Performance analysis of 1D space-time integrated EM field equations computation, with time-domain coordinate-stretched PML termination and periodic boundary conditions

by

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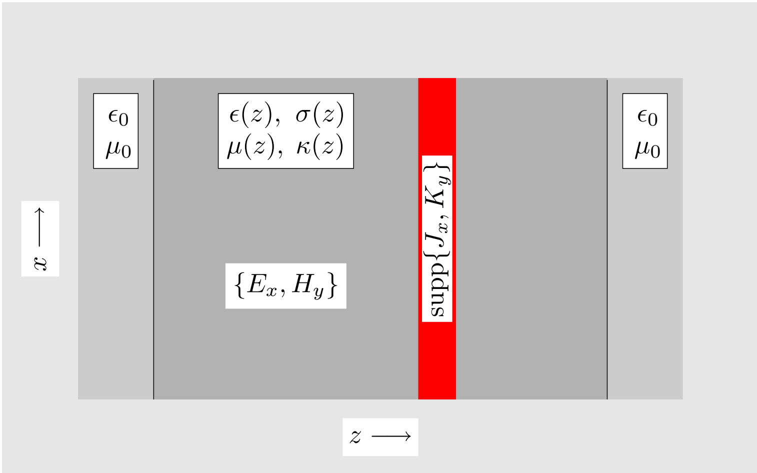
Synopsis:

- Aim and scope
- Description of the configuration (1-D inhomogeneous in space, invariant in time)
- EM field equations, constitutive relations, interface boundary conditions
- Uniqueness of the solution (causality and the time Laplace transformation)
- The 1-D time-domain coordinate stretched PM-embedding
- Truncation of the embedding by periodic boundary conditions
- The space-time discretization procedure
- An analytic benchmark tool
- Performance analysis of the Matlab7.1$^{\mathrm{TM}}$ program

1-D configuration inhomogeneous in space, invariant in time, in homogeneous unbounded embedding:



- $\{x, y, z\}$ = Cartesian position coordinates
- t = time coordinate

$\{E_x(z,t), H_y(z,t)\}$ = EM field quantities

$\{J_x(z,t), K_y(z,t)\}$ = EM source quantities

$\{\epsilon(z), \sigma(z), \mu(z), \kappa(z)\}$ = EM constitutive coefficients inhomogeneous slab

$\{\epsilon_0, \mu_0\}$ = EM constitutive coefficients homogeneous embedding

Electromagnetic field quantities:

- $E_x(z,t) =$ electric field strength (V/m)
- $H_y(z,t) =$ magnetic field strength (A/m)

Electromagnetic source quantities:

- $J_x(z,t) =$ volume source density of electric current (A/m^2)
- $K_y(z,t) =$ volume source density of magnetic current (V/m^2)

Electromagnetic constitutive coefficients:

- $\epsilon(z), \epsilon_0 =$ electric permittivity (F/m)
- $\sigma(z) =$ electric conductivity (S/m)
- $\mu(z), \mu_0 =$ magnetic permeability (H/m)
- $\kappa(z) =$ linear magnetic loss coefficient (Ω/m)

EM field quantities, source quantities and constitutive coefficients

- **Functional uniqueness requirements in space-time ($z \in \mathcal{Z}, t \in \mathcal{T}$):**

$$\mathcal{Z}\times\mathcal{T}\longrightarrow\{E_x(z,t),H_y(z,t)\} = \text{piecewise continuously differentiable}$$

$$\mathcal{Z}\times\mathcal{T}\longrightarrow\{J_x(z,t),K_y(z,t)\} = \text{piecewise continuous}$$

- **Constitutive requirements in space ($z \in \mathcal{Z}$):**

$$\mathcal{Z}\longrightarrow\{\epsilon(z),\sigma(z),\mu(z),\kappa(z)\} = \text{piecewise continuous}$$

$$\epsilon(z) > 0 \ , \ \sigma(z) \geq 0$$

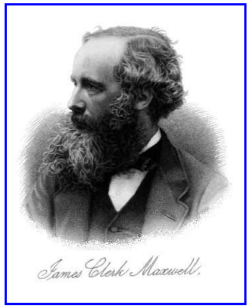
$$\mu(z) > 0 \ , \ \kappa(z) \geq 0$$

$$\epsilon_0 > 0 \ , \ \mu_0 > 0$$

- **CAUSALITY requirements in time ($z \in \mathcal{Z}$):**

If $\{J_x(z,t),K_y(z,t)\} = 0$ for $t < t_0$ then $\{E_x(z,t),H_y(z,t)\} = 0$ for $t < t_0$

EM field equations:



J. C. Maxwell
(1831 – 1879)

- $\partial_z H_y + (\sigma + \epsilon\, \partial_t) E_x = -J_x$ (inhomogeneous slab)
- $\partial_z E_x + (\kappa + \mu\, \partial_t) H_y = -K_y$ (inhomogeneous slab)
- $\partial_z H_y + \epsilon_0\, \partial_t E_x = 0$ (homogeneous embedding)
- $\partial_z E_x + \mu_0\, \partial_t H_y = 0$ (homogeneous embedding)

Interface boundary conditions:

- $E_x(z,t)\big|_{-}^{+} = 0$
- $H_y(z,t)\big|_{-}^{+} = 0$

Time Laplace transformation:

- $\{\hat{E}_x, \hat{H}_y, \hat{J}_x, \hat{K}_y\}(z,s) = \int_{t=0}^{\infty} \exp(-st)\,\{E_x, H_y, J_x, K_y\}(z,t)\,\mathrm{d}t$ for $s \in \mathcal{L}$,
  - $\mathcal{L} = \{s \in \mathbb{R};\, s = s_0 + nh,\, s_0 > 0,\, h > 0,\, n = 0, 1, 2, \ldots\}$

    (LERCH sequence)
  - $\hat{\partial}_t = s$

Basis for: $\Longrightarrow$

- UNIQUENESS (for general causal, dispersive media) (DeHoop, 2003)
- Construction of time-domain coordinate-stretched

  PERFECTLY MATCHED EMBEDDINGS (DeHoop *et al.*, 2007)

Contrast-source, coordinate-stretched EM field equations (Time Laplace-transform domain):

- $\hat{\chi}_z(z,s)$ = time Laplace-transformed z-coordinate stretching function
  - $\hat{\chi}_z(z,s) > 0$ for $s \in \mathcal{L}$, $z \in$ domain of computation
  - $\hat{\chi}_z(z,s) = 1$ for $z \in$ domain of interest

- $\dfrac{1}{\hat{\chi}_z(z,s)}\partial_z \hat{H}_y^{\chi} + s\epsilon_0 \hat{E}_x^{\chi} = -\hat{J}_x - \hat{J}_x^{\mathrm{s};\chi}$ for $z \in \mathbb{R}$
- $\dfrac{1}{\hat{\chi}_z(z,s)}\partial_z \hat{E}_x^{\chi} + s\mu_0 \hat{H}_y^{\chi} = -\hat{K}_y - \hat{K}_y^{\mathrm{s};\chi}$ for $z \in \mathbb{R}$
- $\hat{J}_x^{\mathrm{s};\chi} = (\sigma + s\epsilon - s\epsilon_0)\hat{E}_x^{\chi}$ (volume source density of contrast electric current)
- $\hat{K}_y^{\mathrm{s};\chi} = (\kappa + s\mu - s\mu_0)\hat{H}_y^{\chi}$ (volume source density of contrast magnetic current)

Contrast-source EM field representations (Time Laplace-transform domain):

- $$\hat{E}_x^{\chi}(z,s) = -s\mu_0 \int_{z'\in\mathbb{R}} \hat{G}^{\chi}(z,z',s)\,[\hat{J}_x(z',s) + \hat{J}_x^{s;\chi}(z',s)]\mathrm{d}z' + \frac{1}{\hat{\chi}_z(z,s)}\partial_z \int_{z'\in\mathbb{R}} \hat{G}^{\chi}(z,z',s)\,[\hat{K}_y(z',s) + \hat{K}_y^{s;\chi}(z',s)]\mathrm{d}z' \text{ for } z\in\mathbb{R}$$
- $$\hat{H}_y^{\chi}(z,s) = \frac{1}{\hat{\chi}_z(z,s)}\partial_z \int_{z'\in\mathbb{R}} \hat{G}^{\chi}(z,z',s)\,[\hat{J}_x(z',s) + \hat{J}_x^{s;\chi}(z',s)]\mathrm{d}z' - s\epsilon_0 \int_{z'\in\mathbb{R}} \hat{G}^{\chi}(z,z',s)\,[\hat{K}_y(z',s) + \hat{K}_y^{s;\chi}(z',s)]\mathrm{d}z' \text{ for } z\in\mathbb{R}$$
- $\hat{G}^{\chi}(z,z',s)$ = 1D coordinate-stretched Green's function of the Helmholtz equation (Time Laplace-transform domain)

Contrast-source EM field representations (Time Laplace-transform domain)

Green's function 1D coordinate-stretched Helmholtz equation (Time Laplace-transform domain):

- $$\frac{1}{\hat{\chi}_z(z,s)}\partial_z\left[\frac{1}{\hat{\chi}_z(z,s)}\partial_z \hat{G}^{\chi}(z,z',s)\right] - (s/c_0)^2 \hat{G}^{\chi}(z,z',s) = -\delta(z-z')$$
  for $z \in \mathbb{R}, z' \in \mathbb{R}$
- $$\hat{G}^{\chi}(z,z',s) = \frac{c_0}{2s}\exp\left[-(s/c_0)\left|\int_{\zeta=z'}^{z}\hat{\chi}_z(\zeta,s)\,\mathrm{d}\zeta\right|\right]$$ for $z \in \mathbb{R}, z' \in \mathbb{R}$

$\Longrightarrow$ • $\hat{G}^{\chi}(z,z',s) = \hat{G}(z,z',s)$ **for** $(z,z') \in$ **domain of interest**

$\Longrightarrow$ • $\{\hat{E}^{\chi}_x, \hat{H}^{\chi}_y\}(z,s) = \{\hat{E}_x, \hat{H}_y\}(z,s)$ **for** $z \in$ **domain of interest**

COORDINATE-STRETCHED EMBEDDING IS PERFECTLY MATCHED

Absorptive and time-delaying coordinate-stretching profile:

- $\hat{\chi}_z(z,s) = 1 + N_z(z) + \dfrac{\boldsymbol{\alpha}_z(z)}{s}$
- $N_z(z) = 0$, $\boldsymbol{\alpha}_z(z) = 0$ for $z \in$ domain of interest
- $N_z(z) \geq 0$, $\boldsymbol{\alpha}_z(z) > 0$ for $z \in$ perfectly matched embedding

Time Laplace transform Green's function:

- $\hat{G}^{\chi} = \dfrac{c_0}{2s}\exp[-\boldsymbol{\Gamma}_z(z,z') - sT_z(z,z')]$
- $\boldsymbol{\Gamma}_z(z,z') = \dfrac{1}{c_0}\left|\displaystyle\int_{\zeta=z'}^{z}\boldsymbol{\alpha}_z(\zeta)\mathrm{d}\zeta\right|$
- $T_z(z,z') = \dfrac{1}{c_0}\left|\displaystyle\int_{\zeta=z'}^{z}[1+N_z(\zeta)]\mathrm{d}\zeta\right|$

Time-domain Green's function:

- $G^{\chi}(z,z',t) = \dfrac{c_0}{2}\exp[-\boldsymbol{\Gamma}_z(z,z')]H[t - T_z(z,z')]$
- $\{\boldsymbol{\Gamma}_z(z,z'), T_z(z,z')\} = \{$**excess absorption**, excess time delay$\}$

Absorptive and time-delaying coordinate-stretching profile

Bounded domain of computation

(domain of interest + truncated perfectly matched embedding):

- $\overline{\mathcal{Z}} = \{z \in \mathbb{R}; \boldsymbol{z}_{\min} < z < \boldsymbol{z}_{\max}\}$

Periodic boundary condition:

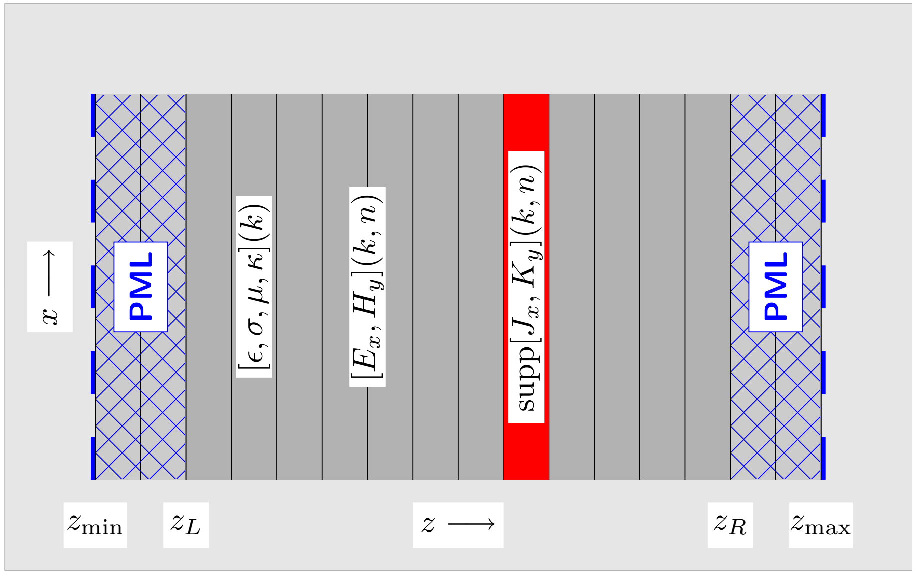
- $\{E_x, H_y\}(\boldsymbol{z}_{\min}, t) = \{E_x, H_y\}(\boldsymbol{z}_{\max}, t)$ for all $t \in \mathbb{R}$

Spurious truncation reverberations suppressed by attenuation in absorptive 'paddings' (truncated perfectly matched embedding):

- $\boldsymbol{\Gamma}_{\text{period}} = \dfrac{1}{c_0} \displaystyle\int_{\zeta=\boldsymbol{z}_{\min}}^{\boldsymbol{z}_{\max}} \boldsymbol{\alpha}_z(\boldsymbol{\zeta}) \mathrm{d}\zeta$

Discretized domain of computation

(domain of interest + truncated perfectly matched embedding):



Constitutive parameters $\Longrightarrow$ piecewise constant:

- $\{\sigma(z), \epsilon(z), \kappa(z), \mu(z)\} = \{\sigma_k, \epsilon_k, \kappa_k, \mu_k\}$ for $z \in \mathcal{Z}_k, \;\; k = 1, \ldots, N_{\mathcal{Z}}$

Discretized domain of computation

Spatial discretization of domain of computation:

- $\overline{\mathcal{Z}} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{N_{\mathcal{Z}}}$
- $\mathcal{Z}_k = \{z \in \mathbb{R}; z_{k-1} < z < z_k\}$

Temporal discretization of time window of interest:

- $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{N_{\mathcal{T}}}$
- $\mathcal{T}_k = \{t \in \mathbb{R}; t_{n-1} < t < t_n\}$

Piecewise linear discretized field and source representations:

- $E_x^{\Delta}(k,n) = E_x(z_k, t_n), \cdots$
- $$E_x(z,t) \simeq \left[E_x^{\Delta}(k-1,n-1)\frac{z_k - z}{z_k - z_{k-1}} + E_x^{\Delta}(k,n-1)\frac{z - z_{k-1}}{z_k - z_{k-1}}\right]\frac{t_n - t}{t_n - t_{n-1}} + \left[E_x^{\Delta}(k-1,n)\frac{z_k - z}{z_k - z_{k-1}} + E_x^{\Delta}(k,n)\frac{z - z_{k-1}}{z_k - z_{k-1}}\right]\frac{t - t_{n-1}}{t_n - t_{n-1}}, \cdots$$

Space-time integrated discretized field equations:

- Apply operation

  - $$\int_{\mathcal{T}_n} \mathrm{d}t \int_{\mathcal{Z}_k} \cdots \mathrm{d}z$$

  to EM field equations in domain of computation
- Substitute local field and source expansions and carry out the integrations
- Invoke periodic boundary conditions (in space)
- Invoke initial conditions (in time)
- Solve system of linear algebraic equations in expansion coefficients

Benchmark problem: homogeneous medium (known analytical solution):

- $\partial_z H_y + \epsilon_0\, \partial_t E_x = -J_x$ for $z \in \mathbb{R}$
- $\partial_z E_x + \mu_0\, \partial_t H_y = -K_y$ for $z \in \mathbb{R}$
- $J_x(z,t) = A_0^J[H(z - z_L^J) - H(z - z_R^J)][H(t) - H(t - t_\mathrm{w}^J)]$
- $K_y(z,t) = A_0^K[H(z - z_L^K) - H(z - z_R^K)][H(t) - H(t - t_\mathrm{w}^K)]$

One-sided field excitation (homogeneous medium):

- $K_y(z,t) = \mu_0 c_0 J_x(z,t)$

  $\Longrightarrow$ one-sided field propagating in the direction of increasing z

- $K_y(z,t) = -\mu_0 c_0 J_x(z,t)$

  $\Longrightarrow$ one-sided field propagating in the direction of decreasing z

Parametrization of excess absorptive profile:

- $$\alpha_z(z) = \begin{cases} 0 & \text{for } z < z_{\min}, \\ \alpha^L_{z,\min} + (\alpha^L_{z,\max} - \alpha^L_{z,\min}) \left(\dfrac{z_L - z}{z_L - z_{\min}}\right)^{p_L} & \text{for } z_{\min} \le z < z_L, \\ 0 & \text{for } z_L \le z < z_R, \\ \alpha^R_{z,\min} + (\alpha^R_{z,\max} - \alpha^R_{z,\min}) \left(\dfrac{z - z_R}{z_{\max} - z_R}\right)^{p_R} & \text{for } z_R \le z < z_{\max}, \\ 0 & \text{for } z_{\max} \le z. \end{cases}$$

Perfectly-matched layer attenuation per spatial period:

- $$\Gamma_{\text{per}} = \frac{1}{c_0}\left\{\left[\frac{1}{p^L+1}\alpha^L_{z,\max} + \frac{p^L}{p^L+1}\alpha^L_{z,\min}\right](z_L - z_{\min}) + \left[\frac{p^R}{p^R+1}\alpha^R_{z,\min} + \frac{1}{p^R+1}\alpha^R_{z,\max}\right](z_{\max} - z_R)\right\}.$$

Illustrative numerical results (space-time source distributions):

- $J_x(z,t) = A_0^J[H(z - z_L^J) - H(z - z_R^J)][H(t) - H(t - t_{\rm w}^J)]$
- $K_y(z,t) = A_0^K[H(z - z_L^K) - H(z - z_R^K)][H(t) - H(t - t_{\rm w}^K)]$

Illustrative numerical results (step sizes in space and time):

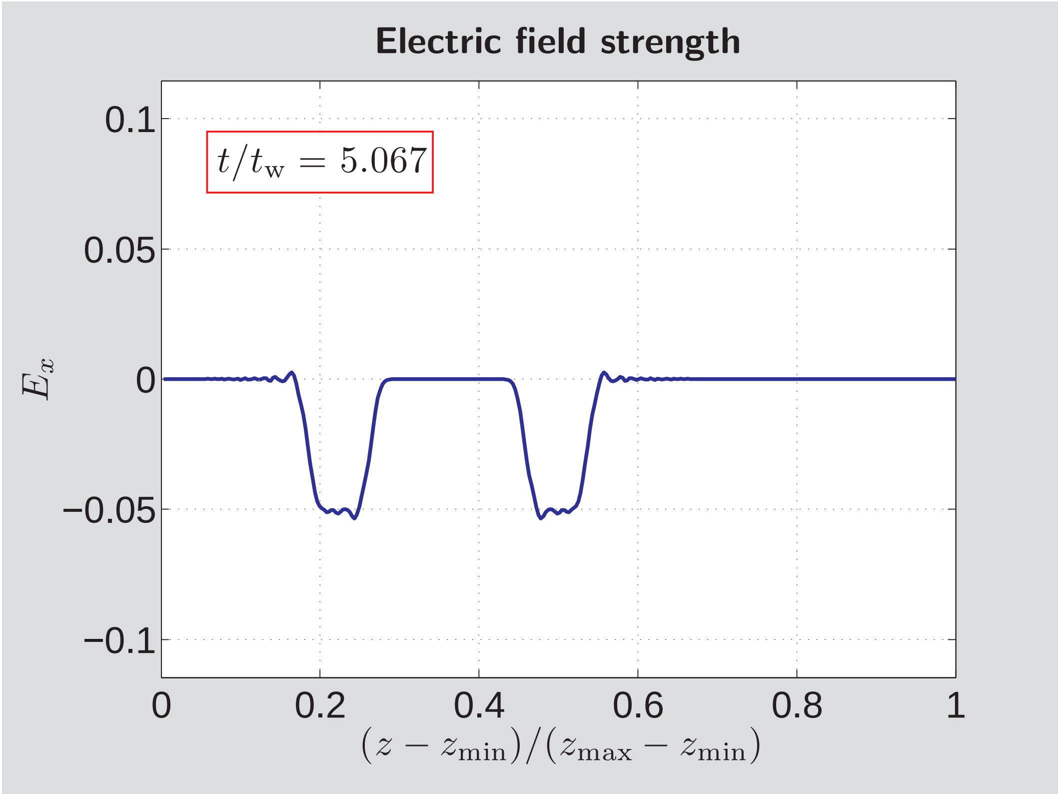
- $\Delta z = c_0 t_{\rm width}/10$

  $=$ spatial extent of rectangular pulse $/10$
- $\Delta t = t_{\rm width}/15$

  $=$ temporal extent of rectangular pulse $/15$

Comparison with analytic benchmark:

- $A_0^J = 1$
- $A_0^K = 0$



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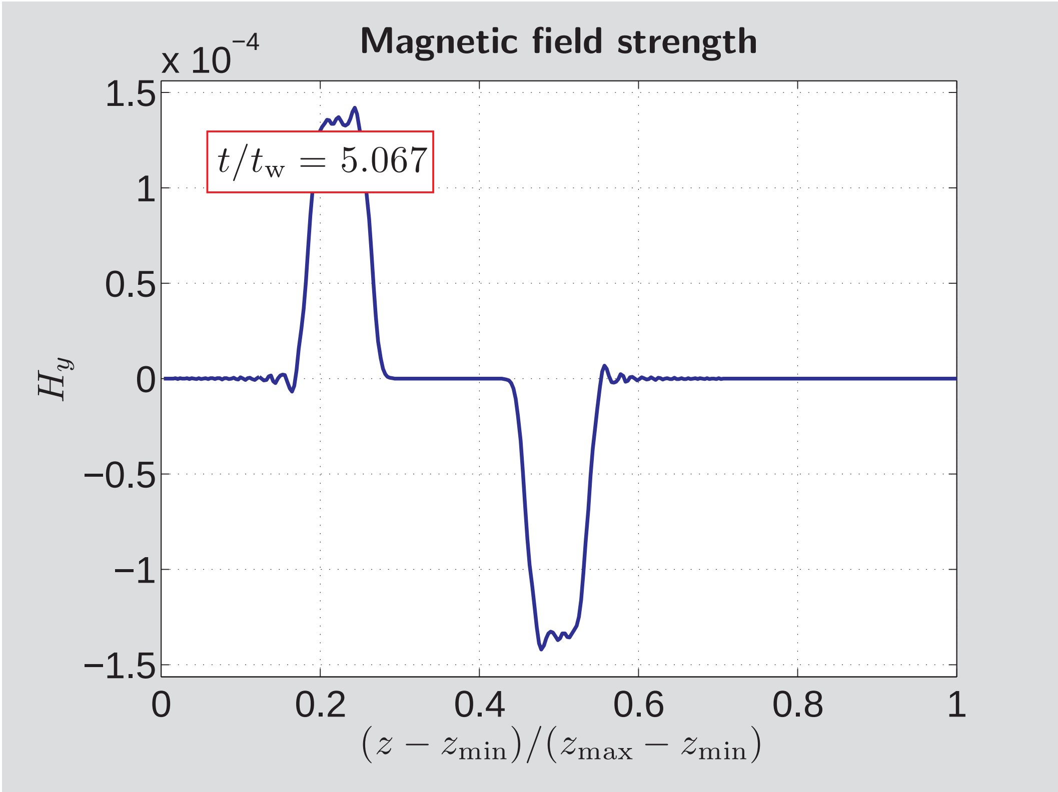

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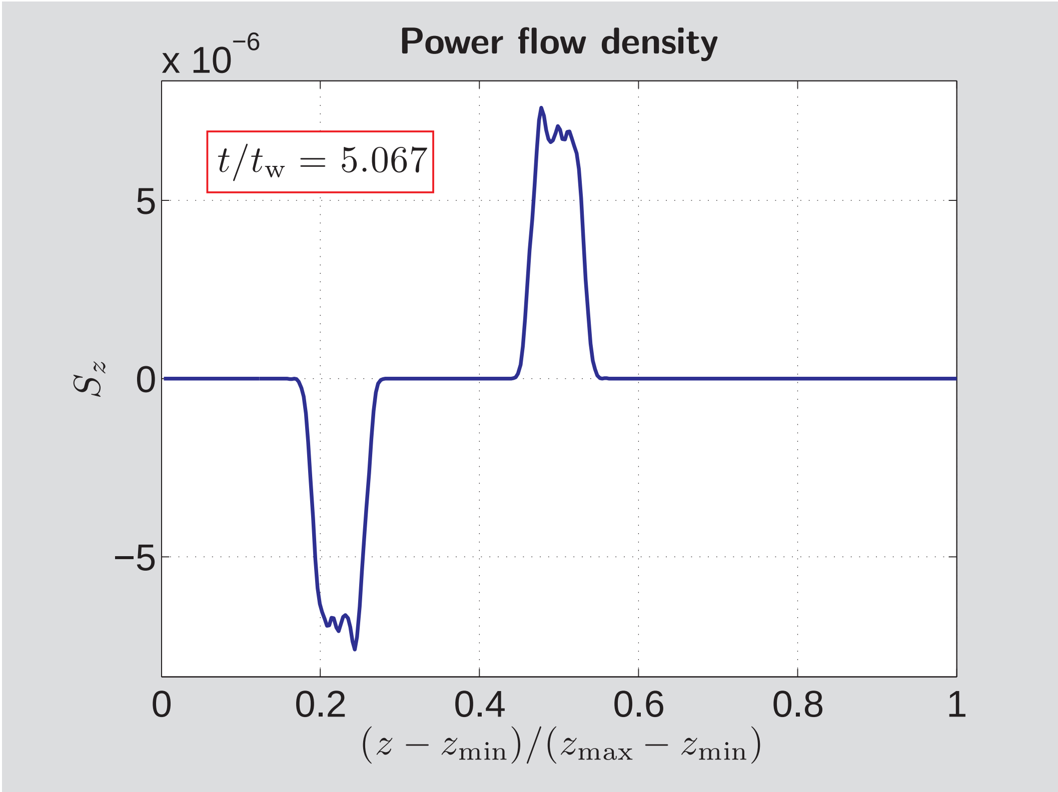

Illustrative numerical results

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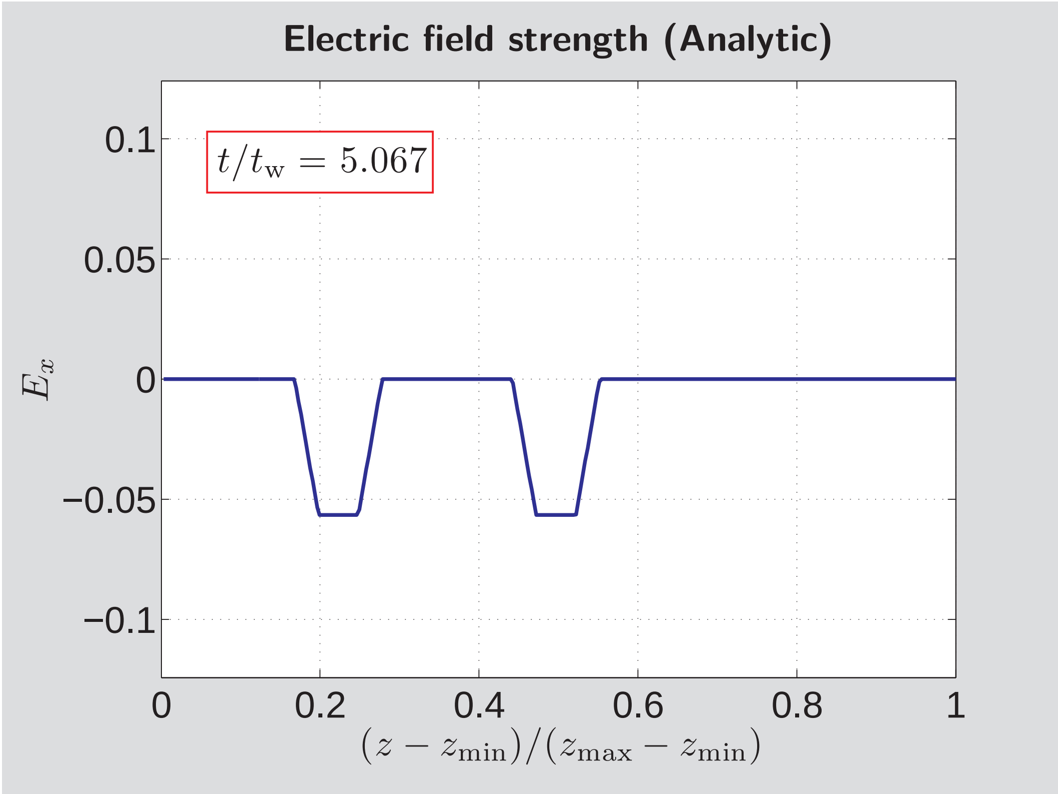

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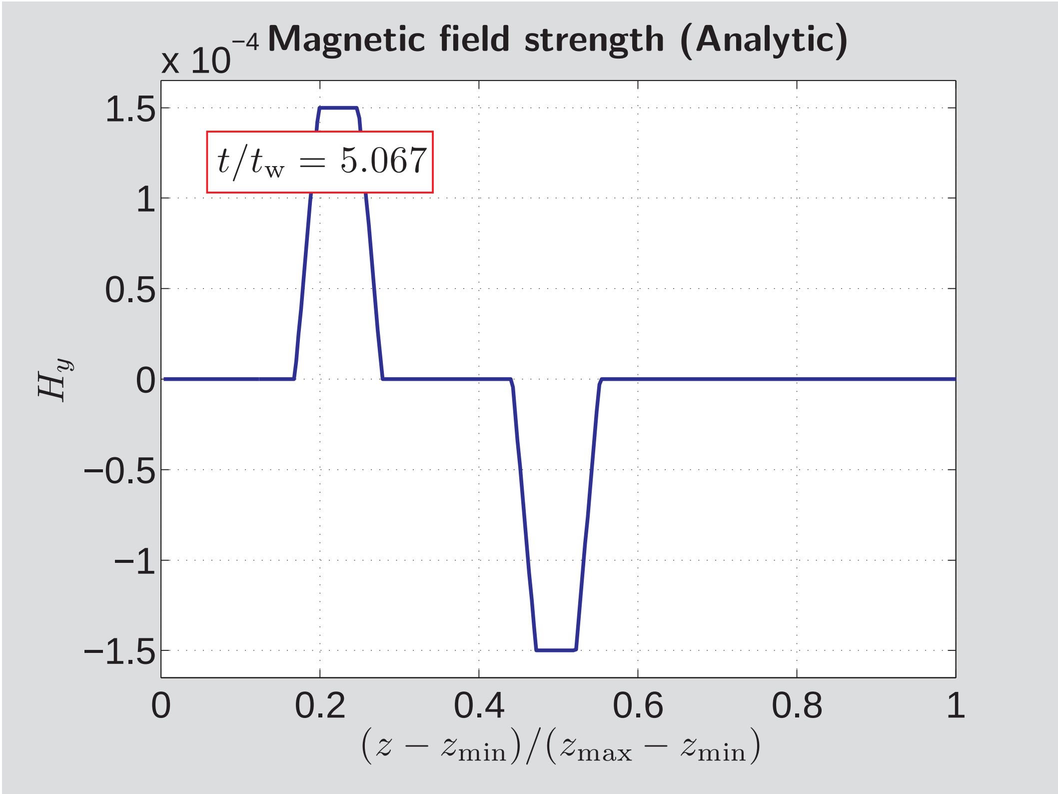

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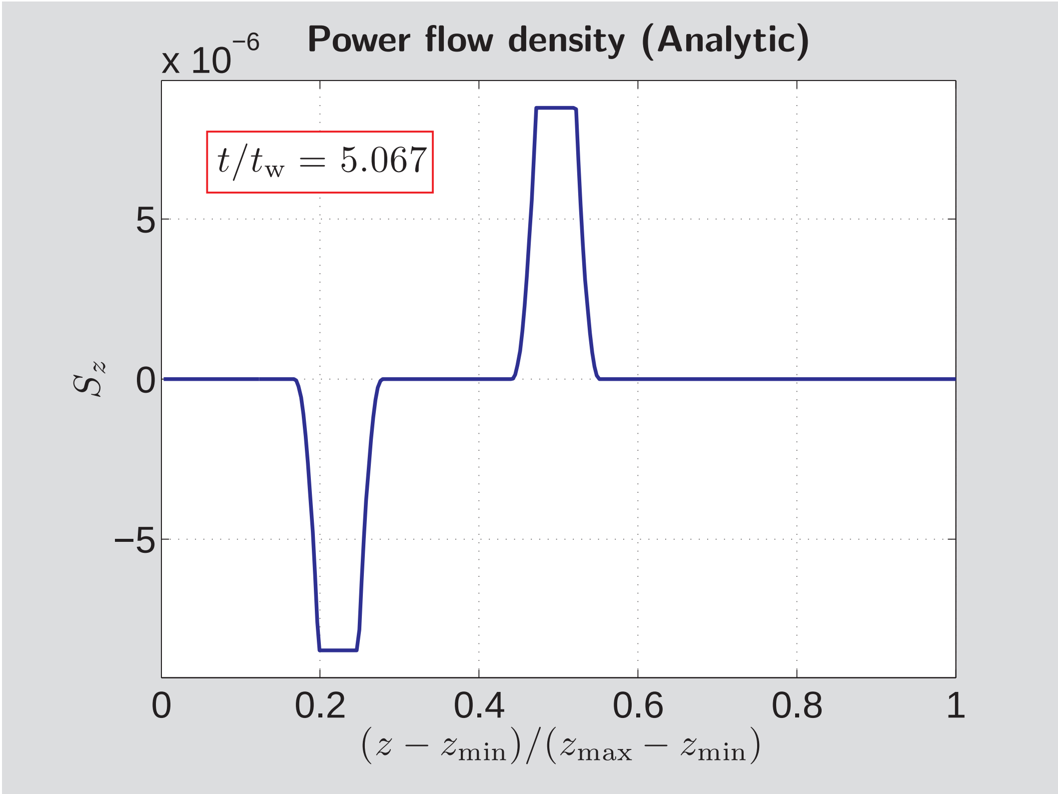
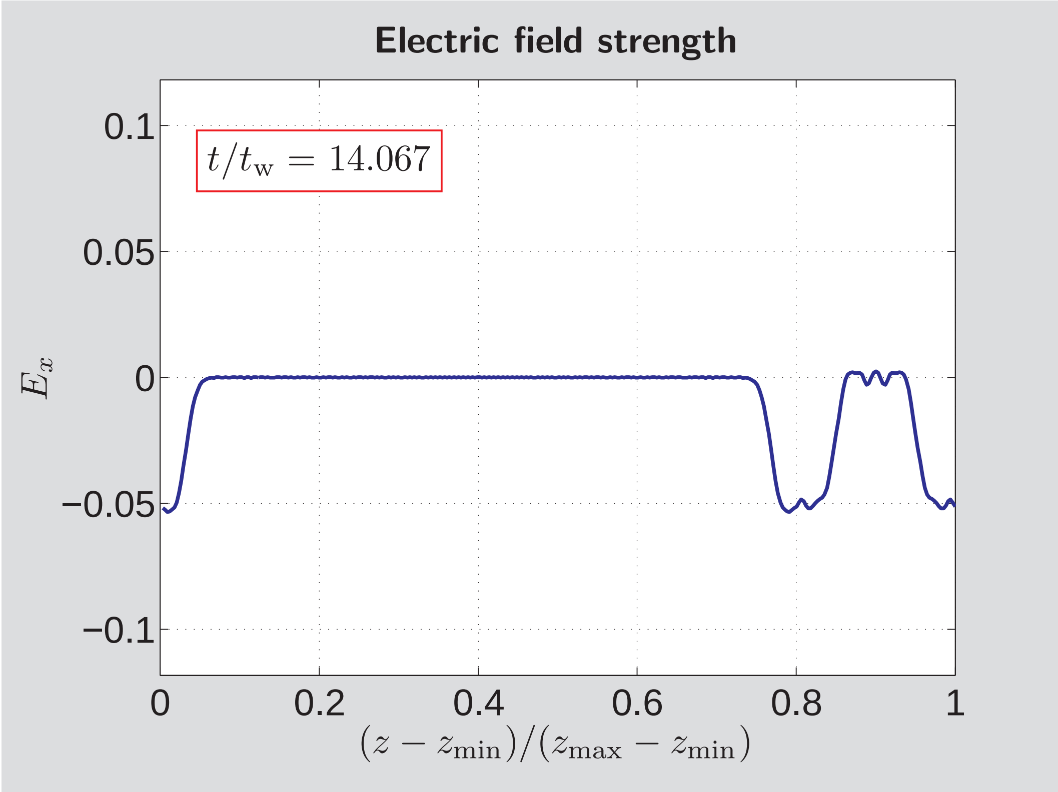


The periodic boundary condition:

- $A_0^J = 1$
- $A_0^K = 0$



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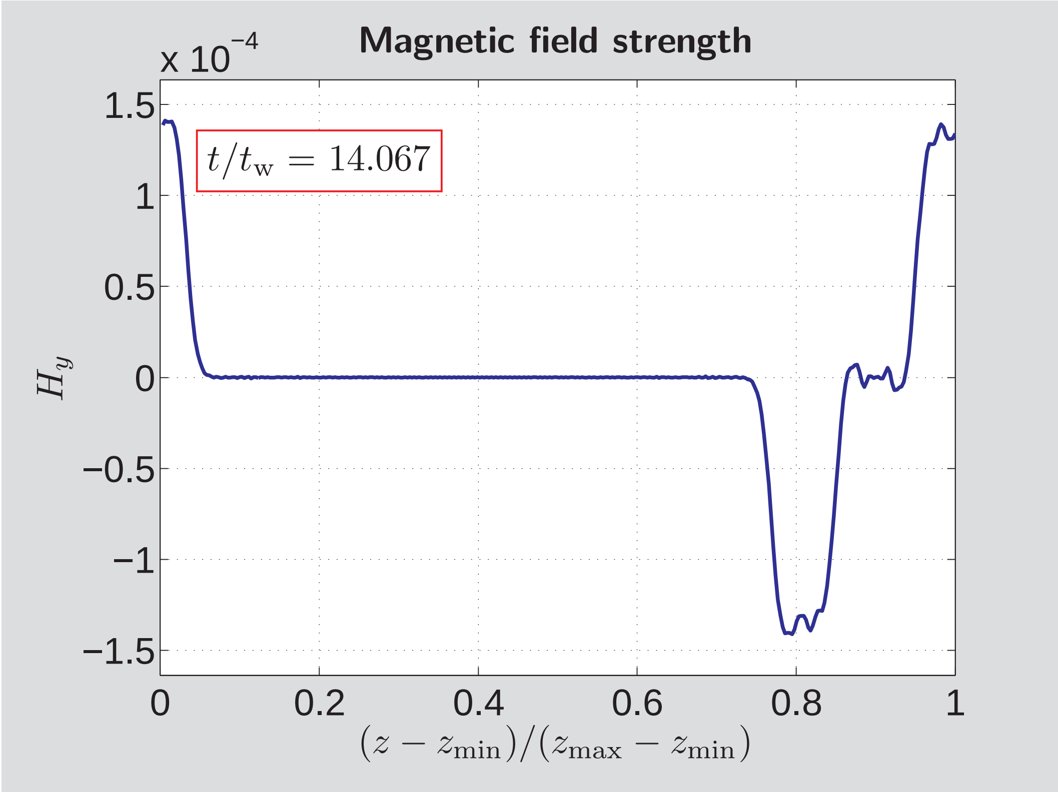

Illustrative numerical results

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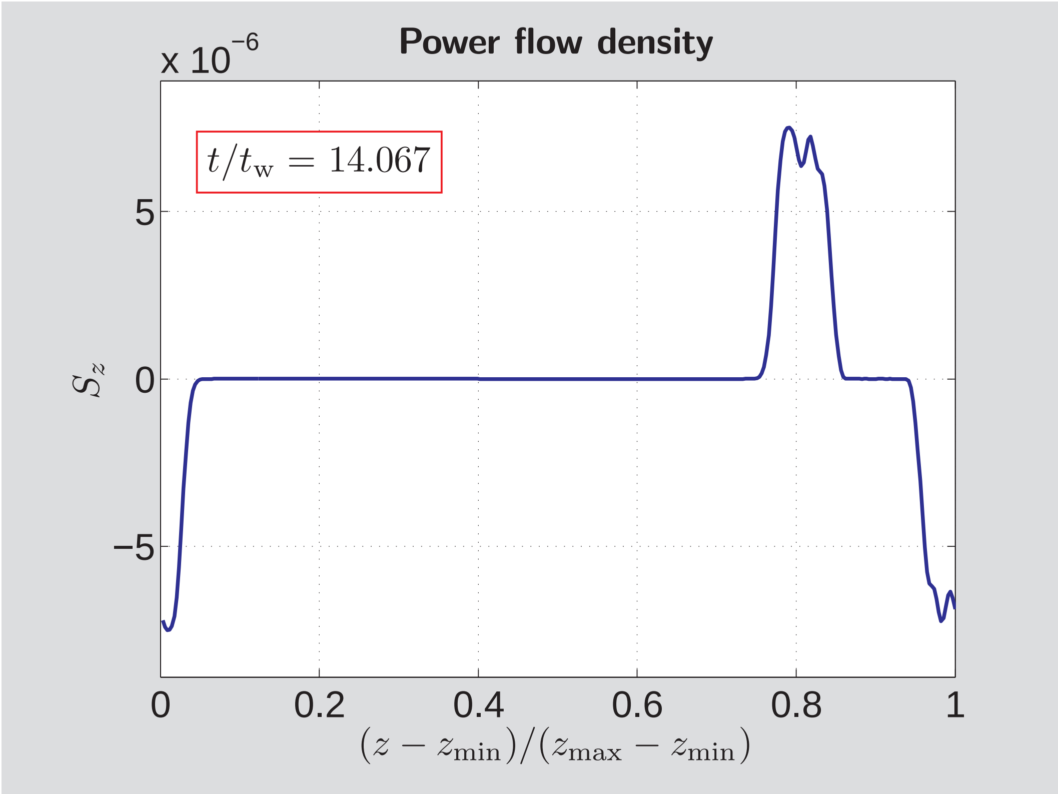
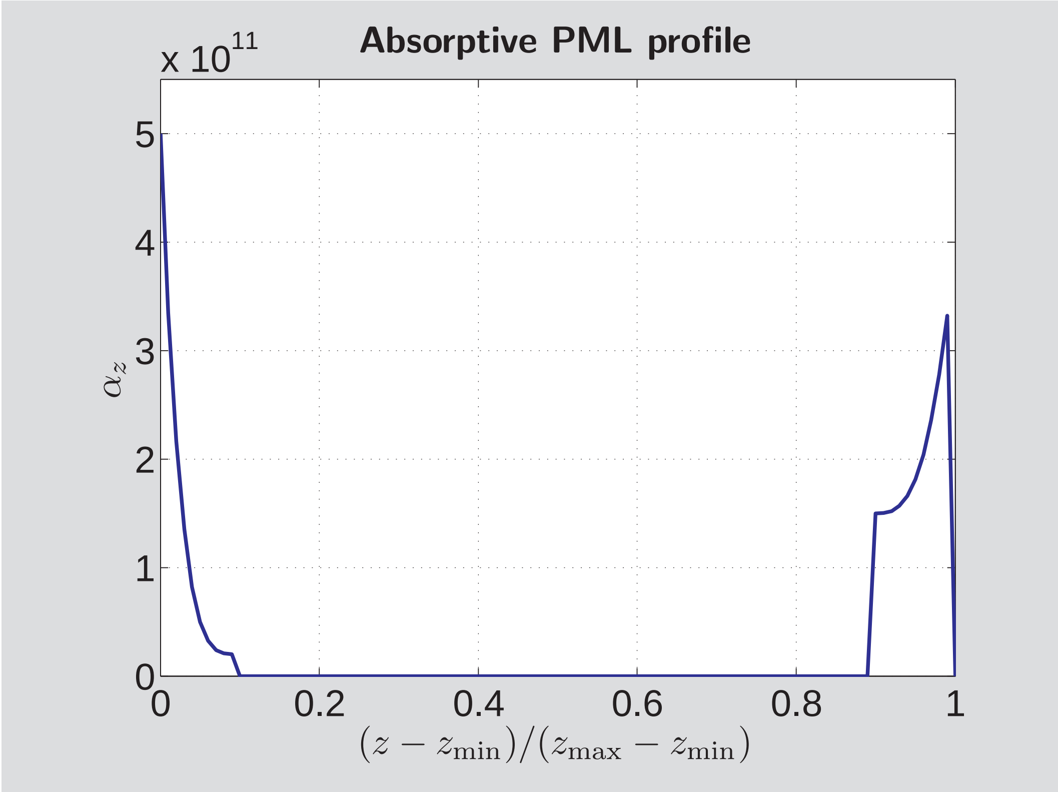


The absorptive perfectly matched embedding (low absorption):

- $A_0^J = 1$ • $A_0^K = 0$ • $\Gamma_{\text{per}} = 1.1$



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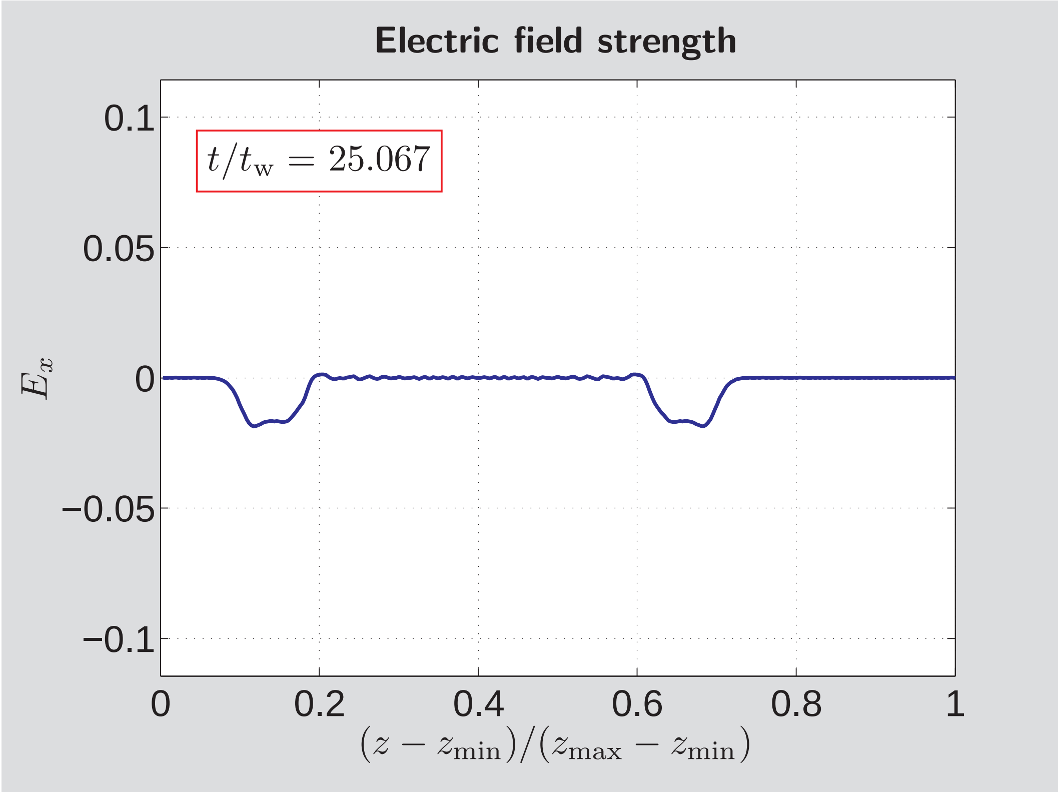

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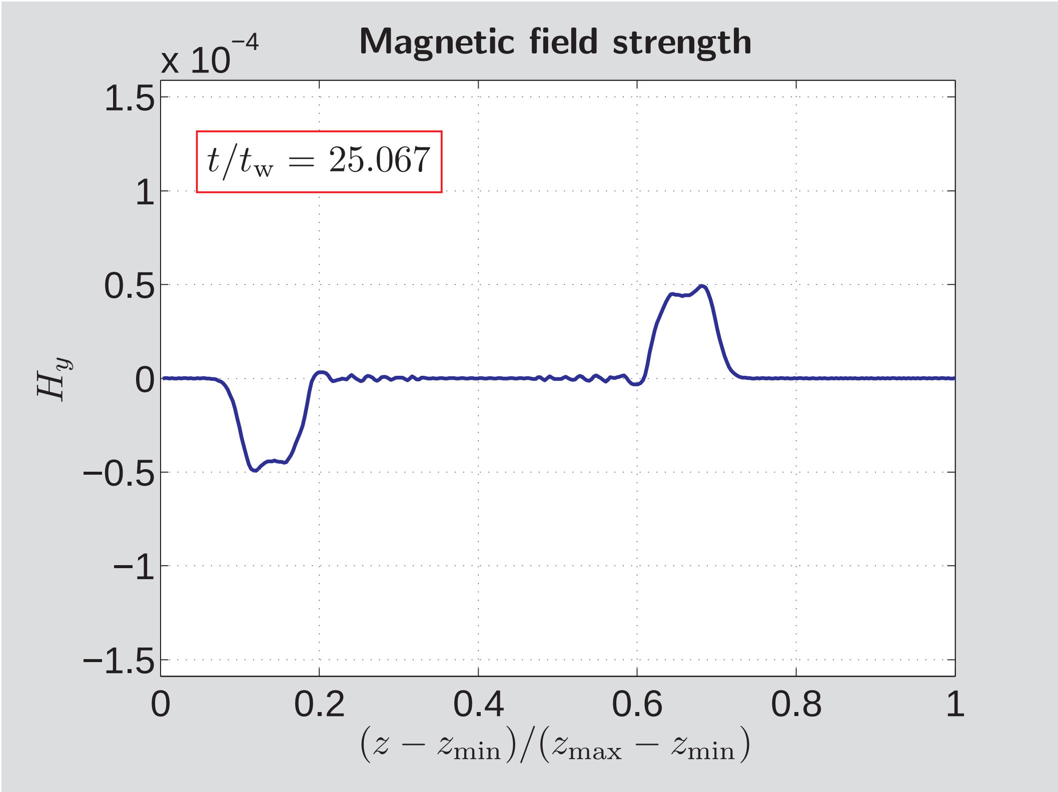

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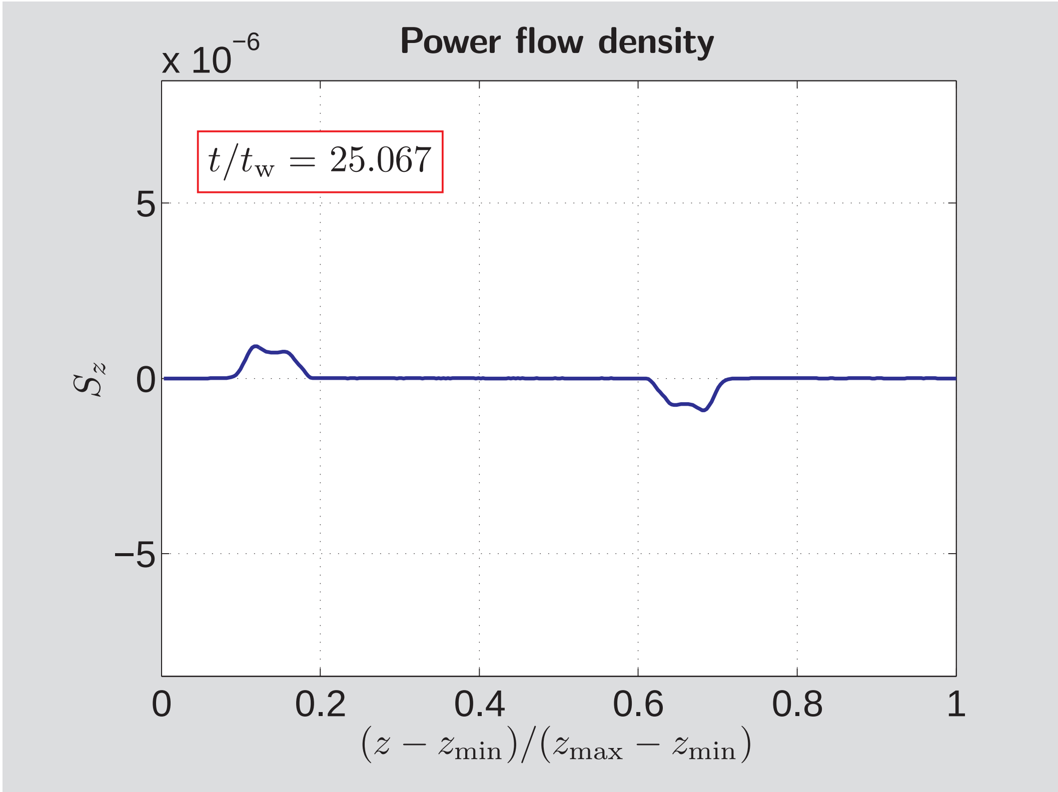
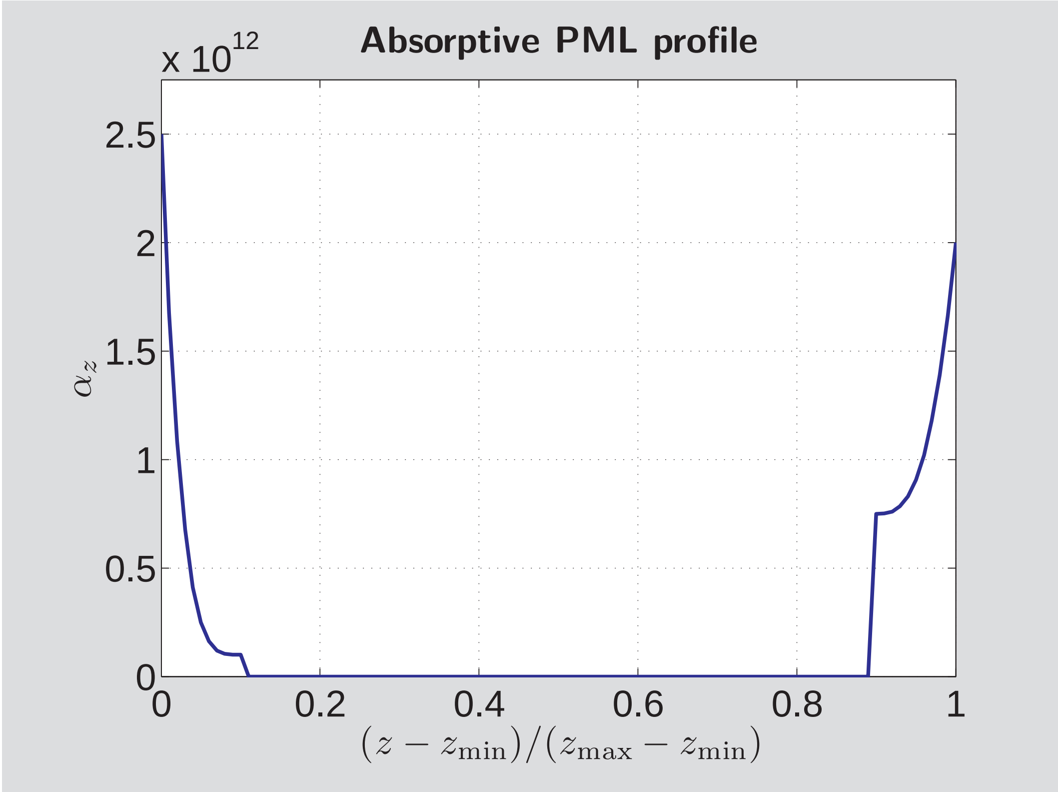


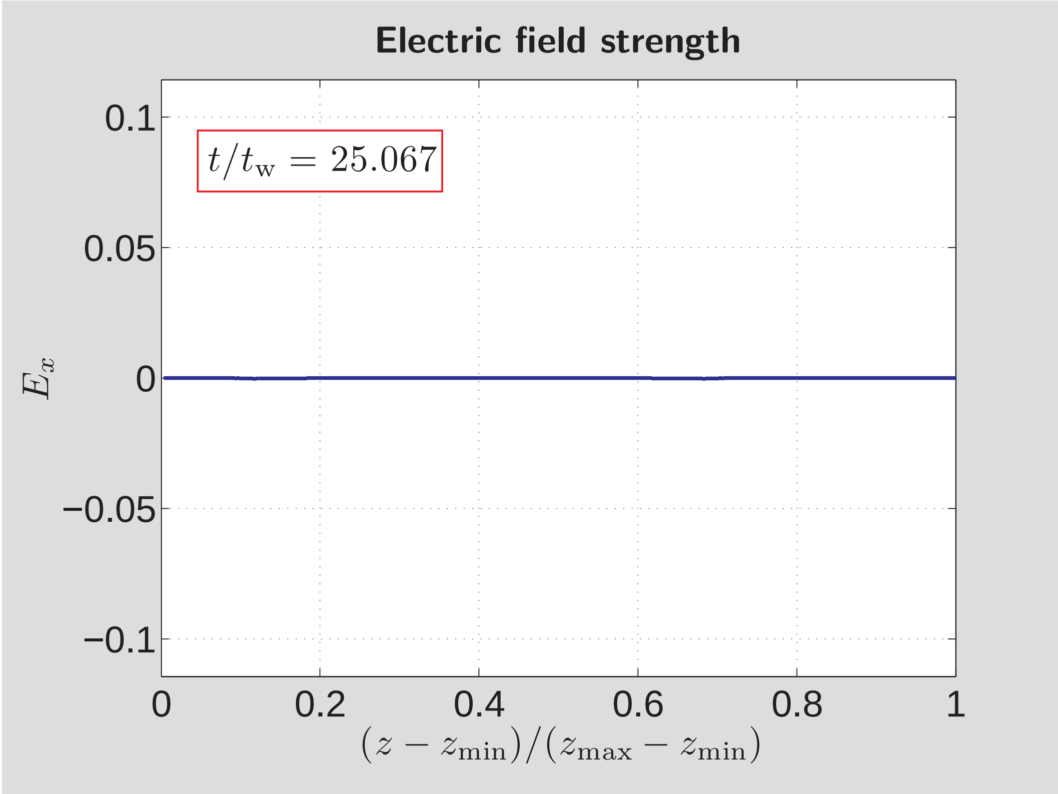
The absorptive perfectly matched embedding (high absorption):

- $A_0^J = 1$
- $A_0^K = 0$
- $\Gamma_{\text{per}} = 5.5$



The absorptive perfectly matched embedding (high absorption):

- $A_0^J = 1$ • $A_0^K = 0$ • $\Gamma_{\text{per}} = 5.5$



The absorptive perfectly matched embedding (high absorption):

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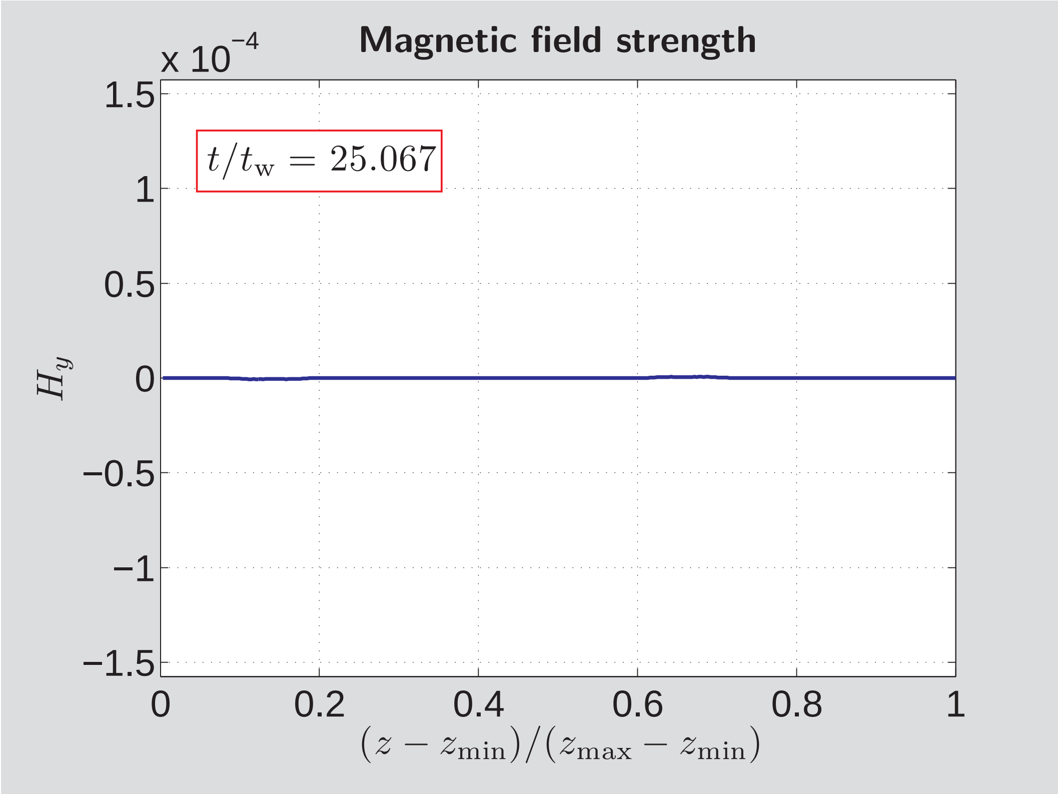

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