Electromagnetic Field Equations in Canonical Form for Space-Time Integrated Field Computation

by

Adrianus T. de Hoop

Delft University of Technology
Laboratory of Electromagnetic Research
Faculty of Electrical Engineering, Mathematics and Computer Science
Mekelweg 4 • 2628 CD Delft • the Netherlands
T: +31 15 2785203 / +31 15 2786620
F: +31 15 2786194
E: a.t.dehoop@tudelft.nl

Laboratory of Electromagnetic Research

Faculty of Electrical Engineering, Mathematics and Computer Science

Delft University of Technology

Mekelweg 4

2628 CD Delft, the Netherlands

T: +31 15 2786620 • **F:** +31 15 2786194 • **W:** www.emlab.ewi.tudelft.nl

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Synopsis:

- Space-time configuration
- EM field quantities in (Cartesian) tensor notation (EINSTEIN)
- Maxwell fields as a thermodynamic system. Area density of EM power flow. Volume density of EM momentum
- Maxwell fields in vacuum (pure radiation fields)
- Maxwell fields in matter. Constitutive relations
- Maxwell fields in 2D configurations
- Construction of space-time integrated field equations
- Construction of a class of benchmark problems

- **Space-Time: • 4D: $\mathbb{R}^3 \times \mathbb{R}$**
- **Space:**
  - $\mathbb{R}^3$: 3D Euclidean space
  - $x_m \in \mathbb{R}^3$: position vector
  - ∂_m : partial differentiation with respect to x_m $(m = 1, 2, 3)$
  - $\delta_{m,n}$: symmetrical unit tensor of rank 2 (Kronecker tensor)
  - $\varepsilon_{i,j,k}$: completely antisymmetric unit tensor of rank 3 (Levi-Cività tensor)
- **Time:**
  - $t \in \mathbb{R}$: time
  - ∂_t : partial differentiation with respect to t

- **EM field quantities (tensor functions on $\mathbb{R}^3 \times \mathbb{R}$):**
- **EM field strengths:**
  - E_r : electric field strength (V/m)
  - H_p : magnetic field strength (A/m)
- **EM flux densities:**
  - D_k : electric flux density (C/m^2)
  - B_j : magnetic flux density (T)

- **Local EM field equations (differentiable fields):**
  - $\varepsilon_{k,m,p}\partial_m H_p = \partial_t D_k$ for $x_m \in \mathbb{R}^3, t \in \mathbb{R}$ **(Maxwell 1)**
  - $\varepsilon_{j,n,r}\partial_n E_r = -\partial_t B_j$ for $x_m \in \mathbb{R}^3, t \in \mathbb{R}$ **(Maxwell 2)**
- **Local EM compatibility relations (differentiable fields) (existence conditions for solutions to Maxwell 1 & Maxwell 2):**
  - $\partial_k D_k = 0$ for $x_m \in \mathbb{R}^3, t \in \mathbb{R}$ $\Longleftarrow$ **(Maxwell 1)**
  - $\partial_j B_j = 0$ for $x_m \in \mathbb{R}^3, t \in \mathbb{R}$ $\Longleftarrow$ **(Maxwell 2)**

The local EM field equations and compatibility relations (differentiable field quantities)

- **Intrinsic field quantities / Area density of EM power flow:**
  - E_r : electric field strength (V/m)
  - H_p : magnetic field strength (A/m)
  - $S_m = \varepsilon_{m,r,p} E_r H_p$ **(area density of EM power flow (W/m^2))**
- **Extrinsic field quantities / Volume density of EM momentum:**
  - D_k : electric flux density (C/m^2)
  - B_j : magnetic flux density (T)
  - $G_n = \varepsilon_{n,k,j} D_k B_j$ **(volume density of EM momentum (W·s/m^4))**
- **Maxwell's equations** $\Longrightarrow$ **time rate of change in extrinsic field quantities is balanced by spatial rates of change in intrinsic field quantities**

- **Local EM constitutive relations in vacuum (SI):**
  - $D_k(x_m,t) = \epsilon_0 \delta_{k,r} E_r(x_m,t)$
  - $B_j(x_m,t) = \mu_0 \delta_{j,p} H_p(x_m,t)$
  - $c_0 = 299792458$ m/s (EM wavespeed in vacuum)
  - $\mu_0 = 4\pi * 10^{-7}$ H/m (permeability of vacuum)
  - $\epsilon_0 = 1/\mu_0 c_0^2$ F/m (permittivity of vacuum)

Constitutive relations for time-invariant, linearly, locally and causally reacting media ($\overset{(t)}{*}$ = time convolution):

- $D_k(x_m,t) = \int_{t'=0}^{\infty} \eta_{k,r}(x_m,t') E_r(x_m,t-t')\mathrm{d}t' + P_k^{\mathrm{ext}}(x_m,t)$

  $= \eta_{k,r}(x_m,t) \overset{(t)}{*} E_r(x_m,t) + P_k^{\mathrm{ext}}(x_m,t)$
- $\eta_{k,r}$ = (electric) permittivity relaxation tensor (F/m*s)
- P_k^{ext} = (field-independent) external electric polarization (C/m^2)
- $B_j(x_m,t) = \int_{t'=0}^{\infty} \zeta_{j,p}(x_m,t') H_p(x_m,t-t')\mathrm{d}t' + M_j^{\mathrm{ext}}(x_m,t)$

  $= \zeta_{j,p}(x_m,t) \overset{(t)}{*} H_p(x_m,t) + M_j^{\mathrm{ext}}(x_m,t)$
- $\zeta_{j,p}$ = (magnetic) permeability relaxation tensor (H/m*s)
- M_j^{ext} = (field-independent) external magnetization (T)

Constitutive relations for time-invariant, linear, locally and causally reacting media

Constitutive relations for simple media:

- $\eta_{k,r}(x_m,t) = \epsilon_{k,r}(x_m)\delta(t) + \sigma_{k,r}(x_m), H(t)$
  - $\epsilon_{k,r} =$ (electric) permittivity tensor (F/m)
  - $\sigma_{k,r} =$ (electric) conductivity tensor (S/m)
- $\zeta_{j,p}(x_m,t) = \mu_{j,p}(x_m)\delta(t) + \kappa_{j,p}(x_m)H(t)$
  - $\mu_{j,p} =$ (magnetic) permeability tensor (H/m)
  - $\kappa_{j,p} =$ linear magnetic hysteresis tensor (H/m·s)

- $\delta(t) =$ Dirac delta distribution
- $H(t) =$ Heaviside unit step function

2D configurations:

- x_μ : position coordinates in transverse plane $(\mu = 1, 2)$
- x_3 : position coordinate in axial direction

  $\partial_3 = 0$

Decoupling of transverse constitutive properties from axial ones:

- $\eta_{\alpha,3}(x_\mu, t) = 0$
- $\eta_{3,\beta}(x_\mu, t) = 0$
- $\zeta_{\gamma,3}(x_\mu, t) = 0$
- $\zeta_{3,\delta}(x_\mu, t) = 0$

Transverse Magnetic (TM-)fields:

- $\{B_3, H_3, M_3\} = 0$ • $\{D_\alpha, E_\beta, P_\alpha\} = 0$

Transverse Magnetic (TM-)field equations:

- $\varepsilon_{3,\mu,\delta}\partial_\mu H_\delta - \partial_t D_3 = 0$
- $\partial_\nu E_3 + \varepsilon_{\nu,3,\gamma}\partial_t B_\gamma = 0$

Transverse Magnetic (TM-)field compatibility relation:

- $\partial_\gamma B_\gamma = 0$

Transverse Magnetic (TM-)field constitutive relations:

- $D_3(x_\mu, t) = \eta_{3,3}(x_\mu, t) \overset{(t)}{*} E_3(x_\mu, t) + P_3^{\text{ext}}(x_\mu, t)$
- $B_\gamma(x_\mu, t) = \zeta_{\gamma,\delta}(x_\mu, t) \overset{(t)}{*} H_\delta(x_\mu, t) + M_\gamma^{\text{ext}}(x_\mu, t)$

Transverse Electric (TE-)fields:

- $\{D_3, E_3, P_3\} = 0$ • $\{B_\gamma, H_\delta, M_\gamma\} = 0$

Transverse Electric (TE-)field equations:

- $\partial_\mu H_3 - \varepsilon_{\mu,3,\alpha}\partial_t D_\alpha = 0$
- $\varepsilon_{3,\nu,\beta}\partial_\nu E_\beta + \partial_t B_3 = 0$

Transverse Electric (TE-)field compatibility relation:

- $\partial_\alpha D_\alpha = 0$

Transverse Electric (TE-)field constitutive relations:

- $D_\alpha(x_\mu, t) = \eta_{\alpha,\beta}(x_\mu, t) \overset{(t)}{*} E_\beta(x_\mu, t) + P_\alpha^{\text{ext}}(x_\mu, t)$
- $B_3(x_\mu, t) = \zeta_{3,3}(x_\mu, t) \overset{(t)}{*} H_3(x_\mu, t) + M_3^{\text{ext}}(x_\mu, t)$

Construction of space-time integrated EM field equations

$\Longrightarrow$ Eliminate ∂_t, ∂_m (3D), ∂_μ (2D) by means of:

- $\int_{\mathcal{T}} \partial_t D_k(x_m,t)\mathrm{d}t = D_k(x_m,t)\Big|_{\partial\mathcal{T}}$ **(3D)** • $\partial\mathcal{T}$ = boundary of $\mathcal{T}$
- $\int_{\mathcal{S}} \nu_k \varepsilon_{k,m,p}\partial_m H_p(x_m,t)\mathrm{d}A = \oint_{\partial\mathcal{S}} \tau_p H_p(x_m,t)\mathrm{d}\lambda$ **(3D)** • $\partial\mathcal{S}$ = boundary of $\mathcal{S}$
- $\int_{\mathcal{T}} \partial_t D_\alpha(x_\mu,t)\mathrm{d}t = D_\alpha(x_\mu,t)\Big|_{\partial\mathcal{T}}$ **(2D)** • $\partial\mathcal{T}$ = boundary of $\mathcal{T}$
- $\int_{\mathcal{S}} \varepsilon_{3,\mu,\delta}\partial_\mu H_\delta(x_\mu,t)\mathrm{d}A = \oint_{\partial\mathcal{S}} \tau_\delta H_\delta(x_m,t)\mathrm{d}\lambda$ **(2D)** • $\partial\mathcal{S}$ = boundary of $\mathcal{S}$
- $\int_{\mathcal{L}} \partial_\nu E_3(x_\mu,t)\mathrm{d}\lambda = E_3(x_\mu,t)\Big|_{\partial\mathcal{L}}$ **(2D)** • $\partial\mathcal{L}$ = boundary of $\mathcal{L}$

Construction of a class of benchmark problems:

- **Start from an analytically known EM field in the domain $\mathcal{D} \subset \mathbb{R}^3$ with boundary $\partial\mathcal{D}$**

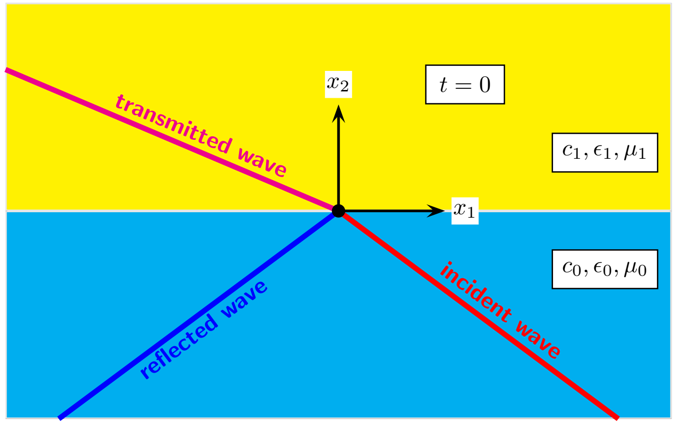
  - $\varepsilon_{k,m,p}\partial_m H_p^* = \partial_t D_k^*$
  - $D_k^* = \eta_{k,r}^* \overset{(t)}{*} E_r^* + P_k^{\text{ext}*}$
  - $\varepsilon_{j,n,r}\partial_n E_r^* = -\partial_t B_j^*$
  - $B_j^* = \zeta_{j,p}^* \overset{(t)}{*} H_p^* + M_j^{\text{ext}*}$

- **Rewrite as a contrast-source problem in the actual medium with known boundary values of either $\boldsymbol{\varepsilon_{k,m,p}\nu_m H_p}$ or $\boldsymbol{\varepsilon_{j,n,r}\nu_n E_r}$**

  - $\varepsilon_{k,m,p}\partial_m H_p = \partial_t D_k$
  - $D_k = \boldsymbol{\eta_{k,r}} \overset{(t)}{*} E_r + (\boldsymbol{\eta_{k,r}^*} - \boldsymbol{\eta_{k,r}}) \overset{(t)}{*} E_r^* + P_k^{\text{ext}*}$
  - $\varepsilon_{j,n,r}\partial_n E_r = -\partial_t B_j$
  - $B_j = \boldsymbol{\zeta_{j,p}} \overset{(t)}{*} H_p + (\boldsymbol{\zeta_{j,p}^*} - \boldsymbol{\zeta_{j,p}}) \overset{(t)}{*} H_p^* + M_j^{\text{ext}*}$

- **Evidently: computed $\{\boldsymbol{E_r, H_p, D_k, B_j}\}$ = known $\{\boldsymbol{E_r^*, H_p^*, D_k^*, B_j^*}\}$**

Pulsed EM wave reflection/transmission as a benchmark problem:



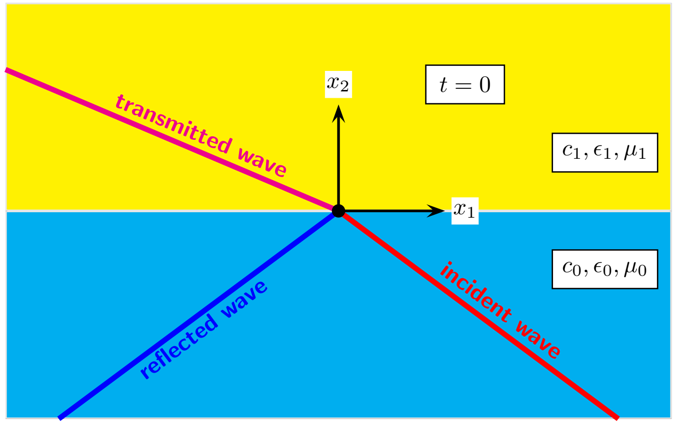
- **$^{\rm i}$: incident wave**
- **$^{\rm r}$: reflected wave**
- **$^{\rm t}$: transmitted wave**

- $\{H_1^{\rm i}, H_2^{\rm i}, E_3^{\rm i}\} = \{\alpha_2 Y_0, -\alpha_1 Y_0, 1\} E_0(t - \alpha_1 x_1/c_0 - \alpha_2 x_2/c_0)$
- $\{H_1^{\rm r}, H_2^{\rm r}, E_3^{\rm r}\} = \{-\alpha_2 Y_0, -\alpha_1 Y_0, 1\}\, R_E\, E_0(t - \alpha_1 x_1/c_0 + \alpha_2 x_2/c_0)$
- $\{H_1^{\rm t}, H_2^{\rm t}, E_3^{\rm t}\} = \{\beta_2 Y_1, -\beta_1 Y_1, 1\}\, T_E\, E_0(t - \beta_1 x_1/c_1 - \beta_2 x_2/c_1)$

- $Y_0 = \epsilon_0 c_0$ • $Y_1 = \epsilon_1 c_1$ • $\alpha_2 = (1-\alpha_1^2)^{1/2}$ • $\alpha_1/c_0 = \beta_1/c_1$ • $\beta_2 = (1-\beta_1^2)^{1/2}$
- $\boldsymbol{c_1 < c_0}$ • $R_E = \dfrac{\alpha_1 Y_0 - \beta_1 Y_1}{\alpha_1 Y_0 + \beta_1 Y_1}$ • $T_E = \dfrac{2\alpha_1 Y_0}{\alpha_1 Y_0 + \beta_1 Y_1}$

Plane EM pulse reflection/transmission at planar interface (TM-case)

Pulsed EM wave reflection/transmission as a benchmark problem:



$^{\mathrm{i}}$: incident wave

$^{\mathrm{r}}$: reflected wave

$^{\mathrm{t}}$: transmitted wave

- $\{E_1^{\mathrm{i}}, E_2^{\mathrm{i}}, H_3^{\mathrm{i}}\} = \{\alpha_2 Z_0, -\alpha_1 Z_0, 1\} H_0(t - \alpha_1 x_1/c_0 - \alpha_2 x_2/c_0)$
- $\{E_1^{\mathrm{r}}, E_2^{\mathrm{r}}, H_3^{\mathrm{r}}\} = \{-\alpha_2 Z_0, -\alpha_1 Z_0, 1\}\, R_H \, H_0(t - \alpha_1 x_1/c_0 + \alpha_2 x_2/c_0)$
- $\{E_1^{\mathrm{t}}, E_2^{\mathrm{t}}, H_3^{\mathrm{t}}\} = \{\beta_2 Z_1, -\beta_1 Z_1, 1\}\, T_H \, H_0(t - \beta_1 x_1/c_1 - \beta_2 x_2/c_1)$

- $Z_0 = \mu_0 c_0$ • $Z_1 = \mu_1 c_1$ • $\alpha_2 = (1 - \alpha_1^2)^{1/2}$ • $\alpha_1/c_0 = \beta_1/c_1$ • $\beta_2 = (1 - \beta_1^2)^{1/2}$
- $\boldsymbol{c_1 < c_0}$ • $R_H = \dfrac{\alpha_1 Z_0 - \beta_1 Z_1}{\alpha_1 Z_0 + \beta_1 Z_1}$ • $T_H = \dfrac{2\alpha_1 Z_0}{\alpha_1 Z_0 + \beta_1 Z_1}$

Plane EM pulse reflection/transmission at planar interface (TE-case)

Pulsed axial electric line current excited TM-fields:

- **Field equations**
  (electric source current time signature $I_0(t) = 0$ for $t < 0$):
  - $\partial_1 H_2 - \partial_2 H_1 - \epsilon_0 \partial_t E_3 = \boldsymbol{I_0(t)} \delta(x_1 - X_1, x_2 - X_2)$
  - $\partial_2 E_3 + \mu_0 \partial_t H_1 = 0$
  - $-\partial_1 E_3 + \mu_0 \partial_t H_2 = 0$
- **2D wave equation for E_3:**
  - $\partial_1^2 E_3 + \partial_2^2 E_3 - \epsilon_0 \mu_0 \partial_t^2 E_3 = \mu_0 \delta(x_1 - X_1, x_2 - X_2) \partial_t I_0(t)$

- **Green's function 2D scalar wave equation:**

  - $\partial_1^2 \boldsymbol{G} + \partial_2^2 \boldsymbol{G} - c_0^{-2}\partial_t^2 \boldsymbol{G} = -\delta(x_1 - X_1, x_2 - X_2)\boldsymbol{\delta(t)}$

- $\boldsymbol{G(D,t)} = \dfrac{1}{2\pi}\dfrac{H(t - D/c_0)}{(t^2 - D^2/c_0^2)^{1/2}}$ • $\boldsymbol{D} = \left[(x_1 - X_1)^2 + (x_2 - X_2)^2\right]^{1/2}$
- $\boldsymbol{\delta(t)} =$ **Dirac delta function** • $H(t) =$ Heaviside unit step function

- **TM-field components:**

  - $$E_3(x_1, x_2, t) = -\mu_0 \partial_t \int_{\tau = D/c_0}^{t} G(\boldsymbol{x}, \boldsymbol{X}, \tau) I_0(t - \tau)\,\mathrm{d}\tau$$
    $$H_1(x_1, x_2, t) = -\mu_0^{-1} \partial_2 \int_{\tau=0}^{t} E_3(x_1, x_2, \tau)\,\mathrm{d}\tau$$
    $$H_2(x_1, x_2, t) = \mu_0^{-1} \partial_1 \int_{\tau=0}^{t} E_3(x_1, x_2, \tau)\,\mathrm{d}\tau$$

Pulsed axial electric line current excited TM fields – A benchmark problem (contd.)

Electric-current source signature (Rectangular pulse):

- $\boldsymbol{I_0(t)} = A\ H(t) - A\ H(t - t_{\rm w})$

- $t_{\rm w}$: pulse time width A : amplitude $H(t)$: Heaviside unit step function

Electric-current source signature (Trapezoidal pulse):

- $$\boldsymbol{I_0(t)} = \frac{A}{t_{\rm r}}R(t) - \frac{A}{t_{\rm r}}R(t - t_{\rm r}) - \frac{A}{t_{\rm f}}R(t - t_{\rm r}/2 + t_{\rm w} + t_{\rm f}/2) + \frac{A}{t_{\rm f}}R(t - t_{\rm r}/2 - t_{\rm w} - t_{\rm f}/2)$$

-

$t_{\rm r}$: pulse rise time	A : amplitude
$t_{\rm w}$: pulse time width	$H(t)$: Heaviside unit step function
$t_{\rm f}$: pulse fall time	$R(t)$: unit ramp function

Electric current source signatures

Step function source signature :

- $\boldsymbol{I_0(t)} = A\ H(t)$

TM-field components

- $E_3 = -\dfrac{\mu_0\ A}{2\pi}\ \dfrac{1}{(t^2 - D^2/c_0^2)^{1/2}}\ H(t - D/c_0)$
- $H_1 = -\dfrac{A}{2\pi}\ \dfrac{t}{(t^2 - D^2/c_0^2)^{1/2}}\ \dfrac{x_2 - X_2}{D}\ H(t - D/c_0)$
- $H_2 = \dfrac{A}{2\pi}\ \dfrac{t}{(t^2 - D^2/c_0^2)^{1/2}}\ \dfrac{x_1 - X_1}{D}\ H(t - D/c_0)$

TM-field components (step function source signature)

Ramp function source signature :

- $\boldsymbol{I_0(t)} = \dfrac{A}{t_\mathrm{r}}\, R(t)$

TM-field components

- $E_3 = -\dfrac{\mu_0\ A}{2\pi t_\mathrm{r}}\ \log\left\{\dfrac{c_0 t}{D} + \left[\left(\dfrac{c_0 t}{D}\right)^2 - 1\right]^{1/2}\right\}\ H(t - D/c_0)$
- $H_1 = -\dfrac{A}{2\pi t_\mathrm{r}}\ (t^2 - D^2/c_0^2)^{1/2}\ \dfrac{x_2 - X_2}{D}\ H(t - D/c_0)$
- $H_2 = \dfrac{A}{2\pi t_\mathrm{r}}\ (t^2 - D^2/c_0^2)^{1/2}\ \dfrac{x_1 - X_1}{D}\ H(t - D/c_0)$

TM-field components (ramp function source signature)

Spatially periodic boundary conditions on $\{(x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < L_1, 0 < x_2 < L_2\}$:

- $E_3(x_1 + L_1, x_2, t) = E_3(x_1, x_2, t)$ for $0 < x_2 < L_2$

&

- $E_3(x_1, x_2 + L_2, t) = E_3(x_1, x_2, t)$ for $0 < x_1 < L_1$

OR

- $H_2(x_1 + L_1, x_2, t) = H_2(x_1, x_2, t)$ for $0 < x_2 < L_2$

&

- $H_1(x_1, x_2 + L_2, t) = H_1(x_1, x_2, t)$ for $0 < x_1 < L_1$

$\Longrightarrow$ • **Check on LATE-TIME BEHAVIOR on spatial mesh of moderate size**

TM-field components (late time behavior)

The space-time EM field integrated method with circulation and flux integrals (all four electric and magnetic field quantities tensors of rank 1 (vectors) in $\mathbb{R}^3$):

- leads after discretization to equal numbers of relations between 'edge' ($\boldsymbol{E}, \boldsymbol{H}$) and 'face' ($\boldsymbol{D}, \boldsymbol{B}$) unknowns (G. De Rham's cohomology , W. V. D. Hodge's theory in topology)
- leads to a square system of linear equations upon supplementing Maxwell's equations with any kind of linear constitutive relations of the type 'edge' ($\boldsymbol{E}$) $\leftrightarrow$ 'face' ($\boldsymbol{D}$) and 'edge' ($\boldsymbol{H}$) $\leftrightarrow$ 'face' ($\boldsymbol{B}$)
- automatically satisfies the compatibility relations on 'face' ($\boldsymbol{D}$) and 'face' ($\boldsymbol{B}$)
- handles strongly heterogeneous media

The constitutive relations in the space-time EM field integrated method with circulation and flux integrals can be implemented by:

- employing simplicial discretization in $\mathbb{R}^3$
- linearly interpolating the integrands (= application of the trapezoidal integration rule (= definition of Riemann integration of a piecewise continuous function))
- assigning constant values of the constitutive coefficients in the simplices bounded by the 'faces'

The space-time EM field integrated method with circulation and flux integrals (all four electric and magnetic field quantities tensors of rank 1 (vectors) in $\mathbb{R}^3$):

- is restricted to a space with Riemannian metric $\Longrightarrow$ does not fit in the Lorentzian metric of 4D relativistic electrodynamics
- does not incorporate the magnetic field as an antisymmetric tensor of rank 2 (as needed in 4D relativistic electrodynamics)

BUT • $\Longrightarrow$ consistently handles field values and power flow densities at wavefronts and interfaces (which standard FDTD methods don't)