overset{(t)}{*} v^A_r]\mathrm{d}A$$

$$= \int_{\mathcal{D}}[-\partial_t\Phi^A_k \overset{(t)}{*} v^Z_k + \partial_t\Phi^Z_r \overset{(t)}{*} v^A_r$$

$$+\partial_t e^A_{i,j} \overset{(t)}{*} \tau^Z_{i,j} - \partial_t e^Z_{p,q} \overset{(t)}{*} \tau^A_{p,q}]\mathrm{d}V$$

$$+ \int_{\mathcal{D}}[f^A_k \overset{(t)}{*} v^Z_k + h^A_{i,j} \overset{(t)}{*} \tau^Z_{i,j}$$

$$-f^Z_r \overset{(t)}{*} v^A_r - h^Z_{p,q} \overset{(t)}{*} \tau^A_{p,q}]\mathrm{d}V$$

- **The reciprocity relation is a "weak" formulation of the field problem: for the theorem to hold for arbitrary States Z, State A must satisfy the field equations**

- **Electromagnetic wave quantities**:

$$E_r = \text{electric field strength (V/m)}$$
$$H_p = \text{magnetic field strength (A/m)}$$
$$D_k = \text{electric flux density (C/m}^2\text{)}$$
$$B_j = \text{magnetic flux density (T)}$$

- **Electromagnetic source quantities**:

$$J_k = \text{volume source density of electric current (A/m}^2\text{)}$$
$$K_j = \text{volume source density of magnetic current (V/m}^2\text{)}$$

Electromagnetic wave and source quantities

- **Electromagnetic medium parameters for an inhomogeneous, anisotropic medium with relaxation**:

$\kappa^{\mathrm{e}}_{k,r}$ = electric relaxation function (F/m·s)

$\kappa^{\mathrm{m}}_{j,p}$ = magnetic relaxation function (H/m·s)

- **Electromagnetic constitutive relations**:

$$D_k = \kappa^{\mathrm{e}}_{k,r} \overset{(t)}{*} E_r$$
$$B_j = \kappa^{\mathrm{m}}_{j,p} \overset{(t)}{*} H_p$$

where

$\overset{(t)}{*}$ = time convolution

- **Electromagnetic constitutive relations for a simple inhomogeneous, anisotropic medium**:

$$\partial_t D_k = \sigma_{k,r} E_r + \epsilon_{k,r} \partial_t E_r$$
$$\partial_t B_j = \chi_{j,p} H_p + \mu_{j,p} \partial_t H_p$$

where

$\sigma_{k,r}$ = conductivity (S/m)

$\epsilon_{k,r}$ = permittivity (F/m)

$\chi_{j,p}$ = linear magnetic hysteresis loss coefficient (H/m·s)

$\mu_{j,p}$ = permeability (H/m)

Electromagnetic medium properties (continued)

- **Electromagnetic field equations**:

$$-\epsilon_{k,m,p}\partial_m H_p + \partial_t D_k = -J_k$$
$$\epsilon_{j,n,r}\partial_n E_r + \partial_t B_j = -K_j$$

- **Electromagnetic compatibility relations**:

$$\int_{\mathcal{S}} \nu_k \partial_t D_k \mathrm{d}A = -\int_{\mathcal{S}} \nu_k J_k \mathrm{d}A$$
$$\int_{\mathcal{S}} \nu_j \partial_t B_j \mathrm{d}A = -\int_{\mathcal{S}} \nu_j K_j \mathrm{d}A$$

for any closed surface $\mathcal{S}$ (ν_m = unit vector along outward normal to $\mathcal{S}$)

- **Levi-Civita tensor**:

$\epsilon_{k,m,p} = \{+1, -1, 0\}$ if $\{k, m, p\}$ is {even, odd, no} permutation of $\{1, 2, 3\}$

Electromagnetic field equations, compatibility relations

- **Interface boundary conditions**: across any surface of discontinuity in medium properties

$$\epsilon_{k,m,p}\nu_m H_p = \text{continuous}$$
$$\epsilon_{j,n,r}\nu_n E_r = \text{continuous}$$

$\nu_m =$ unit vector along the normal to the interface

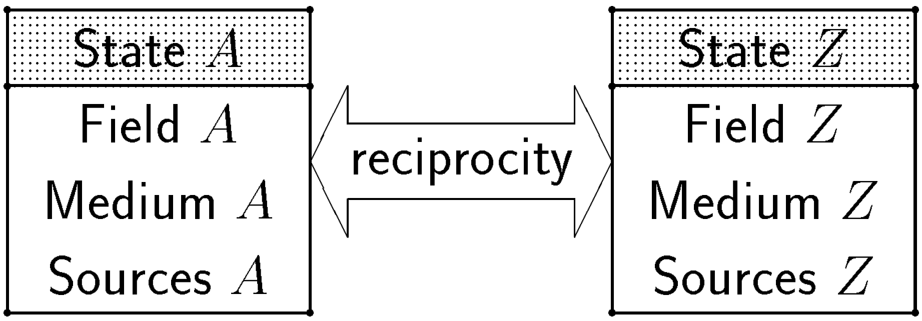
- **Compatibility boundary conditions**: across any surface of discontinuity in medium properties and/or volume source distributions

$$\nu_k(\partial_t D_k + J_k) = \text{continuous}$$
$$\nu_j(\partial_t B_j + K_j) = \text{continuous}$$

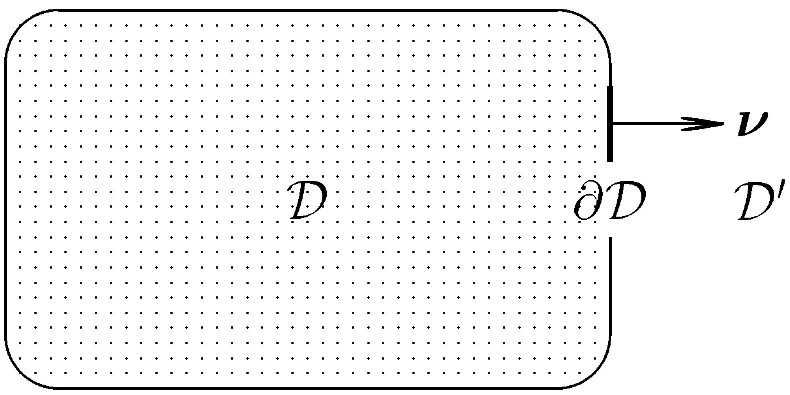
$\nu_k =$ unit vector along the normal to the surface of discontinuity

Electromagnetic boundary conditions

- **The two States A and Z:**



- **Domain of application:**



- **Electromagnetic reciprocity theorem of the time-convolution type**:

$$\epsilon_{m,r,p} \int_{\partial\mathcal{D}} \nu_m [E_r^A \overset{(t)}{*} H_p^Z - E_r^Z \overset{(t)}{*} H_p^A] \mathrm{d}A$$

$$= \int_{\mathcal{D}} [\partial_t D_k^A \overset{(t)}{*} E_k^Z - \partial_t D_r^Z \overset{(t)}{*} E_r^A$$

$$-\partial_t B_j^A \overset{(t)}{*} H_j^Z + \partial_t B_p^Z \overset{(t)}{*} H_p^A] \mathrm{d}V$$

$$+ \int_{\mathcal{D}} [J_k^A \overset{(t)}{*} E_k^Z - K_j^A \overset{(t)}{*} H_j^Z$$

$$-J_r^Z \overset{(t)}{*} E_r^A + K_p^Z \overset{(t)}{*} H_p^A] \mathrm{d}V$$

- **The reciprocity relation is a "weak" formulation of the field problem: for the theorem to hold for arbitrary States Z, State A must satisfy the field equations**

Reciprocity theorem of the time-convolution type

- **Reciprocity theorem**

$\int_{\partial\mathcal{D}}$ [surface interaction term]$_m \nu_m \mathrm{d}A$

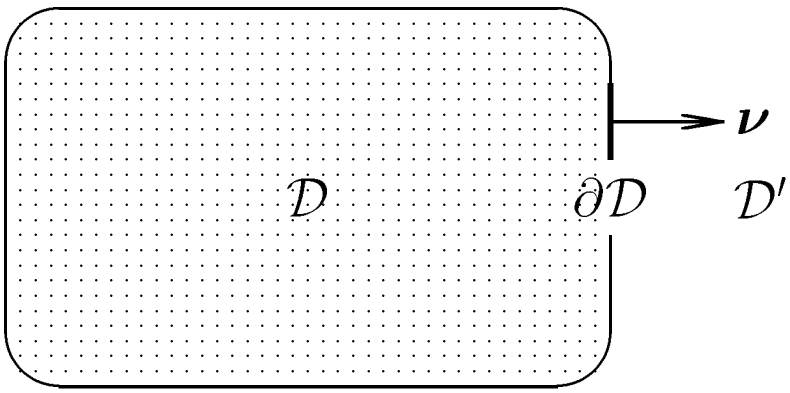
$= \int_{\mathcal{D}}$ [domain interaction terms]$\mathrm{d}V$;

[domain interaction terms]

$=$ [contrast in media terms]

$+$[source/field interaction terms]

- **Domain of application:**



Reciprocity theorem (general structure)

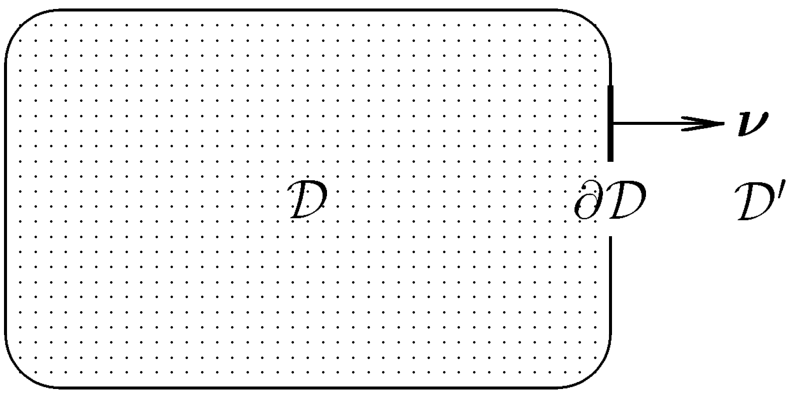
- $\int_{\partial\mathcal{D}}$ [surface interaction term]$_m \nu_m \mathrm{d}A = 0$

if

- exterior domain $\mathcal{D}'$ is sourcefree

&

- State $\boldsymbol{A}$ and State $\boldsymbol{Z}$ are both causal

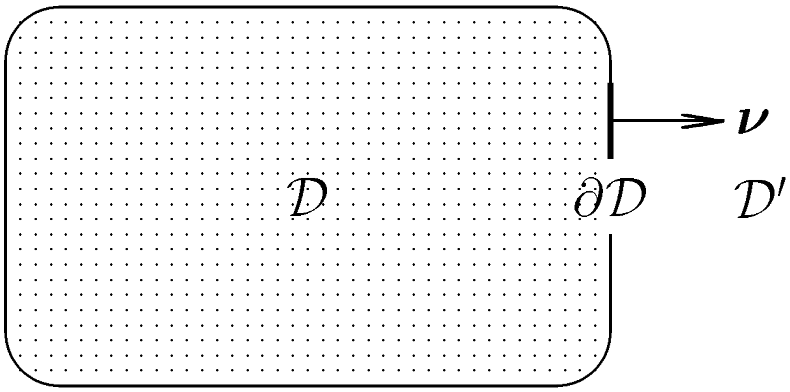


Reciprocity theorem (general properties)

- $\int_{\mathcal{D}}$ [contrast in media terms]$\mathrm{d}V = 0$

if

- Medium A/Z = Adjoint(Medium Z/A) in $\mathcal{D}$

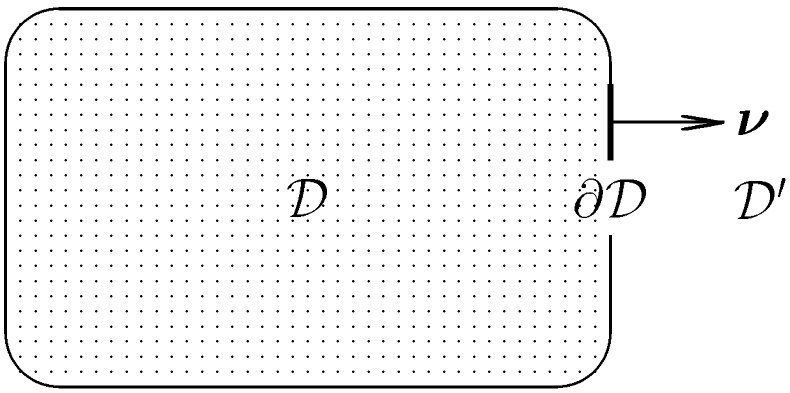


Reciprocity theorem (general properties)

- $\int_{\mathcal{D}}$ [source/field interaction terms]$\mathrm{d}V = 0$

if

- $\mathcal{D}$ is sourcefree



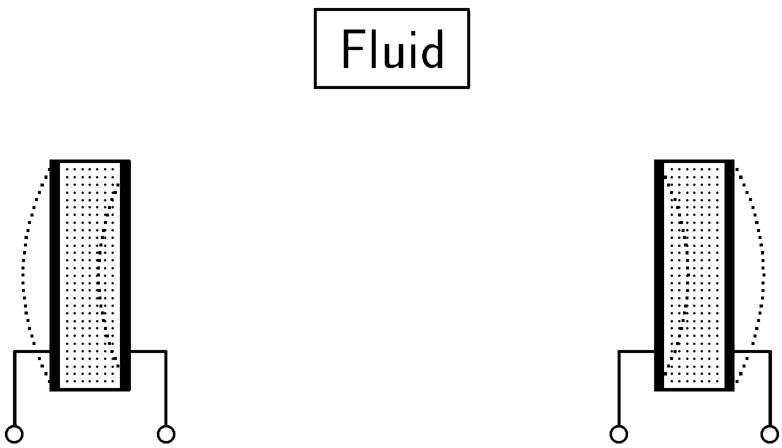
Reciprocity theorem (general properties)

Fluid

- **Volume injection transducer (monopole transducer)**:

$$q \neq 0 \ ; \ f_k = 0$$

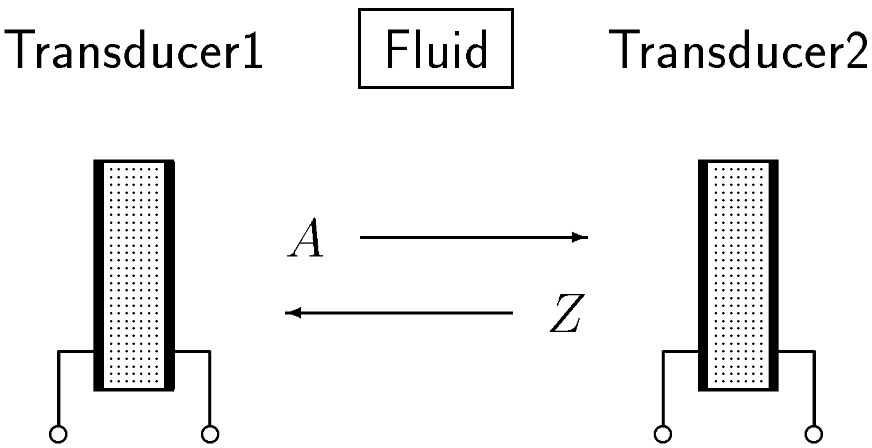
Acoustic transducer (monopole transducer)



- **Force action transducer (dipole transducer)**:

$$q = 0 \ ; \ f_k \neq 0$$

Acoustic transducer (dipole transducer)



	State A	State Z
Transducer1	Transmitting	Receiving
Transducer2	Receiving	Transmitting

Transmission/Reception reciprocity (acoustic waves)

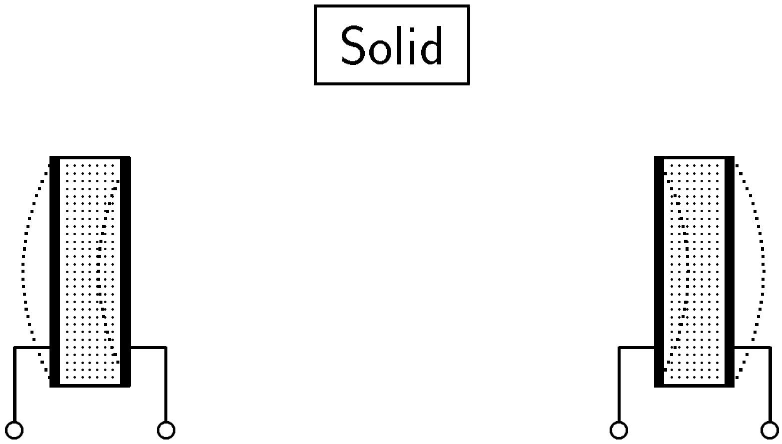
Transmission/reception reciprocity (volume action transducers):

$$\int_{\mathrm{Transducer1}} [f_k^A \overset{(t)}{*} v_k^Z - q^A \overset{(t)}{*} p^Z] \mathrm{d}V$$

$$= \int_{\mathrm{Transducer2}} [f_r^Z \overset{(t)}{*} v_r^A - q^Z \overset{(t)}{*} p^A] \mathrm{d}V$$

$\Longrightarrow$

- **Monopole transducer (volume injection transducer;** $q \neq 0; f_k = 0$**) is sensitive to acoustic pressure** p
- **Dipole transducer (force action transducer;** $q = 0; f_k \neq 0$**) is sensitive to particle velocity** v_r



- **Force action transducer**:

$$f_k \neq 0 \;\; ; \;\; h_{i,j} = 0$$

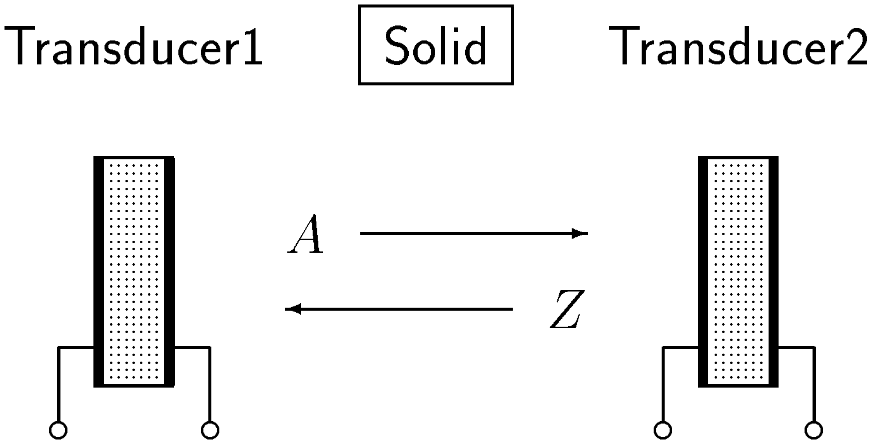
Elastodynamic force action transducer

Solid

- **Deformation transducer**:

$$f_k = 0 \ ; \ h_{i,j} \neq 0$$

Elastodynamic deformation transducer



	State A	State Z
Transducer1	Transmitting	Receiving
Transducer2	Receiving	Transmitting

Transmission/Reception reciprocity (elastic waves)

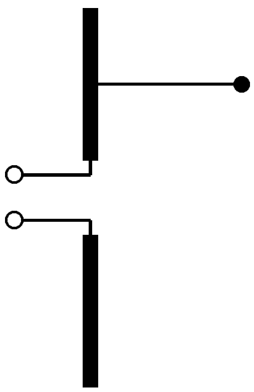
Transmission/reception reciprocity (volume action transducers):

$$\int_{\text{Transducer1}}[f_k^A \overset{(t)}{*} v_k^Z + h_{i,j}^A \overset{(t)}{*} \tau_{i,j}^Z]\mathrm{d}V$$

$$= \int_{\text{Transducer2}}[f_r^Z \overset{(t)}{*} v_r^A + h_{p,q}^Z \overset{(t)}{*} \tau_{p,q}^A]\mathrm{d}V$$

$\Longrightarrow$

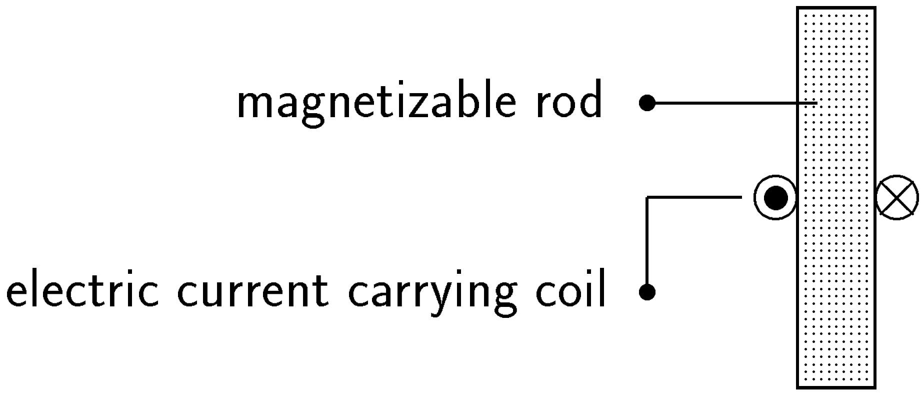
- **Force action transducer (**$f_k \neq 0; h_{i,j} = 0$**) is sensitive to particle velocity** v_r
- **Deformation transducer (**$f_k = 0; h_{i,j} \neq 0$**) is sensitive to dynamic stress** $\tau_{p,q}$



- **Electric dipole antenna**:

$$J_k \neq 0 \ ; \ K_j = 0$$

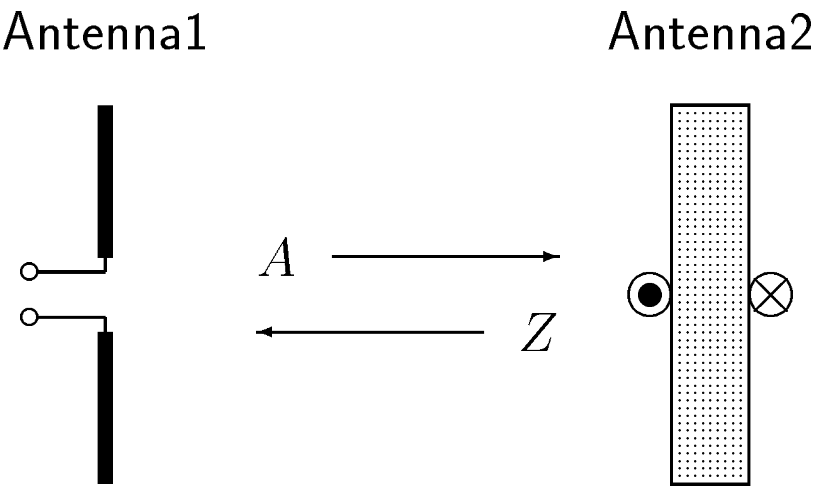
Electric dipole antenna



- **Magnetic dipole antenna**:

$$J_k = 0 \;\; ; \;\; K_j \neq 0$$

Magnetic dipole antenna



	State A	State Z
Antenna1	Transmitting	Receiving
Antenna2	Receiving	Transmitting

Electromagnetic antenna reciprocity

Transmission/reception reciprocity (volume action antennas):

$$\int_{\text{Antenna1}} [J_k^A \overset{(t)}{*} E_k^Z - K_j^A \overset{(t)}{*} H_j^Z] \mathrm{d}V$$
$$= \int_{\text{Antenna2}} [J_r^Z \overset{(t)}{*} E_r^A - K_p^Z \overset{(t)}{*} H_p^A] \mathrm{d}V$$

$\Longrightarrow$

- **Electric dipole antenna (**$J_k \neq 0; K_j = 0$**) is sensitive to electric field strength** E_r
- **Magnetic dipole antenna (**$J_k = 0; K_j \neq 0$**) is sensitive to magnetic field strength** H_p

Transmission/reception reciprocity describes quantitatively

- the **interaction** of an observer's **measuring apparatus ('receiver')** with an **interrogated wavefield**
- the type of **data** the observer can **extract from the wavefield**

This interaction is FUNDAMENTAL to the entire field of macroscopic physics

Reciprocity serves to formulate

- **Direct (forward) source problems**
- **Inverse source problems**
- **Direct (forward) scattering problems**
- **Inverse scattering problems**

in mathematical physics in a unifying manner

Reciprocity leads, when supplemented with

- **superposition principle**
- **error minimization criterion**

to

- **computational algorithms of an iterative nature**

with

- **guaranteed decrease in error at each iteration step**

in problems of computational physics

Wavefield problems:

	Sources	Medium	Field
Direct source problem	known	known	**?**
Direct scattering problem[1]	known	known	**?**
Inverse source problem	**?** in $\mathcal{D}^{\mathrm{T}}$	known	known in $\mathcal{D}^{\Omega}$
Inverse scattering problem[1]	known	**?** in $\mathcal{D}^{\mathrm{S}}$	known in $\mathcal{D}^{\Omega}$

[1]employs superposition and embedding technique

Wavefield problem categories

Direct source problem (computational steps):

- **State $A \Longrightarrow$ discretization (expansion) of the actual field (Field A) to be computed**
- **State $Z \Longrightarrow$ sequence of computational states with Field Z of local (elemental) support and corresponding Sources Z**
- **Reciprocity theorem $\Longrightarrow$ Finite-element formulation**

Direct source problem (computation)

Direct scattering problem (computational steps):

- **Superpositon principle $\Longrightarrow$ Total field = Incident field + Scattered field**
- **Incident field: Field due to (known) sources in known embedding**
- **State A $\Longrightarrow$ discretization (expansion) of the scattered field to be computed**
- **State Z $\Longrightarrow$ Computational state (different choices)**

Direct scattering problem (computational steps) (continued):

- **State $Z \Longrightarrow$ sequence of computational states with Field Z of local (elemental) support and corresponding Sources Z**
- **Reciprocity theorem $\Longrightarrow$ Finite-element formulation**

Direct scattering problem (computational steps) (continued):

- **State $Z \Longrightarrow$ sequence of computational states with Source Z of local (elemental) support and corresponding Field Z radiated in embedding**
- **Reciprocity theorem $\Longrightarrow$ Integral equation formulation**

Inverse source problem (computational steps):

- **State $A \Longrightarrow$ actual wavefield radiated due to sources with assumed support $\mathcal{D}^{\mathrm{T}}$ and observed in $\mathcal{D}^{\Omega}$**
- **State $Z \Longrightarrow$ sequence of computational states with Sources Z with (subdomain of) $\mathcal{D}^{\Omega}$ as support and corresponding Fields Z radiated in embedding**
- **Error minimization + Reciprocity theorem $\Longrightarrow$ minimum norm solution**

Inverse source problem (computation) 1

Inverse source problem (computational procedure):

- The **sequence** $\{$**Sources** $Z\}$ and the **choice of the error criterion** determine the **inversion algorithm** and the resulting minimum norm solution

N.B. The inverse source problem has no unique solution (because of the existence of 'non-radiating sources' with support $\mathcal{D}^{\mathrm{T}}$ whose wavefield has also the support $\mathcal{D}^{\mathrm{T}}$)

Inverse scattering problem (computational steps):

- **State $A \Longrightarrow$ actual scattered wavefield due to contrast sources with assumed support $\mathcal{D}^{\mathrm{s}}$ and observed in $\mathcal{D}^{\Omega}$**
- **State $Z \Longrightarrow$ sequence of computational states with Sources Z with (subdomain of) $\mathcal{D}^{\Omega}$ as support and corresponding Fields Z radiated in embedding**

Inverse scattering problem (computational steps):

- **Simultaneous error minimization on (Linearized) Reciprocity theorem + Constitutive relation $\Longrightarrow$ Iterative procedure leading to a minimum norm solution**
- The **sequence** {**Sources** Z} and the **choice of the error criterion** determine the **inversion algorithm** and the resulting **minimum norm solution**

Inverse scattering problem (computational steps):

N.B. Due to the condition that the constitutive parameters to be reconstructed are to be independent of the interrogating source distributions (of the incident wavefield), the inverse scattering problem can have a unique solution.

Operator equation to be "solved":

- $\mathsf{L}u \simeq q$

Inner product:

- $< v, u >$ with
- $< v, u > = < u, v >^*$
- $< v, \mathsf{L}u > = < \mathsf{L}^{\mathsf{H}} v, u >$

  ($\mathsf{L}^{\mathsf{H}}=$ **Hermitean conjugate** of L)

Associated norm:

- $||u|| = < u, u >^{1/2}$

Residual:

- $r = q - \mathsf{L}u$

Operator equation, inner product, norm, residual

Iterative solution procedure:

- $u^{[n+1]} = u^{[n]} + \delta u^{[n]}$ for $n = 0, 1, 2, \ldots$

Residual after n steps:

- $r^{[n]} = q - \mathsf{L}u^{[n]}$ for $n = 0, 1, 2, \ldots$

$\Longrightarrow r^{[n+1]} = r^{[n]} - \mathsf{L}\delta u^{[n]}$

"Improvement condition":

- $||r^{[n]}||^2 > ||r^{[n+1]}||^2$ for $n = 0, 1, 2, \ldots$

Sufficient condition for improvement:

- $< r^{[n+1]}, \mathsf{L}\delta u^{[n]} > + < \mathsf{L}\delta u^{[n]}, r^{[n+1]} >= 0$

which is satisfied by*

- $\delta u^{[n]} = \alpha^{[n]}\mathsf{L}^{\mathsf{H}} r^{[n]}$ with • $\alpha^{[n]} = \frac{||\mathsf{L}^{\mathsf{H}} r^{[n]}||^2}{||\mathsf{L}\mathsf{L}^{\mathsf{H}} r^{[n]}||^2}$

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*More sophisticated choices for $\delta u^{[n]}$ exist

Iterative solution procedure, "improvement condition"

Owing to its generality, the

- property of **reciprocity**

has

- **multitudinous applications**

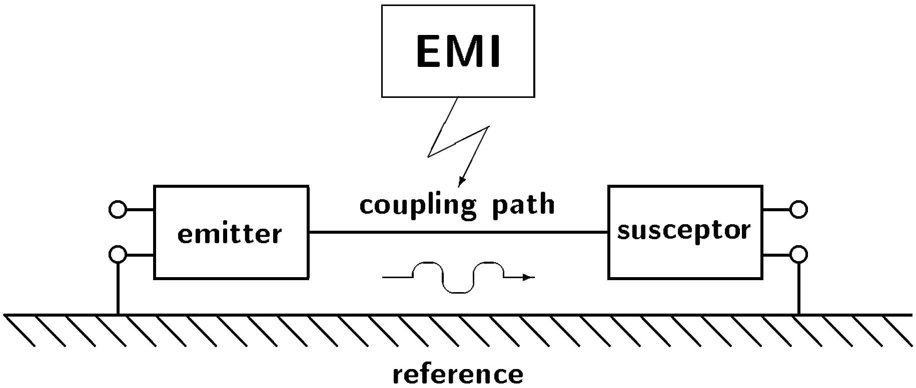
in

- **a variety of fields in the applied sciences**

Some are mentioned on subsequent viewgraphs

Reciprocity - Applications 1

- **Transmission/reception reciprocity** serves as a **necessity check** of the **consistency** and **accuracy** of any **numerical algorithm** for **acoustic, elastic and electromagnetic wavefield computation**
- **Transmission/reception reciprocity** is the **basic tool** for the **ElectroMagnetic Interference (EMI) analysis** in the **ElectroMagnetic Compatibility** of **electric** and **electronic equipment, devices and systems**



ElectroMagnetic Interference problem with emitter, susceptor and coupling path.

ElectroMagnetic Interference (EMI) configuration

Through **inverse source problems**, reciprocity lies at the basis of the analysis of **acoustic emission processes** such as

- **Detection of interior crack formation in mechanical structures**
- **Monitoring of the rock fracturing process for the enhancement of production in oil wells**

Reciprocity - Applications 3

Through **inverse scattering problems**, reciprocity lies at the basis of

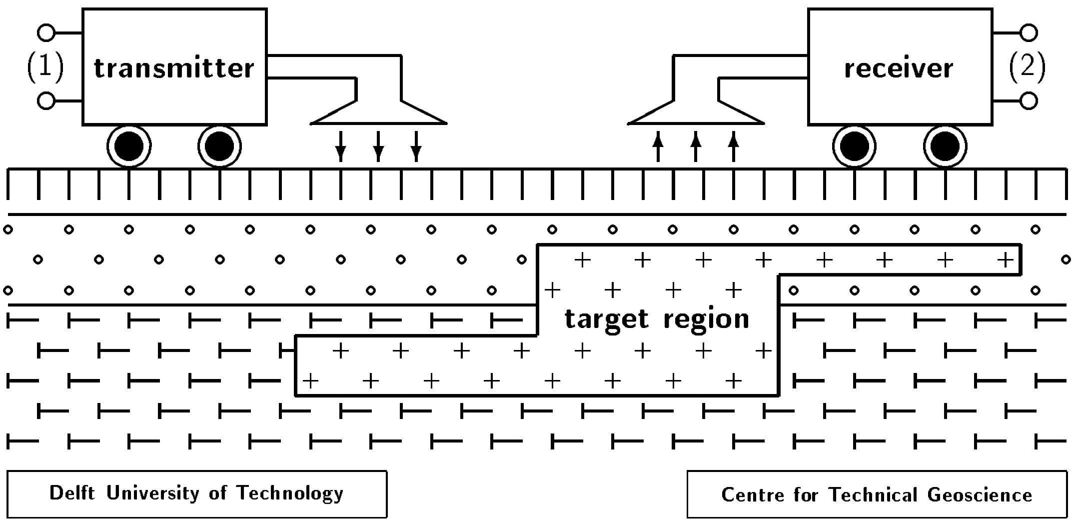
- **Ultrasonic imaging for medical diagnostics**
- **X-ray tomography**
- **Imaging of the subsurface of the Earth in the seismic exploration for fossil energy resources**
- **Borehole electric and electromagnetic profiling of subsurface structures**
- **Cross-borehole seismic and electromagnetic evaluation of oil and gas reservoirs**

Reciprocity - Applications 4

Inverse scattering applications of **reciprocity** (continued):

- **Ground Penetrating Radar systems for shallow imaging of underground cables, pipelines and archeological artifacts**
- **Remote sensing of weeds under agricultural vegetation**
- **Determination of permittivity profiles in graded-indes glass fibers for optical communication**

GPR system



Ground Penetrating Radar system with transmitting antenna, receiving antenna and target region in a layered soil

Ground Penetrating Radar system

A derivation of the reciprocity theorems and more details about their applications can be found in:

- De Hoop, A. T., *Handbook of Radiaton and Scattering of Waves*, London • New York, Academic Press, 1085 + XXX pp., Chapters 7, 15 and 28.

Reciprocity - More details