Impulsive spherical-wave reflection against a planar absorptive and dispersive Dirichlet-to-Neumann boundary – An extension of the modified Cagniard method

by

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Synopsis:

- Introduction
- Description of the configuration
- Formulation of the wave reflection problem
- The complex slowness representation for the wavefield
- Space-time expressions for the wavefield
- Partial-fraction parametrization of the boundary admittance
- Plane-wave admittance parametrization of the boundary admittance
- Discussion of the results: the parameter-steered kernel functions
- Conclusion and Applications

Mathematical ingredients:

- Unilateral time Laplace transformation
- Lerch's uniqueness theorem
- Modified Cagniard type wavefield representation in space-time
- Schouten–Van der Pol theorem (functional change of the time Laplace transform parameter)
- Linear, causal, passive systems theory parametrization of the Dirichlet-to-Neumann boundary admittance
- Construction of the time-domain parameter-steered kernel functions

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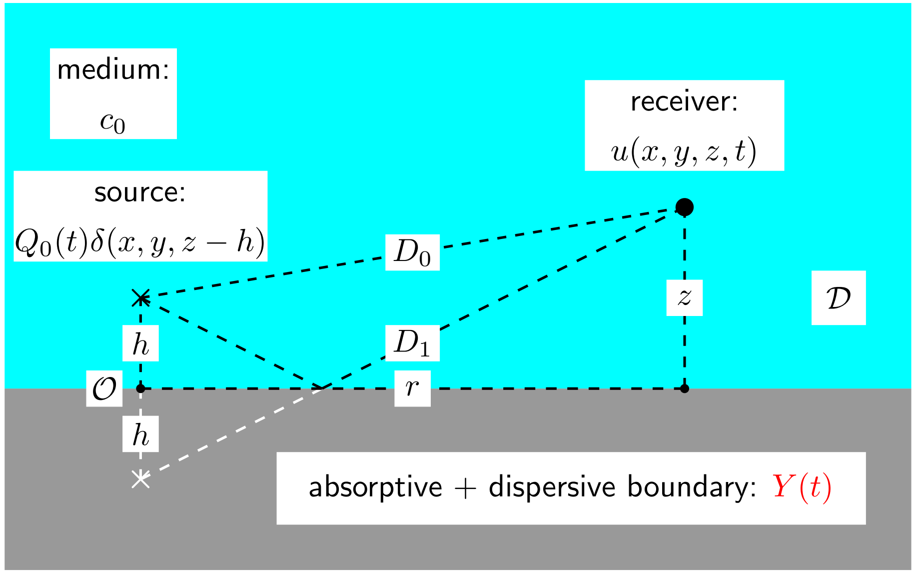
Configuration:



Local Dirichlet-to-Neumann boundary condition:

- $\partial_z u(x,y,0,t) = c_0^{-1} Y(t) \overset{(t)}{*} \partial_t u(x,y,0,t)$ [$\overset{(t)}{*}$ = time convolution]

Wave equation:

- $\nabla^2 u - c_0^{-2} \partial_t^2 u = -Q_0(t)\delta(x,y,z-h)$

Point-source excited spherical-wave reflection configuration

Wavefield quantities and medium parameters:

- $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z =$ position vector
- $u(\boldsymbol{r}, t) =$ wavefunction
- $c_0 =$ wavespeed
- $Q_0(t) =$ point-source signature

Boundary admittance:

- $Y(t) =$ normalized boundary admittance function

Total wavefield, incident wavefield$^{(\mathrm{i})}$, reflected wavefield$^{(\mathrm{r})}$

- $u(\boldsymbol{r}, t) = u^{\mathrm{i}}(\boldsymbol{r}, t) + u^{\mathrm{r}}(\boldsymbol{r}, t)$

Incident field wave equation:

- $\nabla^2 u^{\mathrm{i}} - c_0^{-2}\partial_t^2 u^{\mathrm{i}} = -Q_0(t)\delta(x,y,z-h)$ [$\delta(\boldsymbol{r})$ = 3D Dirac delta distribution]

Incident wavefield:

- $u^{\mathrm{i}} = \partial_t Q_0(t) \overset{(t)}{*} G_0(\boldsymbol{r},t)$ [$\overset{(t)}{*}$ = time convolution]

Space-time scalar Green's function:

- $G_0(\boldsymbol{r},t) = \dfrac{H(t-T_0)}{4\pi D_0}$ for $D_0 > 0$ [$H(\cdot)$ = Heaviside unit step function]

Source/Receiver distance:

- $D_0 = [x^2 + y^2 + (z-h)^2]^{1/2} \geq 0$

Source/Receiver travel time:

- $T_0 = D_0/c_0$

Unilateral time Laplace transformation $[\partial_t \to s]$:

- $$\hat{u}(\boldsymbol{r}, s) = \int_{t=0}^{\infty} \exp(-st)\, u(\boldsymbol{r}, t)\,\mathrm{d}t \quad [s \in \mathbb{R}, s > 0]$$

Complex slowness representation $[\partial_x \to -\mathrm{i}s\alpha, \partial_y \to -\mathrm{i}s\beta]$:

- $$\hat{u}(x, y, z, s) = \frac{s^2}{4\pi^2} \int_{\alpha=-\infty}^{\infty} \mathrm{d}\alpha \int_{\beta=-\infty}^{\infty} \tilde{u}(\alpha, \beta, z, s) \exp[-\mathrm{i}s(\alpha x + \beta y)]\,\mathrm{d}\beta$$

Complex slowness domain incident wavefield:

- $$\tilde{u}^{\mathrm{i}} = \frac{\hat{Q}_0(s)}{2s\gamma_0} \exp(-s\gamma_0 |z - h|) \quad [\gamma_0(\alpha, \beta) = (c_0^{-2} + \alpha^2 + \beta^2)^{1/2},\ \mathrm{Re}(\gamma_0) \geq 0]$$

Complex slowness domain reflected wavefield:

- $$\tilde{u}^{\mathrm{r}} = \frac{\hat{Q}_0(s)}{2s\gamma_0} \tilde{R} \exp[-s\gamma_0(z + h)] \quad [\tilde{R} = \text{reflection coefficient}]$$

Admittance boundary condition $\Longrightarrow$ Reflection coefficient:

- $$\tilde{R} = \frac{c_0\gamma_0 - \hat{Y}(s)}{c_0\gamma_0 + \hat{Y}(s)} = 1 - \frac{2\,\hat{Y}(s)}{c_0\gamma_0 + \hat{Y}(s)}$$

Time Laplace transformed reflected wavefield:

- $$\hat{u}^{\mathrm{r}}(\boldsymbol{r}, s) = s\hat{Q}_0(s)\hat{G}^{\mathrm{r}}(\boldsymbol{r}, s)$$

Time Laplace transformed reflected-wave Green's function:

- $$\hat{G}^{\mathrm{r}}(\boldsymbol{r}, s) = \frac{1}{4\pi^2}\int_{\alpha=-\infty}^{\infty} \mathrm{d}\alpha \int_{\beta=-\infty}^{\infty} \left[1 - \frac{2\,\hat{Y}(s)}{c_0\gamma_0 + \hat{Y}(s)}\right]\frac{1}{2\gamma_0} \exp\{-s[\mathrm{i}(\alpha x + \beta y) + \gamma_0(z+h)]\}\mathrm{d}\beta$$

Time-domain reflected-wave Green's function:

- $$\hat{G}^{\mathrm{r}}(\boldsymbol{r}, s) \stackrel{\text{CAGN}}{\Longrightarrow} G^{\mathrm{r}}(\boldsymbol{r}, t) \qquad \left[\stackrel{\text{CAGN}}{\Longrightarrow} = \text{3D Modified Cagniard method}\right]$$

Time Laplace transformed reflected wavefield

Time Laplace transformed reflected-wave Green's function:

- $$\hat{G}^{\mathrm{r}}(\boldsymbol{r},s)=\frac{1}{4\pi^2}\int_{\alpha=-\infty}^{\infty}\mathrm{d}\alpha\int_{\beta=-\infty}^{\infty}\left[1-\frac{2\,\hat{Y}(s)}{c_0\gamma_0+\hat{Y}(s)}\right]\frac{1}{2\gamma_0}\exp\{-s[\mathrm{i}(\alpha x+\beta y)+\gamma_0(z+h)]\}\mathrm{d}\beta$$

Change variables of integration [$x=r\cos(\theta),y=r\sin(\theta)$] :

- $\alpha=\mathrm{i}p\cos(\theta)-q\sin(\theta)$ • $\beta=\mathrm{i}p\sin(\theta)+q\cos(\theta)$ [$q\in\mathbb{R},\ p\in\mathbb{I}$]
- $\int_{\alpha=-\infty}^{\infty}\mathrm{d}\alpha\int_{\beta=-\infty}^{\infty}(\ldots)\,\mathrm{d}\beta\longmapsto\mathrm{i}^{-1}\int_{q=-\infty}^{\infty}\mathrm{d}q\int_{p=-\mathrm{i}\infty}^{\mathrm{i}\infty}(\ldots)\,\mathrm{d}p$
- $\mathrm{i}(\alpha x+\beta y)+\gamma_0(z+h)\longmapsto pr+\overline{\gamma_0}(q,p)(z+h)$
- $\overline{\gamma_0}(q,p)=[\Omega(q)^2-p^2]^{1/2}$ • $\Omega(q)=(c_0^{-2}+q^2)^{1/2}$

After analytic continuation into $\{p\in\mathbb{C}\}$, deform path of integration from $\mathbb{I}$ to • Modified Cagniard path

Modified Cagniard path (hyperbolic arc in upper half of p-plane) :

- $pr + \overline{\gamma_0}(q,p)(z+h) = \tau$
- $T_1(q) < \tau < \infty$
- $\mathrm{Im}(p) \geq 0$
- $T_1(q) = \Omega(q) D_1$

Image source/receiver distance:

- $D_1 = [x^2 + y^2 + (z+h)^2]^{1/2} \geq 0$

Image source/receiver travel time:

- $T_1(0) = T_1 = D_1/c_0$

Jacobian :

- $\mathrm{d}p = \dfrac{\mathrm{i}\overline{\gamma_0}}{[\tau^2 - T_1^2(q)]^{1/2}} \, \mathrm{d}\tau$

Cauchy's theorem + Jordan's lemma + Schwarz's reflection principle

$$\Longrightarrow$$

- $$\hat{G}^r(\boldsymbol{r}, s) = \frac{1}{2\pi^2} \int_{q=0}^{\infty} \mathrm{d}q \int_{\tau=\Omega(q)}^{\infty} \exp(-s\tau)\, \frac{1}{[\tau^2 - T_1^2(q)]^{1/2}}\, \mathrm{Re}\left[1 - \frac{2\,\hat{Y}(s)}{c_0\overline{\gamma_0} + \hat{Y}(s)}\right] \mathrm{d}\tau$$

Interchange of order of integration :

- $$\int_{q=0}^{\infty} \mathrm{d}q \int_{\tau=\Omega(q)}^{\infty} (\ldots)\, \mathrm{d}\tau \longmapsto \int_{\tau=T_1}^{\infty} \mathrm{d}\tau \int_{q=0}^{(\tau^2/D_1^2-1/c_0^2)^{1/2}} (\ldots)\, \mathrm{d}q$$

Change of variable in q-integration [$q = (\tau^2/D_1^2 - 1/c_0^2)^{1/2} \sin\psi$] :

- $$\int_{q=0}^{(\tau^2/D_1^2-1/c_0^2)^{1/2}} (\ldots)\, \frac{1}{[\tau^2 - T_1^2(q)]^{1/2}}\, \mathrm{d}q \longmapsto \frac{1}{D_1} \int_{\psi=0}^{\pi/2} (\ldots)\, \mathrm{d}\psi$$

Modified Cagniard representation of time Laplace transformed reflected wave Green's function :

- $\hat{G}^r(\boldsymbol{r},s) = \dfrac{1}{2\pi^2 D_1}\displaystyle\int_{\tau=T_1}^{\infty} \exp(-s\tau)\mathrm{d}\tau \int_{\psi=0}^{\pi/2} \mathrm{Re}\left[1 - \dfrac{2\,\hat{Y}(s)}{c_0\overline{\gamma_0} + \hat{Y}(s)}\right]\mathrm{d}\psi$
- $c_0\overline{\gamma_0} = \Gamma_1(\boldsymbol{r},\tau) - \mathrm{i}\Gamma_2(\boldsymbol{r},\tau)\cos(\psi)$
- $\dfrac{2}{\pi}\displaystyle\int_{\psi=0}^{\pi/2} \mathrm{Re}\left[\dfrac{1}{c_0\overline{\gamma_0} + \hat{Y}(s)}\right]\mathrm{d}\psi = \dfrac{1}{\{[\Gamma_1(\boldsymbol{r},\tau) + \hat{Y}(s)]^2 + \Gamma_2^2(\boldsymbol{r},\tau)\}^{1/2}}$

Configurational/medium quantities [$\hat{Y}(s)$ = reflector property]:

- $\Gamma_1(\boldsymbol{r},\tau) = \dfrac{c_0\tau(z+h)}{D_1^2}$
- $\Gamma_2(\boldsymbol{r},\tau) = \dfrac{c_0(\tau^2 - T_1^2)^{1/2} r}{D_1^2}$

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Time Laplace transformed reflected-wave Green's function:

- $$\hat{G}^{\mathrm{r}}(\boldsymbol{r}, s) = \frac{1}{4\pi D_1} \int_{\tau=T_1}^{\infty} \exp(-s\tau) \hat{K}^{\mathrm{r}}(\boldsymbol{r}, \tau, s) \, \mathrm{d}\tau$$

Time Laplace transformed reflected-wave kernel function:

- $$\hat{K}^{\mathrm{r}}(\boldsymbol{r}, \tau, s) = 1 - \frac{-2\,\hat{Y}(s)}{\{[\Gamma_1(\boldsymbol{r}, \tau) + \hat{Y}(s)]^2 + \Gamma_2^2(\boldsymbol{r}, \tau)]\}^{1/2}}$$

Time-domain reflected-wave kernel function:

- $$\hat{K}^{\mathrm{r}}(\boldsymbol{r}, \tau, s) \overset{\text{CAGN}^+}{\Longrightarrow} K^{\mathrm{r}}(\boldsymbol{r}, \tau, t) \qquad \left[\overset{\text{CAGN}^+}{\Longrightarrow} = \textbf{Extended Cagniard method} \right]$$

Time-domain reflected-wave Green's function:

- $$G^{\mathrm{r}}(\boldsymbol{r}, t) = \left[\frac{1}{4\pi D_1} \int_{\tau=T_1}^{t} K^{\mathrm{r}}(\boldsymbol{r}, \tau, t-\tau) \, \mathrm{d}\tau \right] H(t - T_1)$$

Time-domain reflected-wave Green's function

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Remaining issue:

- $\hat{K}^{\mathrm{d}}(\boldsymbol{r},\tau,s) \overset{\text{CAGN}^{+}}{\Longrightarrow} K^{\mathrm{d}}(\boldsymbol{r},\tau,t)$

Ingredients CAGNIARD^{+} method:

- **LERCH's** uniqueness theorem of the unilateral Laplace transformation
- **Modified CAGNIARD's** path of integration in the complex slowness plane
- The **SCHOUTEN–VAN DER POL** theorem (1934) of the unilateral Laplace transformation [Replacement of transform parameter s by some function $\phi(s)$ within a specific class of functions]
- Adequate **Table of Laplace transform pairs** (e.g., Abramowitz, M. & Stegun, I. E., *Handbook of Mathematical Functions*, Chapter 29)

Abramowitz & Stegun, Formula 29.3.55 [$s \to \Gamma_1(\boldsymbol{r},\boldsymbol{\tau}) + \hat{Y}(s)$] $\Longrightarrow$:

- $$\frac{\hat{Y}(s)}{\{[\Gamma_1(\boldsymbol{r},\tau)+\hat{Y}(s)]^2+\Gamma_2^2(\boldsymbol{r},\tau)]\}^{1/2}} = 1 - \int_{w=0}^{\infty} K_M(\boldsymbol{r},\tau,w)\hat{K}_Y(w,s)\,\mathrm{d}w$$

Kernel functions:

- $$K_M(\boldsymbol{r},\tau,w) = \{-\partial_w\,[\exp[-\Gamma_1(\boldsymbol{r},\tau)w]J_0[\Gamma_2(\boldsymbol{r},\tau)w]]\}\,H(\tau-T_1)H(w)$$

[medium/configuration related]

- $\hat{K}_Y(w,s) = \exp[-\hat{Y}(s)w]$ **[boundary properties related]**

$\Longrightarrow$ **Reflected-wave s-domain kernel function:**

- $$\hat{K}^{\mathrm{r}}(\boldsymbol{r},\tau,s) = -1 + 2\int_{w=0}^{\infty} K_M(\boldsymbol{r},\tau,w)\hat{K}_Y(w,s)\,\mathrm{d}w$$

$\Longrightarrow$ **Reflected-wave time-domain kernel function:**

- $$K^{\mathrm{r}}(\boldsymbol{r},\tau,t) = -\delta(t) + 2\left[\int_{w=0}^{\infty} K_M(\boldsymbol{r},\tau,w)K_Y(w,t)\,\mathrm{d}w\right]H(t)$$

Remaining issue:

- $\hat{K}_Y(w,s) = \exp[-\hat{Y}(s)w] \stackrel{\text{CAGN}^+}{\Longrightarrow} K_Y(w,t)$

$\Longrightarrow$ **via parametrization of $\hat{Y}(s)$:**

- **Partial-fraction parametrization of $\hat{Y}(s)$**
- **Plane-wave admittance parametrization of $\hat{Y}(s)$**

N.B. Partial-fraction and/or Plane-wave admittance parametrization offer an alternative to (standard) Padé $\{n,n\}$ parametrizations $(n = 1, 2, \ldots)$

- $\hat{K}_Y(w,s) = \exp[-\hat{Y}(s)w]$

Partial fraction parametrization of $\hat{Y}(s) \Longrightarrow$:

- $\hat{Y}(s) = \sum_{n=0}^{N} \hat{Y}^{(n)}(s)$
- $\hat{Y}^{(0)}(s) = Y^{\infty}$
- $\hat{Y}^{(n)}(s) = \dfrac{A_n}{s+\alpha_n} \quad (n = 1, \ldots, N)$

s-domain admittance kernel function

- $\hat{K}_Y(w,s) = \prod_{n=0}^{N} \hat{K}_Y^{(n)}(w,s)$
- $\hat{K}_Y^{(0)}(w,s) = \exp(-Y^{\infty}w)$
- $\hat{K}_Y^{(n)}(w,s) = \exp[-Y^{(n)}(s)w] \quad (n = 1, \ldots, N)$

s-domain admittance kernel function

- $\hat{K}_Y(w,s) = \hat{K}_Y^{(0)}(w,s) \cdot \hat{K}_Y^{(1)}(w,t) \cdots \hat{K}_Y^{(N)}(w,t)$

Time-domain admittance kernel function

- $K_Y(w,t) = K_Y^{(0)}(w,t) \overset{(t)}{*} K_Y^{(1)}(w,t) \overset{(t)}{*} \cdots \overset{(t)}{*} K_Y^{(N)}(w,t)$

- $K_Y^{(0)}(w,t) = \exp(-Y^\infty w)H(w)\delta(t)$

SCHOUTEN–VAN DER POL THEOREM +

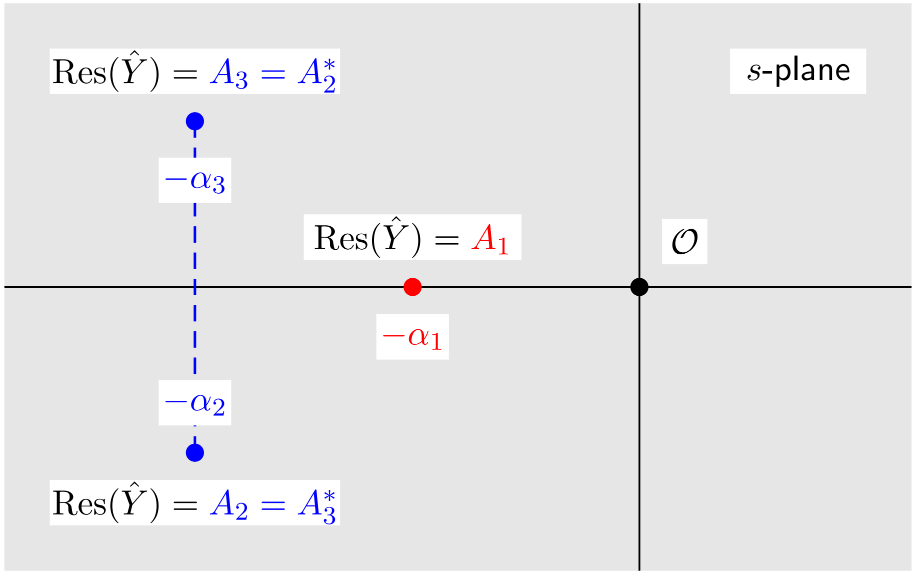
Abramowitz & Stegun, Formula 29.3.75 $[s \rightarrow (s+\alpha_n)^{-1}] \Longrightarrow$:

- $K_Y^{(n)}(w,t) = H(w)\delta(t) - \left[\exp(-\alpha_n t)(A_n w/t)^{1/2} J_1[2(A_n w t)^{1/2}]\right] H(w)H(t)$

  for $n = 1, \ldots, N$

Partial-fraction parametrization of boundary admittance (2)

Location of poles in $\hat{Y}(s)$ $[= Y^{\infty} + \sum_{n=1}^{N} A_n(s+\alpha_n)^{-1}]$:



$Y(t)$:
- linear
- time invariant
- passive
- causal

Conditions on pole location and residue values:

- **Pole on real axis:** $\mathrm{Re}(\alpha_1 > 0),\ A_1 > 0$
- **Two complex conjugate poles:** $\mathrm{Re}(\alpha_{2,3} > 0),\ \mathrm{Re}(A_{2,3}) > 0,$

$$\mathrm{Re}(A_2)\mathrm{Re}(\alpha_2) + \mathrm{Im}(A_2)\mathrm{Im}(\alpha_2) > 0$$

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- $\hat{K}_W(w,s) = \exp[-\hat{Y}_W(s)w]$

Plane-wave admittance parametrization of $\hat{Y}_W(s) \Longrightarrow$:

- $\hat{Y}_W(s) = Y_1^\infty[\hat{X}(s)]^{1/2}$ • $Y_1^\infty = \dfrac{c_0}{c_1}$ • $\hat{X}(s) = \dfrac{s + \hat{\alpha}_1(s)}{s + \hat{\beta}_1(s)}$

c_1 = wavespeed instantaneous response

$\alpha_1(t)$ = compliance relaxation function

$\beta_1(t)$ = inertia relaxation function

SCHOUTEN–VAN DER POL theorem +
Abramowitz and Stegun, Formula (29.3.82) $[s \to s^{1/2}] \Longrightarrow$:

- $$\hat{K}_W(w,s) = \int_{u=0}^{\infty} \exp[-\hat{X}(s)u]\Upsilon(w,u)\,\mathrm{d}u$$
- $$\Upsilon(w,u) = \frac{Y_1^{\infty} w}{(4\pi u^3)^{1/2}} \exp\left[-\frac{(Y_1^{\infty} w)^2}{4u}\right] H(w)H(u)$$

Final steps:

- Partial-fraction parametrization of $\hat{\alpha}_1(s)$, $\hat{\beta}_1(s)$ and $\hat{X}(s)$
- Transformation back to the time domain

Plane-wave admittance parametrization of boundary admittance (2)

Parametrization of boundary absorption and dispersion properties in a time-domain point-source/point-receiver reflection configuration has been achieved through:

- **Partial-fraction** parametrization of the boundary's time Laplace-transformed local admittance function $\hat{Y}(s)$
- **Plane-wave admittance** parametrization of the boundary's time Laplace-transformed local admittance function $\hat{Y}(s)$

incorporating the following properties of the boundary's time-domain local admittance function $Y(t)$:

- Linearity
- Time invariance
- Passivity
- Causality

Applications are found in :

- Outdoor sound propagation
  (influence of natural ground surface, traffic noise analysis)
- Sound absorbing layers in aircraft jet engines
- Architectural acoustics
  (scattering and (re)distribution of sound in concert halls)

Lit. De Hoop, A.T., Lam, C.H., Kooij, B.J., "Parametrization of acoustic boundary absorption and dispersion properties in time-domain source/receiver reflection measurement", The Journal of the Acoustical Society of America, Vol. **118**, 2005, pp.