WAVEFIELD RECIPROCITY AND OPTIMUM DATA FITTING IN REMOTE SENSING

Maarten V. de Hoop

&

Adrianus T. de Hoop

Center for Wave Phenomena

Colorado School of Mines

GOLDEN CO 80401

U.S.A.

The **objective** is to construct a **remote sensing theory** based on

- concepts of **wavefield reciprocity**

and

- techniques of **optimum data fitting**

for

- a **general class of wavefields**

The **class of wavefields considered** encompasses

- **acoustic waves in fluids**
- **elastic waves in condensed matter (solids)**
- **electromagnetic waves**

The physical **wavefield constituents** that characterize a **WAVEFIELD STATE** are:

- the **sources** that generate the field (**excitation**)
- the **medium** in which the field is present (**propagation**)
- the **wavefield state quantities** through which the field is accessible to measurement (**reception**)

RECIPROCITY = TIME-CONVOLVED INTERACTION OF TWO STATES OVER BOUNDED DOMAIN IN SPACE

Wavefield constituents and reciprocity

Remote sensing problems can be subdivided into:

- an **inverse source** part
- an **inverse constituency** part

Remote sensing problems are conveniently formulated as **inverse scattering problems**

Remote sensing problems (categories)

The **remote sensing configuration** consists of

- a **bounded domain to be interrogated**

present in an

- **embedding whose Green's functions (point-source generated wavefields) are - analytically or computationally - known**
- The configuration is **interrogated** by an arbitrary number of **sources**
- The **wavefield** is **measured** by an arbitrary number of **receivers**

The multi-source/multi-receiver remote sensing problem

The physical **wavefield constituents** that characterize a **WAVEFIELD STATE** are:

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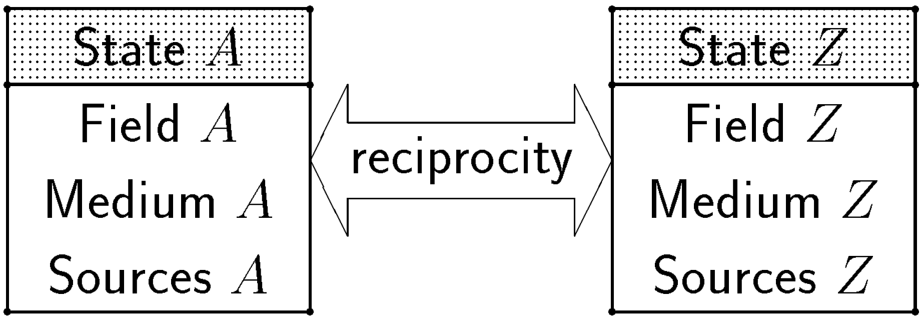
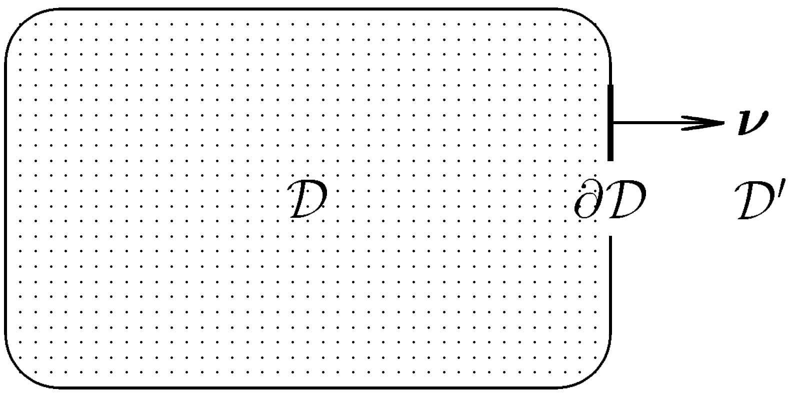
RECIPROCITY = TIME-CONVOLVED INTERACTION OF TWO STATES OVER BOUNDED DOMAIN IN SPACE

- **The two States A and Z:**



- **Domain of application:**



Wave phenomena are phenomena in

- **(three-dimensional) space** ($x \in \mathcal{R}^3$)

and

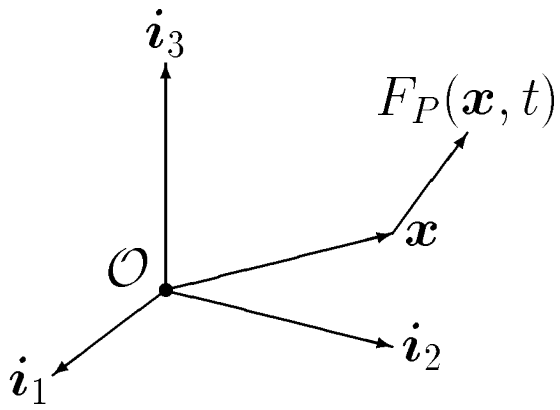
- **time** ($t \in \mathcal{R}$)

They

- are **causally related to the action of their sources**
- **transfer energy from source to receiver**
- **SHARE** the property of **RECIPROCITY**

Wave phenomena (some general characteristics)

Reference frame in $\mathcal{R}^3$:



Position and time:

- $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$
- $\partial_m =$ partial differentiation with respect to x_m
- $\boldsymbol{\partial} = \boldsymbol{i}_1\partial_1 + \boldsymbol{i}_2\partial_2 + \boldsymbol{i}_3\partial_3$
- $\partial_t =$ partial differentiation with respect to time t

Cartesian reference frame, position and time

- **Wave quantities:**

$$F_P(\boldsymbol{x}, t) = \text{'field strength'}$$
$$\Phi_I(\boldsymbol{x}, t) = \text{'flux density'}$$

- **Source quantities:**

$$Q_I(\boldsymbol{x}, t) = \text{'volume source density'}$$

- **Medium properties:**

$M_{I,P}(\boldsymbol{x}, t)$ = constitutive relaxation function (inhomogeneous, anisotropic, lossy media)

Wavefield, source and medium quantities (arrays)

- **Field equations:**

$$\mathsf{D}_{I,P}F_P + \partial_t \Phi_I \;=\; Q_I$$

  - $\mathsf{D}_{I,P} = \mathsf{D}_{I,P}(\boldsymbol{\partial})$ **homogeneous of degree one in $\boldsymbol{\partial}$, with coefficients ± 1**

- **Constitutive relation (linear, time invariant, locally reacting):**

$$\Phi_I \;=\; M_{I,P} \overset{(t)}{*} F_P$$

with

$$\overset{(t)}{*} \;=\; \text{time convolution}$$

- **Summation over repeated subscripts**

Field equations, constitutive relations

The **boundary conditions** across a **passive (= sourcefree) interface** where the **constitutive parameters jump by finite amounts** are:

- $N_{I,P} F_P$ **continuous**

where

- $N_{I,P} = \mathsf{D}_{I,P}(\boldsymbol{\nu})$

in which

- $\boldsymbol{\nu}$ = unit vector along the normal to the interface

A **necessary** and **sufficient condition** for **RECIPROCITY** to hold is

- $\mathsf{D}_{I,P} = -\mathsf{D}_{P,I}$

Local reciprocity relation of the time-convolution type:

- $\mathsf{D}_{I,P}(F_P^A \overset{(t)}{*} F_I^Z)$

  (field/field interaction term)

  $= F_I^Z \overset{(t)}{*} \mathsf{D}_{I,P} F_P^A - F_P^A \overset{(t)}{*} \mathsf{D}_{P,I} F_I^Z$

  $= -(F_I^Z \overset{(t)}{*} \partial_t \Phi_I^A - F_P^A \overset{(t)}{*} \partial_t \Phi_P^Z)$

  (field/flux interaction term)

  $+(F_I^Z \overset{(t)}{*} Q_I^A - F_P^A \overset{(t)}{*} Q_P^Z)$

  (field/source interaction term)

Local reciprocity relation of the time-convolution type

Global reciprocity theorem of the time-convolution type (for the domain $\mathcal{D}$):

- $\int_{\partial\mathcal{D}} N_{I,P}(F_P^A \overset{(t)}{*} F_I^Z)\mathrm{d}A$

  (field/field interaction term)

  $\int_{\mathcal{D}} \mathsf{D}_{I,P}(F_P^A \overset{(t)}{*} F_I^Z)\mathrm{d}V$

  $= \int_{\mathcal{D}}(F_I^Z \overset{(t)}{*} \mathsf{D}_{I,P}F_P^A - F_P^A \overset{(t)}{*} \mathsf{D}_{P,I}F_I^Z)\mathrm{d}V$

  $= -\int_{\mathcal{D}}(F_I^Z \overset{(t)}{*} \partial_t\Phi_I^A - F_P^A \overset{(t)}{*} \partial_t\Phi_P^Z)\mathrm{d}V$

  (field/flux interaction term)

  $+\int_{\mathcal{D}}(F_I^Z \overset{(t)}{*} Q_I^A - F_P^A \overset{(t)}{*} Q_P^Z)\mathrm{d}V$

  (field/source interaction term)

Global reciprocity relation of the time-convolution type

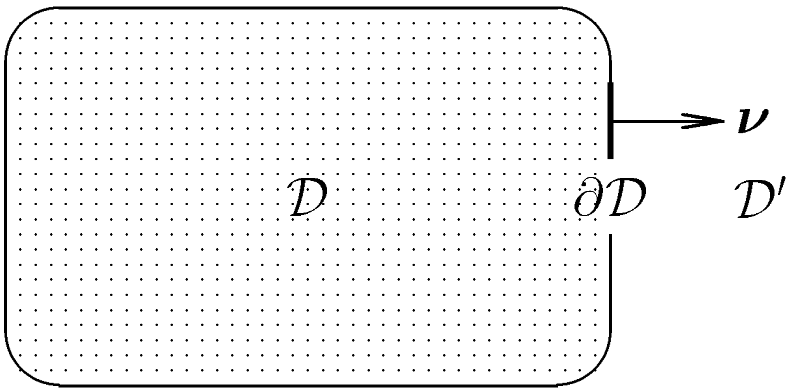
- **Reciprocity theorem:**

  Field/Field interaction term$|_{\partial\mathcal{D}}$

  $=$ **Field/Flux interaction term**$|_{\mathcal{D}}$

  $+$ **Source/Field interaction term**$|_{\mathcal{D}}$

- **Domain of application:**



Reciprocity theorem (general structure)

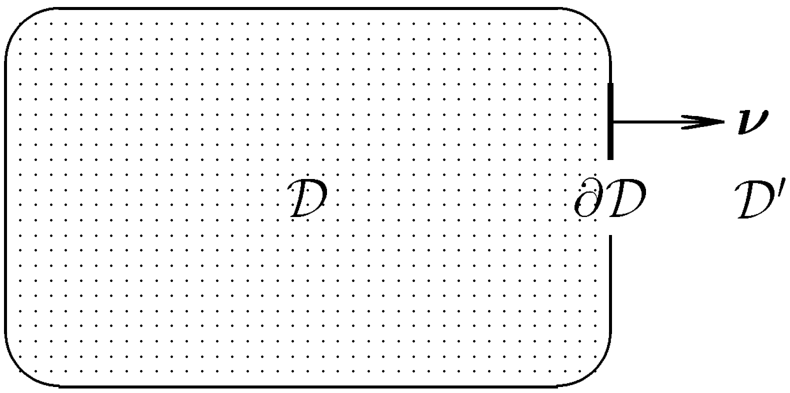
- **Field/Field interaction term$|_{\partial\mathcal{D}}$**
  $= \int_{\partial\mathcal{D}} N_{I,P}(F_P^A \overset{(t)}{*} F_I^Z)\mathrm{d}A$
  $= \mathbf{0}$

if

- exterior domain $\mathcal{D}'$ is sourcefree

&

- State $\boldsymbol{A}$ and State $\boldsymbol{Z}$ are both causal



Reciprocity theorem (general properties)

- **Field/Flux interaction term$|_{\mathcal{D}}$**

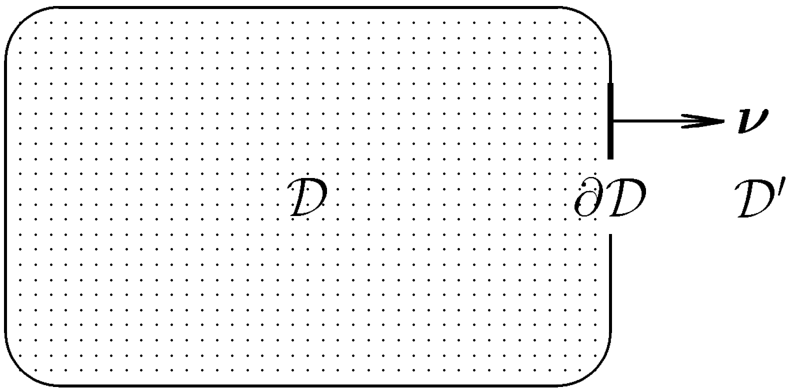
  $-\int_{\mathcal{D}}(F_I^Z \overset{(t)}{*} \partial_t \Phi_I^A - F_P^A \overset{(t)}{*} \partial_t \Phi_P^Z)\mathrm{d}V$

  $= \mathbf{0}$

if

- Medium A/Z = Adjoint(Medium Z/A)

  in $\mathcal{D} \Longrightarrow \bullet\ M_{I,P}^A(\boldsymbol{x}, t) = M_{P,I}^Z(\boldsymbol{x}, t)$



Reciprocity theorem (general properties)

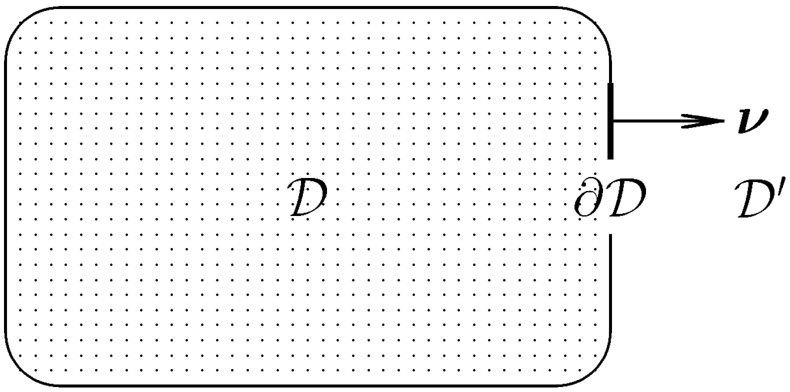
- Source/Field interaction term$|_{\mathcal{D}}$

  $\int_{\mathcal{D}}(F_I^Z \overset{(t)}{*} Q_I^A - F_P^A \overset{(t)}{*} Q_P^Z)\mathrm{d}V$

  $= \mathbf{0}$

if

- $\mathcal{D}$ is sourcefree



Reciprocity theorem (general properties)

Multi-source/multi-receiver remote sensing problem:

- **Given SOURCES** $\{Q^{\mathrm{i}}_{I;n}; n = 1, \ldots, N\}$ **generate in a given embedding (background) with constitutive parameters** $M^{\mathrm{b}}_{I,P}$ **the INCIDENT WAVEFIELDS** $\{F^{\mathrm{i}}_{P;n}; n = 1, \ldots, N\}$ **to interrogate the configuration for the presence of a contrasting (scattering) domain of bounded support** $\mathcal{D}^{\mathrm{s}}$

The multi-source/multi-receiver remote sensing problem

Remote sensing problem (continued):

- **The PRESENCE of $\mathcal{D}^{\mathrm{s}}$ with constitutive parameters $M^{\mathrm{s}}_{I,P}$ manifests itself by the presence of SCATTERED WAVEFIELDS $\{F^{\mathrm{s}}_{P;n}; n = 1, \ldots, N\}$ that are associated with CONTRAST SOURCES $\{Q^{\mathrm{s}}_{I;n}; n = 1, \ldots, N\}$ that radiate into the embedding**
- $Q^{\mathrm{s}}_{I;n} = -\partial_t (M^{\mathrm{s}}_{I,P} - M^{\mathrm{b}}_{I,P}) \overset{(t)}{*} F_{P;n}$
- $F_{P;n} = F^{\mathrm{i}}_{P;n} + F^{\mathrm{s}}_{P;n}$

The remote sensing problem (continued)

Remote sensing problem (continued):

- **The scattered wavefields are measured in M observational domains $\{\mathcal{D}^{\Omega}_{m}; m = 1, \ldots, M\}$ by RECEIVERS whose RECEIVING PROPERTIES are characterized by their EQUIVALENT SOURCE DENSITIES $\{Q^{\Omega}_{P;m}\}$ in their TRANSMITTING STATE**

- **The collection of observed data (time traces) is**

  - $d_{m,n}(t) = \int_{\mathcal{D}^{\Omega}_{m}} Q^{\Omega}_{P;m} \overset{(t)}{*} F^{\mathrm{s}}_{P;n} \mathrm{d}V;$
    $(m = 1, \ldots, M;$
    $n = 1, \ldots, N)$

The remote sensing problem (continued)

Remote sensing problem (continued):

- **The AIM is to RECONSTRUCT $M^{s}_{I,P}$ for $x \in \mathcal{D}^{s}$ from the collection of observed data**

- **The reconstruction takes place via an OPTIMUM DATA FITTING TECHNIQUE where the unknown contrast source and constituency distributions are ITERATIVELY UPDATED such that the MISMATCH between the observed and the calculated data - according to an appropriate ERROR CRITERION - DECREASES at each step**

The remote sensing problem (continued)

Scattered wavefield source-type integral representation:

- $F^{\mathrm{s}}_{P;n}(\boldsymbol{x}, t)$
  $= \int_{\mathcal{D}^{\mathrm{s}}} G^{\mathrm{b}}_{P,I}(\boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} Q^{\mathrm{s}}_{I;n}(\boldsymbol{x}', t) \mathrm{d}V(\boldsymbol{x}')$

in which

$G^{\mathrm{b}}_{P,I}(\boldsymbol{x}, \boldsymbol{x}', t)$ = Green's function (causal wavefield at $\{\boldsymbol{x}, t\}$ due to point source at $\{\boldsymbol{x}, 0\}$)

$Q^{\mathrm{s}}_{I;n}(\boldsymbol{x}', t)$ = contrast volume source density (support $\mathcal{D}^{\mathrm{s}}$)

Scattered wavefield source-type integral representation

Operator notation:

$$\mathsf{L}_{m,n} \; : \; Q^{\mathrm{s}}_{I;n} \xrightarrow{\mathsf{L}_{m,n}} d_{m,n}(t)$$
$$(m = 1, \ldots, M;$$
$$n = 1, \ldots, N)$$

Operator:

$$\mathsf{L}_{m,n}(Q^{\mathrm{s}}_{I;n}) \; = \; \int_{\mathcal{D}^{\Omega}_{m}} Q^{\Omega}_{P;m}(x,t)\mathrm{d}V(x)$$
$$\overset{(t)}{*} \int_{\mathcal{D}^{\mathrm{s}}} G^{\mathrm{b}}_{P,I}(x,x',t)$$
$$\overset{(t)}{*} \; Q^{\mathrm{s}}_{I;n}(x',t)\mathrm{d}V(x')$$
$$(m = 1, \ldots, M;$$
$$n = 1, \ldots, N)$$

Operator notation: contrast source densities $\longrightarrow$ data

Determine $\boldsymbol{Q^{\mathrm{s}}_{I;n}}$ such that

- $\mathsf{L}_{m,n}(Q^{\mathrm{s}}_{I;n}) \simeq d_{m,n}$

  $(m = 1, \ldots, M; n$ **fixed)**

i.e., MINIMIZE the norm of the residual:

- $$\int_{-\infty}^{\infty}\{\sum_{m=1}^{M}\sum_{\mu=1}^{M} g_{m,\mu;n}[d_{m,n} - \mathsf{L}_{m,n}(Q^{\mathrm{s}}_{I;n})] [d_{\mu,n} - \mathsf{L}_{\mu,n}(Q^{\mathrm{s}}_{I;n})]\}\mathrm{d}t$$

  (n fixed)

where

- $\{g_{m,\mu;n}; m = 1, \ldots, M; \mu = 1, \ldots, M; n \text{ fixed}\}$

defines a **positive definite form**

Inverse contrast source problem (fixed excitation)

Determine $\Delta M_{I,P} = M^{\rm s}_{I,P} - M^{\rm b}_{I,P}$ **such that**

- $Q^{\rm s}_{I;n}) \;\simeq\; -\partial_t \Delta M_{I,P} \overset{(t)}{*} (F^{\rm i}_{P;n} + F^{\rm s}_{P;n})$

  $(n = 1, \ldots, N)$

i.e., MINIMIZE the norm of the residual:

- $\int_{-\infty}^{\infty} {\rm d}t \int_{\mathcal{D}^{\rm s}}$
  $\{\sum_{n=1}^{N} \sum_{\nu=1}^{N} \gamma_{n,\nu} [Q^{\rm s}_{I;n} + \partial_t \Delta M_{I,P} \overset{(t)}{*} (F^{\rm i}_{P;n} + F^{\rm s}_{P;n})]$
  $[Q^{\rm s}_{I;\nu} + \partial_t \Delta M_{I,P} \overset{(t)}{*} (F^{\rm i}_{P;\nu} + F^{\rm s}_{P;\nu})]\}{\rm d}V$

where

- $\{\gamma_{n,\nu}; n = 1, \ldots, N; \nu = 1, \ldots, N;\}$

defines a **positive definite form**

Operator equation to be "solved":

- $\mathsf{L}u \simeq q$

Inner product:

- $< v, u >$ with
- $< v, u > = < u, v >^*$
- $< v, \mathsf{L}u > = < \mathsf{L}^{\mathsf{H}} v, u >$

  ($\mathsf{L}^{\mathsf{H}} =$ **Hermitean conjugate** of L)

Associated norm:

- $||u|| = < u, u >^{1/2}$

Residual:

- $r = q - \mathsf{L}u$

Operator equation, inner product, norm, residual

Iterative solution procedure:

- $u^{[n+1]} = u^{[n]} + \delta u^{[n]}$ for $n = 0, 1, 2, \ldots$

Residual after n steps:

- $r^{[n]} = q - \mathsf{L} u^{[n]}$ for $n = 0, 1, 2, \ldots$

$\Longrightarrow r^{[n+1]} = r^{[n]} - \mathsf{L}\delta u^{[n]}$

"Improvement condition":

- $||r^{[n]}||^2 > ||r^{[n+1]}||^2$ for $n = 0, 1, 2, \ldots$

Sufficient condition for improvement:

- $< r^{[n+1]}, \mathsf{L}\delta u^{[n]} > + < \mathsf{L}\delta u^{[n]}, r^{[n+1]} >= 0$

which is satisfied by*

- $\delta u^{[n]} = \alpha^{[n]} \mathsf{L}^{\mathsf{H}} r^{[n]}$ with • $\alpha^{[n]} = \frac{||\mathsf{L}^{\mathsf{H}} r^{[n]}||^2}{||\mathsf{L}\mathsf{L}^{\mathsf{H}} r^{[n]}||^2}$

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*More sophisticated choices for $\delta u^{[n]}$ exist

Iterative solution procedure, "improvement condition"