

**The mathematics that models wavefield physics in
engineering applications - A voyage through
the landscape of fundamentals**

by

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Lecture presented at IRCTR, TU Delft, NL, on 2010 March 04

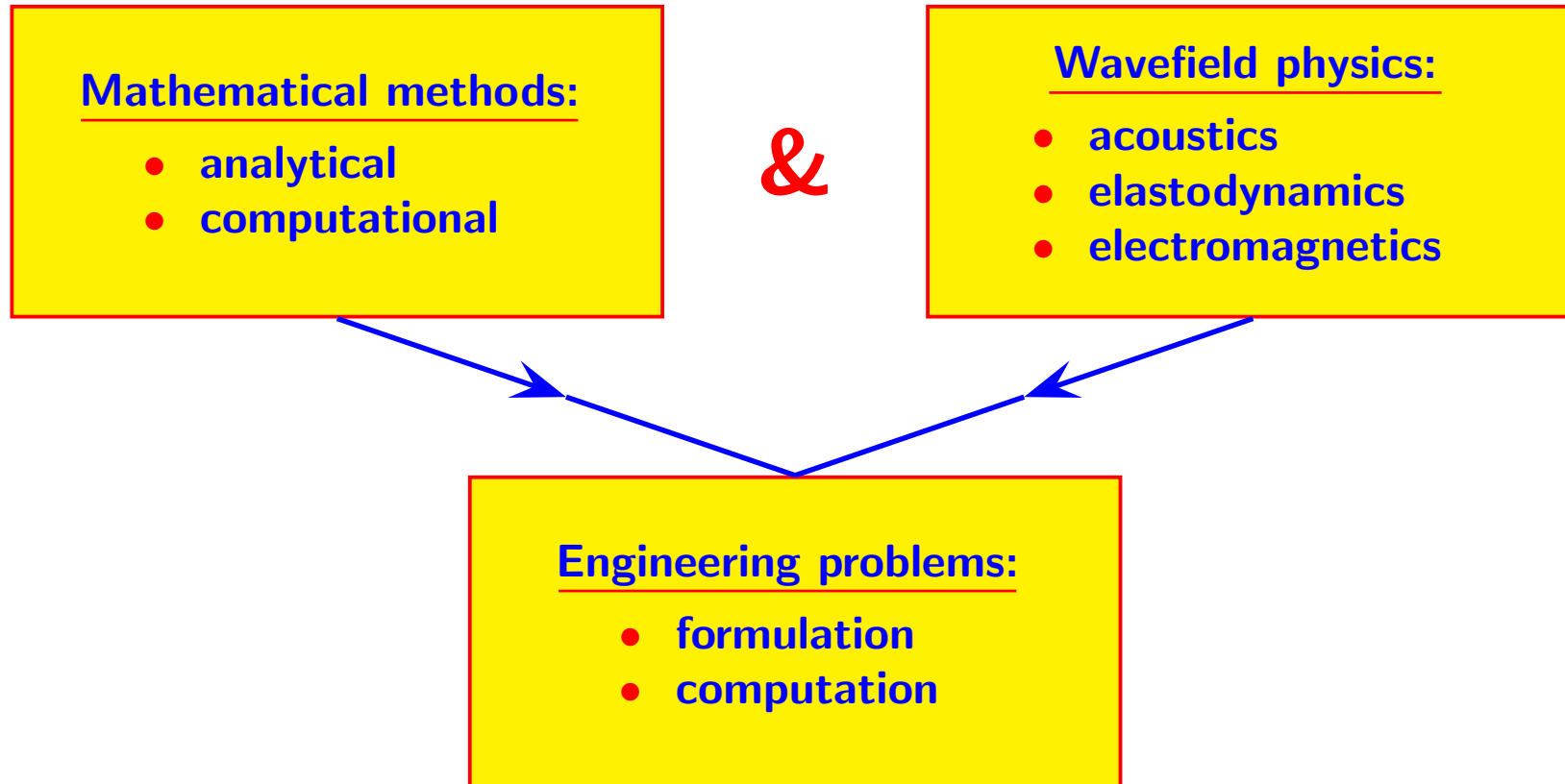
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Title and affiliation

Synopsis:

- System of wavefield equations in canonical form
- Field compatibility relations
- Constitutive relations in media with relaxation (absorption + dispersion)
- Spatial interface boundary conditions
- The initial-value problem and the time Laplace transformation (causality)
- Computational wavefield discretization
- The space-time integrated field equations + computational properties
- The 3D Perfectly Matched Embedding
- A simple application and benchmark problem

An intimate triangle



Structure of any macroscopic physical configuration:

- **BACKGROUND + CONTRAST** (Lorentz, 1916)

Universal **BACKGROUND:**

- **empty universe** (homogeneous + isotropic) (**vacuum**)
- **observer** $\left\{ \begin{array}{l} (N\text{-dimensional}) \text{ **space** \\ (1\text{-dimensional}) \text{ **time** } \end{array} \right\}$ decomposition
- capable of carrying **phenomena** satisfying **Lorentz-invariant** equations
(**electromagnetics, gravitation (?)**)

CONTRAST:

- **matter** interacting with **phenomena in empty space** (**electromagnetics, gravitation (?)**) + carrying **material phenomena** (**acoustics, elastodynamics**)

Each (macroscopic) **FIELD** is represented by:

FOUR FIELD QUANTITIES (FLDQ's):

- {intensive FLDQ 1, intensive FLDQ 2}
- {extensive FLDQ 1, extensive FLDQ 2}

PROPERTIES:

- (intensive FLDQ 1) * (intensive FLDQ 2) =
 - area density of power flow
- (extensive FLDQ 1) * (extensive FLDQ 2) =
 - volume density of flow of momentum

$$\Rightarrow \text{FLDQ} = \text{FLDQ}(\boldsymbol{x}, t)$$

with $\boldsymbol{x} = \{x_1, \dots, x_N\} \in \mathbb{R}^N$ (**space**), $t \in \mathbb{R}$ (time)

FIELD EQUATIONS

couple • **RATES OF CHANGE IN SPACE (∂_x) of intensive FLDQ's**

with • **RATES OF CHANGE IN TIME (∂_t) of extensive FLDQ's**

\Rightarrow **WAVE MOTION** \Leftarrow

CANONICAL (TENSOR) FORM:

(Poincaré, 1905; Einstein, 1905; Minkowski, 1908)

- $D(\partial_x)$ (**intensive FLDQ 1**) + ∂_t (**extensive FLDQ 2**) = 0
- $D(\partial_x)$ (**intensive FLDQ 2**) + ∂_t (**extensive FLDQ 1**) = 0
- **D**: array composed of unit tensors (De Hoop, 1995, 2008)

• **COMPATIBILITY RELATIONS:** • $\partial_{x_1} \partial_{x_2}$ (FLDQ) = $\partial_{x_2} \partial_{x_1}$ (FLDQ)

CONSTITUTIVE RELATIONS:

- **extensive FLDQ = CONSTITUTIVE OPERATOR (intensive FLDQ)**

CONSTITUTIVE OPERATOR:

- linear
- local \implies ~~SPATIAL DISPERSION~~ (\implies ~~infinite wavespeed~~)
- time-invariant
- **active** (field-independent) part (= external sources) +
passive (field-dependent) part (= medium response)
- medium response = **instantaneous response** +
(Boltzmann) relaxation (absorption + dispersion)

CONSTITUTIVE RELATIONS:

- (extensive FLDQ 1,2)(x, t) =

$$\underbrace{(\text{COEFF } 1,2)(x) * (\text{intensive FLDQ } 1,2)(x, t)}_{\text{instantaneous response}} +$$

$$\underbrace{\int_{\tau=0}^{\infty} (\text{RELAXF } 1,2)(x, \tau) * (\text{intensive FLDQ } 1,2)(x, t - \tau) d\tau}_{\text{Boltzmann relaxation}}$$

- **BOLTZMANN RELAXATION** (Boltzmann, 1876) \implies **CAUSALITY**

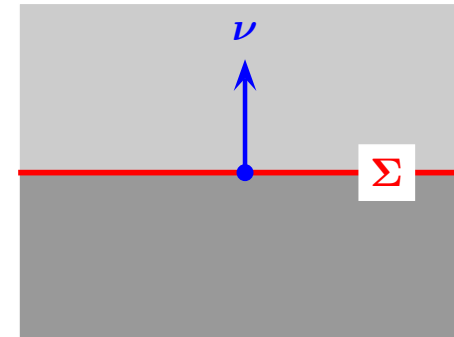
(PASSIVE) INTERFACE BOUNDARY CONDITIONS:

- Across passive interface Σ between two different media

NO JUMPS ALLOWED in certain FLDQ's: WHICH ONES?

- DECOMPOSITION OF: ∂_x about $x \in \Sigma$:

$$\bullet \partial_x = \underbrace{\nu(\nu \cdot \partial_x)}_{\text{normal to } \Sigma} + \underbrace{[\partial_x - \nu(\nu \cdot \partial_x)]}_{\text{tangential to } \Sigma}$$



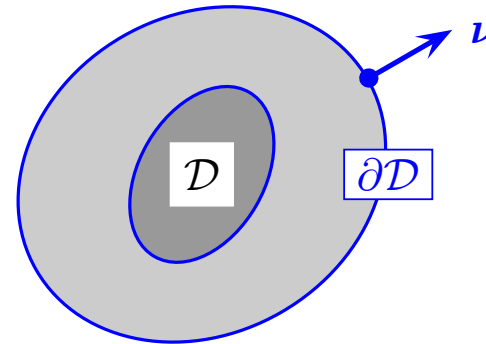
CONTINUITY CONDITIONS:

- $\mathbf{D}(\nu)(\text{intensive FLDQ } 1,2)|_{-}^{+} = 0$

INITIAL-VALUE PROBLEM FIELD EQUATIONS: $(t \in \mathbb{R}; t_0 \leq t < \infty)$

• UNIQUENESS DATA:

- **Field on bounded support $\mathcal{D} \subset \mathbb{R}^N$**

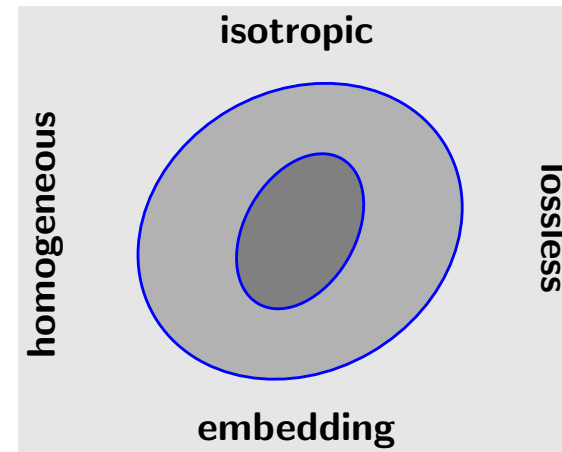


- **Initial field values: FLDQ's (x, t_0) for $x \in \mathcal{D}$**
- **FIELD EQUATIONS** for $x \in \mathcal{D}; t_0 < t < \infty$
- **BOUNDARY VALUES:**
 - **D**(ν)(**intensive FLDQ 1**)(x, t) for $x \in \partial\mathcal{D}, t_0 \leq t < \infty$ **OR**
 - **D**(ν)(**intensive FLDQ 2**)(x, t) for $x \in \partial\mathcal{D}, t_0 \leq t < \infty$

INITIAL-VALUE PROBLEM FIELD EQUATIONS: $(t \in \mathbb{R}; t_0 \leq t < \infty)$

• UNIQUENESS DATA:

- **Field on unbounded support \mathbb{R}^N**



- **Initial field values: FLDQ's (x, t_0) for $x \in \mathbb{R}^N$**
- **FIELD EQUATIONS** for $x \in \mathbb{R}^N; t_0 < t < \infty$
- **OUTGOING WAVES** in **homogeneous, isotropic, lossless embedding**

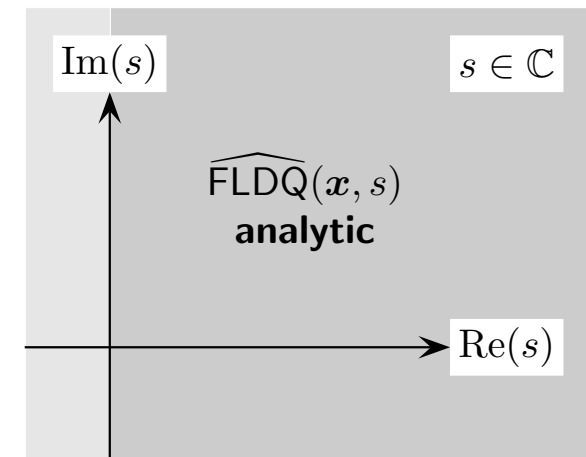
INITIAL-VALUE PROBLEM FIELD EQUATIONS: ($t \in \mathbb{R}; t_0 \leq t < \infty$)

• MATHEMATICAL PROOF:

- **Via shifted time Laplace transformation:** (De Hoop, 2003, 2004)

- $\widehat{\text{FLDQ}}(\mathbf{x}, s) = \int_{t=t_0}^{\infty} \exp(-st) \text{FLDQ}(\mathbf{x}, t) dt$
for $s \in \mathbb{C}, \text{Re}(s) > 0$

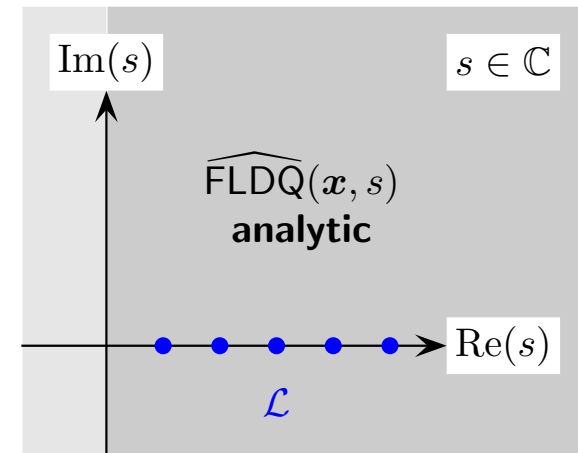
- $\partial_t \text{FLDQ}(\mathbf{x}, t) \mapsto s \widehat{\text{FLDQ}}(\mathbf{x}, s) - \underbrace{\text{FLDQ}(\mathbf{x}, t_0)}_{\text{initial value}}$



INVERSE TIME LAPLACE TRANSFORMATION: $(t \in \mathbb{R}; 0 \leq t < \infty)$

- use of Lerch's uniqueness theorem:

$$\bullet \{ \widehat{\text{FLDQ}}(\mathbf{x}, s) |_{s \in \mathcal{L}} \} \xrightarrow{\text{1-to-1}} \text{FLDQ}(\mathbf{x}, t)H(t)$$



$$\bullet \mathcal{L} = \{ s \in \mathbb{C}; \text{Im}(s) = 0, \text{Re}(s) = s_0 + nh, s_0 > 0, h > 0, n = 0, 1, 2, 3, \dots \}$$

- via **INSPECTION** (Tables of Laplace Transforms)

- use of **Schouten-Van der Pol theorem**: (Schouten, 1934; Van der Pol, 1934)

$$\bullet \text{For } \exp[-\hat{\Phi}(\mathbf{x}, s)\tau] \xrightarrow{\quad} \Psi(\mathbf{x}, t, \tau)H(t)$$

$$\bullet \widehat{\text{FLDQ}}[\mathbf{x}, \hat{\Phi}(\mathbf{x}, s)] \xrightarrow{\quad} \left[\int_{\tau=0}^{\infty} \Psi(\mathbf{x}, t, \tau) \text{FLDQ}(\mathbf{x}, \tau) d\tau \right] H(t)$$

WAVEFIELD COMPUTATION

- Select (bounded) **spatial domain of computation** $[\mathcal{D}] \subset \mathbb{R}^N$
- Select (bounded) **time window of computation** $[\mathcal{T}] \subset \mathbb{R}$
- Construct (unbounded) **Perfectly Matched Embedding (PME)**
 $[\mathcal{D}]^\infty = \mathbb{R}^N \setminus [\mathcal{D}]$ **via time-dependent orthogonal Cartesian coordinate stretching** (De Hoop, Remis, Van den Berg, 2007)
- **Terminate PME with periodic boundary conditions**
 (De Hoop, Remis, Van den Berg, 2007)
- **Discretize** $[\mathcal{D}]$ **into union of adjacent simplices**
- **Discretize** $[\mathcal{T}]$ **into union of successive intervals**
- **Test with** **BENCHMARK PROBLEMS** \Leftarrow **(AMUSING !)**

WAVEFIELD COMPUTATION

- **Discretize** $\text{FLDQ}(\boldsymbol{x}, t)$ **using:**
 - piecewise linear interpolation on spatial grid
 - piecewise linear interpolation on temporal grid
 - nodal values of **CONTINUOUS** field components as (nodal, edge, face) expansion coefficients
- **Substitute discretized field in space-time integrated field equations**
- **Compute integrations via simplicial ('trapezoidal') rule**
- **Discretize constitutive relations (piecewise constant in $[\mathcal{D}]$)**
- **Solve system of equations in space-time expansion coefficients**
- **Apply SUCCESSIVE GRID REFINEMENT (CONVERGES !)**

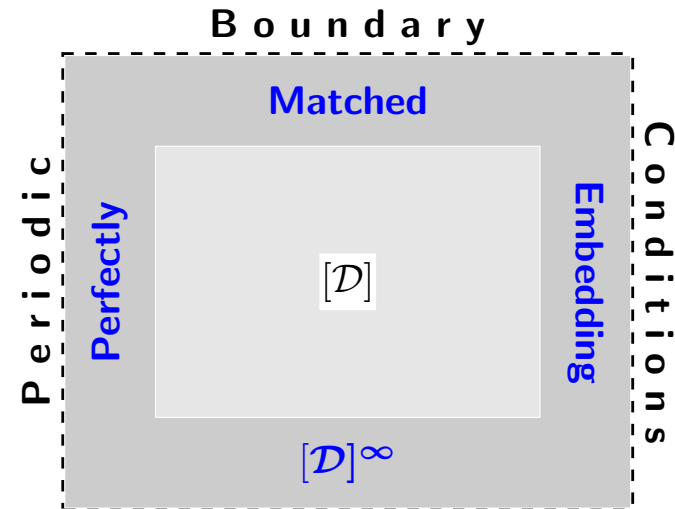
CONSTRUCTION OF PERFECTLY MATCHED EMBEDDING (PME)

- Time Laplace-transform Cartesian coordinate stretching:

$$\bullet \partial_{x_n} \mapsto \partial_{\hat{X}_n} = \frac{1}{\hat{\chi}_n(x_n, s)} \partial_{x_n} \implies$$

$$\hat{X}_n(x_n, s) = \int_{\xi_n=a_n}^{x_n} \hat{\chi}_n(\xi_n, s) d\xi_n$$

$(n = 1, \dots, N)$



- $\{\hat{\chi}_n(x_n, s); n = 1, \dots, N\}$ **analytic** for $s \in \mathbb{C}, \text{Re}(s) > 0, \mathbf{x} \in [D]^\infty$

- $\{\hat{\chi}_n(x_n, s); n = 1, \dots, N\} > \mathbf{0}$ for $s \in \mathbb{C}, \text{Re}(s) > 0, \text{Im}(s) = 0, \mathbf{x} \in [D]^\infty$

- $\{\hat{\chi}_n(x_n, s); n = 1, \dots, N\} = \mathbf{1}$ for $\mathbf{x} \in [D] \implies$ **field unchanged in [D]**

Boundary Conditions \implies spurious field (De Hoop, Remis, Van den Berg, 2007)

THE SPACE-TIME INTEGRATED FIELD EQUATIONS

- **Apply operators**

- $\int_{\mathbf{x} \in \mathcal{D}} \dots dV$ and
 - $\int_{t \in \mathcal{T}} \dots dt$ to **FIELD EQUATIONS**

- **Use**

- $\int_{\mathbf{x} \in \mathcal{D}} \mathbf{D}(\partial \mathbf{x}) [\text{intensive FLDQ}(\mathbf{x}, t)] dV = \int_{\mathbf{x} \in \partial \mathcal{D}} \mathbf{D}(\boldsymbol{\nu}) [\text{intensive FLDQ}(\mathbf{x}, t)] dA$
(Gauss in \mathbb{R}^N)

- $\int_{t \in \mathcal{T}} \partial_t [\text{extensive FLDQ}(\mathbf{x}, t)] dt = [\text{extensive FLDQ}(\mathbf{x}, t)] \Big|_{t \in \partial \mathcal{T}}$
(Gauss in \mathbb{R})

\Rightarrow **In RHS's only continuous quantities occur**

THE SIMPLICIAL INTEGRATION RULE

- **Simplicial integration rule in \mathbb{R}^N (= trapezoidal rule in \mathbb{R}):**

Let $\Sigma^N \subset \mathbb{R}^N = N$ -simplex on vertices $\{\mathbf{x}(0), \dots, \mathbf{x}(N)\}$, then

- $\int_{\mathbf{x} \in \Sigma^N} [\text{discretized FLDQ}(\mathbf{x}, t)] dV \simeq$

$$\frac{V^N}{N+1} \left[\text{FLDQ}[(\mathbf{x}(0), t)] + \dots + \text{FLDQ}[(\mathbf{x}(N), t)] \right]$$
- $V^N = \text{volume of } \Sigma^N$

(De Hoop, 1995, 2008)

UNIT TENSORS IN WAVEFIELD PHYSICS

Symmetrical **unit tensor of rank two**: (Kronecker tensor)

- $\delta_{i,p} = 1$ for $i = p$, $\delta_{i,p} = 0$ for $i \neq p$

Unit tensors of rank four:

- $\Delta_{i,j,p,q} = \delta_{i,p}\delta_{j,q}$ (**reproduction**)
- $\Delta_{i,j,p,q}^- = (1/2)(\Delta_{i,j,p,q} - \Delta_{i,j,q,p})$ (**electromagnetics**)
- $\Delta_{i,j,p,q}^+ = (1/2)(\Delta_{i,j,p,q} + \Delta_{i,j,q,p})$ (**elastodynamics**)
- $\Delta_{i,j,p,q}^\delta = (1/N)\delta_{i,j}\delta_{p,q}$ (**acoustics**)
- $\Delta_{i,j,p,q}^\Delta = \Delta_{i,j,p,q} - \Delta_{i,j,p,q}^\delta$ (**elastodynamics**)
- $\Delta_{i,j,p,q}^{\Delta,+} = \Delta_{i,j,p,q}^+ - \Delta_{i,j,p,q}^\delta$ (**elastodynamics**)

(De Hoop, 1995, 2008)

TEST PULSES IN TIME FOR BENCHMARKING:

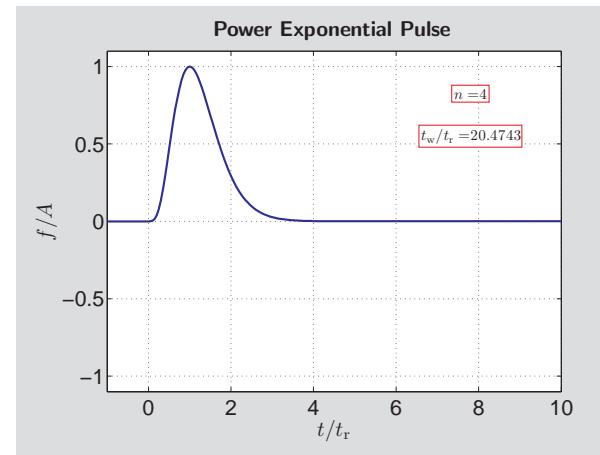
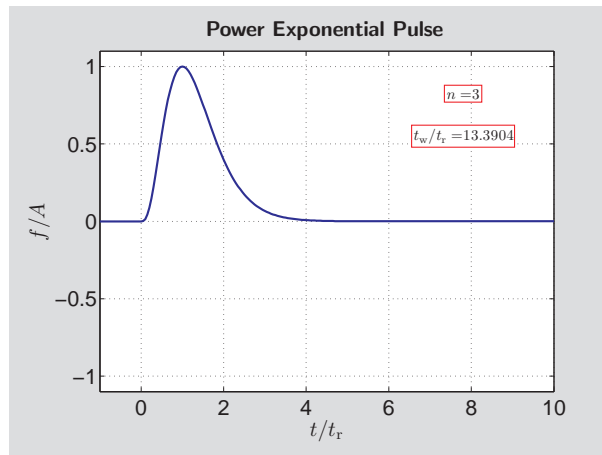
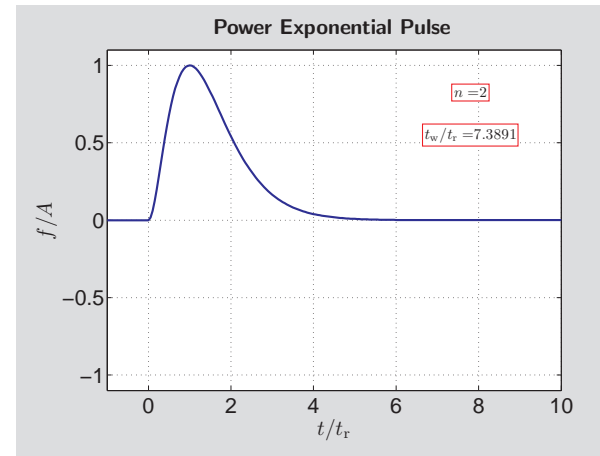
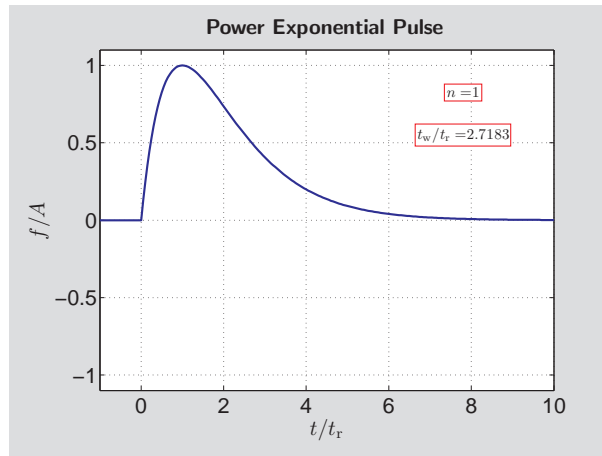
The unipolar pulse :

- $f(t) \geq 0$ for $t \geq 0$
- $\partial_t f(t)|_{t=t_r} = 0 \implies t_r$ (pulse rise time)
- $A = f(t_r)$ (pulse amplitude)
- $t_w = \frac{1}{A} \int_{t=0}^{\infty} f(t) dt$ (pulse time width)

The power exponential pulse :

- $f(t) = A \left(\frac{t}{t_r} \right)^n \exp \left[-n \left(\frac{t}{t_r} - 1 \right) \right] H(t)$ for $n = 1, 2, 3, \dots$
- $t_w = \frac{n!}{n^{n+1}} \exp(n) t_r$
- $\hat{f}(s) = A \frac{n!}{(s + n/t_r)^{n+1}} \frac{\exp(n)}{t_r^n}$ for $s \in \mathbb{C}, \text{Re}(s) > 0$

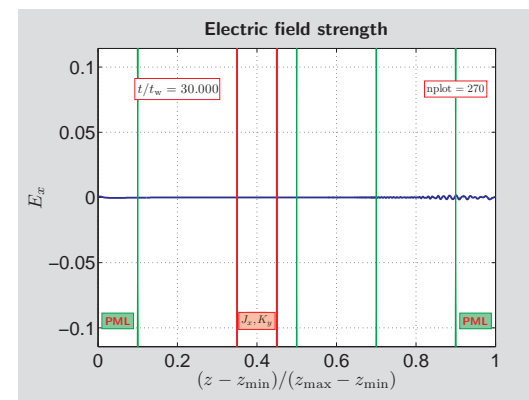
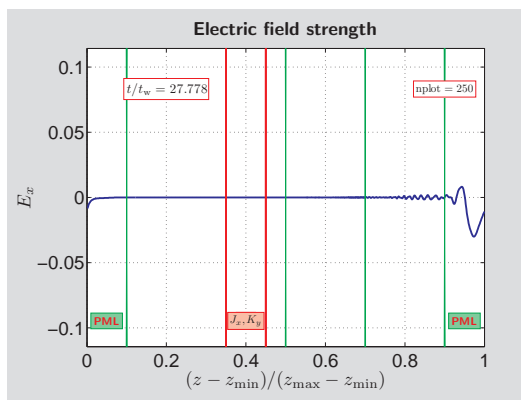
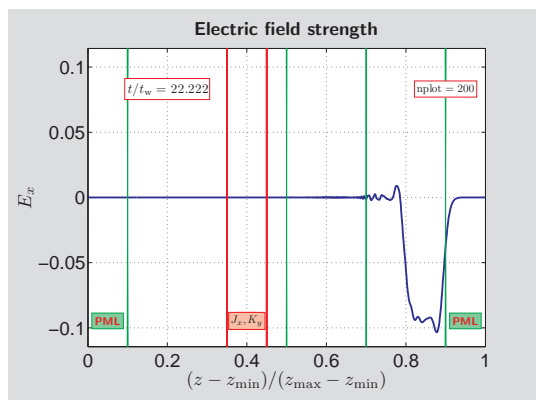
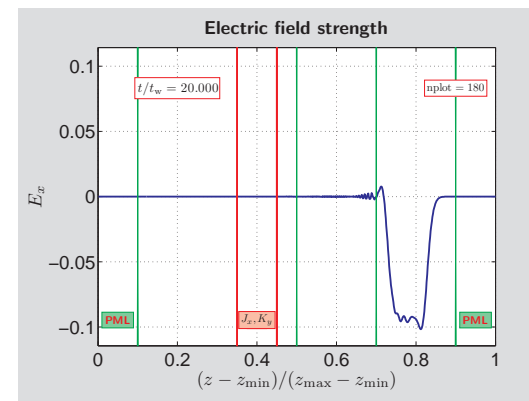
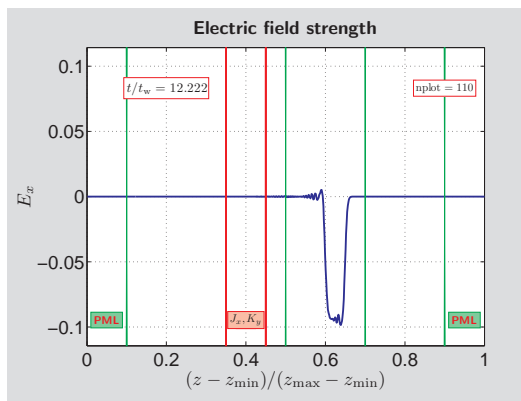
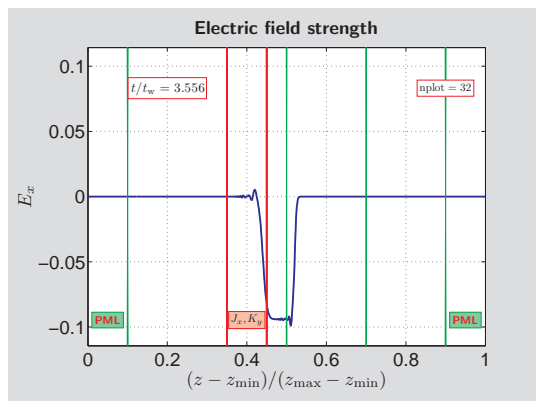
POWER EXPONENTIAL PULSES:



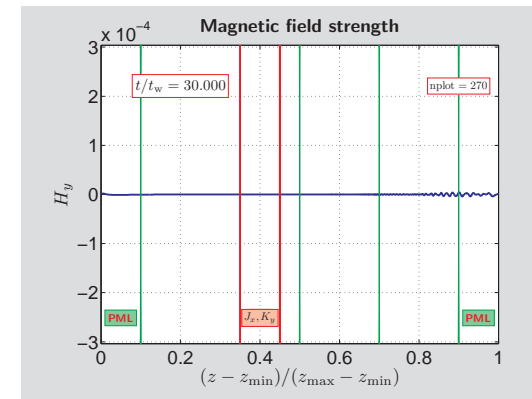
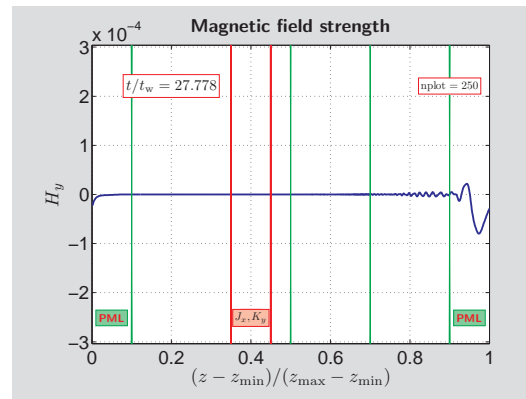
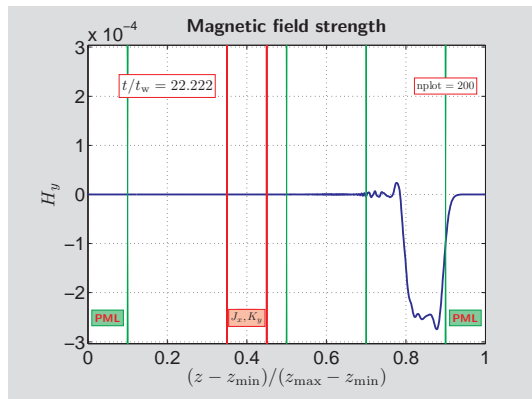
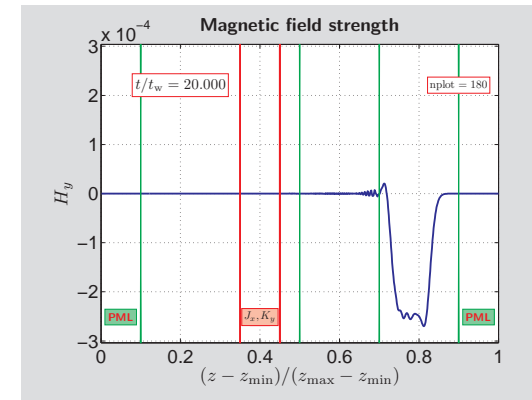
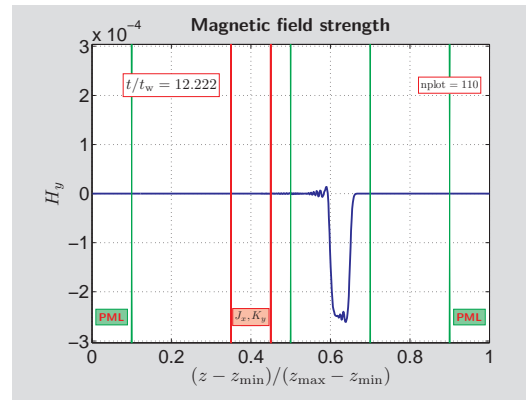
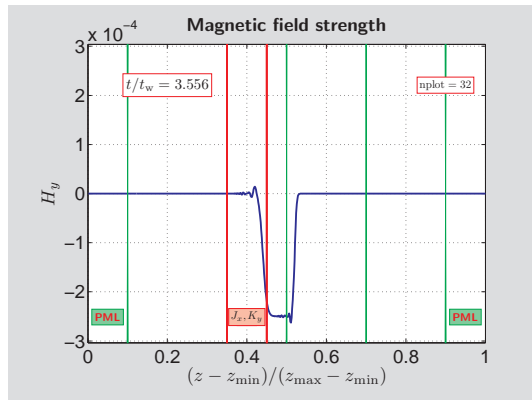
1D EM TD benchmark problem:

- **Electric-/Magnetic-current source excitation** \implies One-sided field
- **Propagation across slab with contrasting wave speed, no contrast in wave impedance** \implies Pulse narrowing in space, no reflection
- **Absorptive PML-padding with jump discontinuity at interface** \implies Absorption, no reflection
- **Periodic boundary condition** \implies Uniformity in PML absorption
- **Discretization data**
 - $\Delta z = (\text{spatial pulse width})/10$
 - $\Delta t = (\text{pulse time width})/9$
 - $N_{\text{cells}} = 681$
 - $\Gamma_{\text{period}} = 6.17$

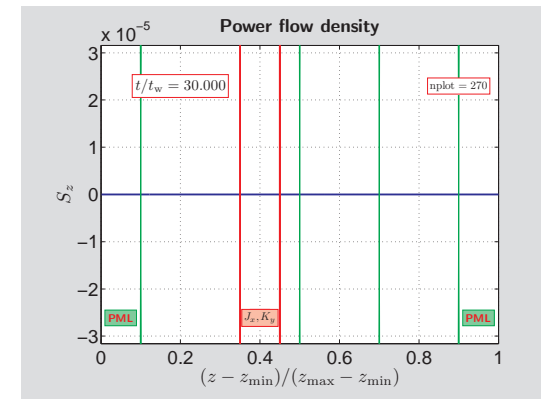
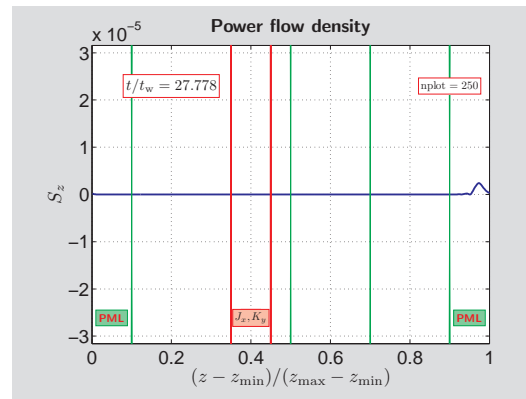
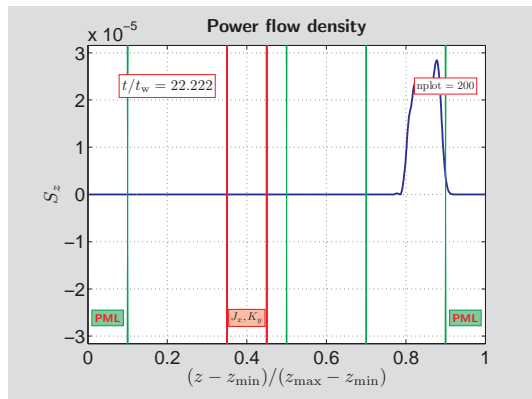
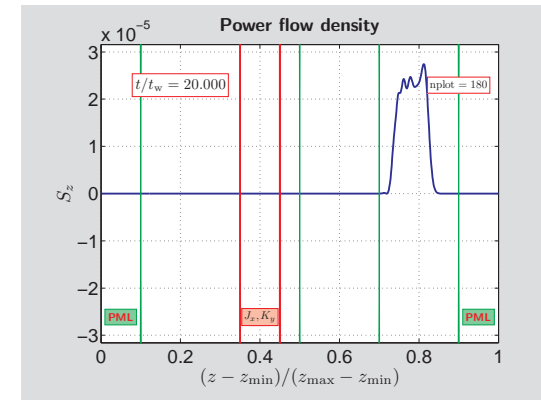
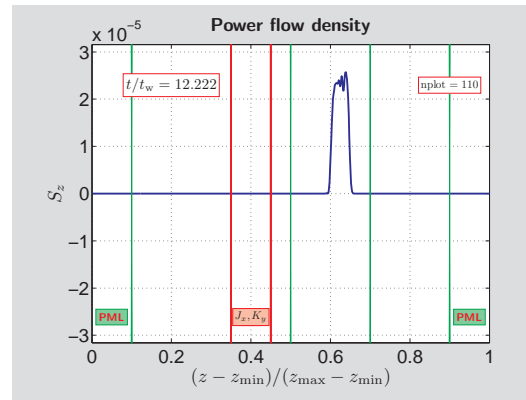
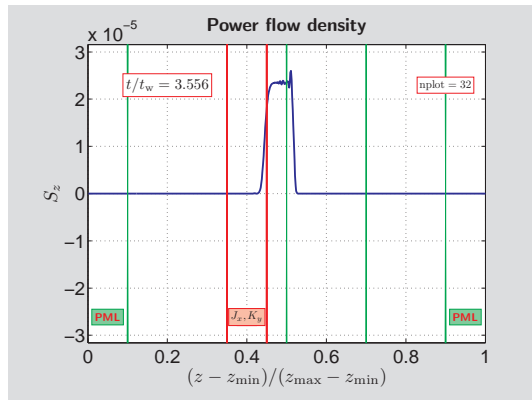
1D EM TD benchmark problem (Electric field):



1D EM TD benchmark problem (Magnetic field):



1D EM TD benchmark problem (Poynting vector):



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