Electromagnetic field theory - A modern tensorial approach

by

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- **EM THEORY:**

- WHY ~~vectorial~~ tensorial ? $\Longrightarrow$
  - Leads to (considerable) simplifications
  - Directly matches array handling in computational coding
  - Generalizes to $(N+1)$-spacetime (relativity, cosmology)

EM theory: why tensorial rather than vectorial?

• TENSORIAL EM THEORY: simplifications $\Longrightarrow$

- Involves only ELEMENTARY MATHEMATICS:
  - ∂ : (partial) differentiation
  - $\overset{(t)}{*}$: time convolution
  - $\overset{(\boldsymbol{x})}{*}$: spatial convolution
- Does away with standard ~~VECTOR CALCULUS~~
- Does away with the condition of ~~RIGHT-HANDEDNESS~~ in orientation of spatial reference frame

• TENSORIAL EM THEORY: implications $\Longrightarrow$

- Reformulates the notion of the ~~'MAGNETIC MONOPOLE'~~
  (= tensor of rank 3, ~~scalar~~)
- Considers SpaceTime as an 'affine' space (not a 'metric' space) $\Longrightarrow$
  ~~4D LORENTZ 'METRIC'~~ (Weyl, H., *Space, Time, Matter*, 1922):
  Observer (biologically?) decomposes
  'spacetime' into 'space × time (causality**!**)'
- Has implications for (Dirac) 'string theory' in quantum electrodynamics
  (no 'vector potential' as the basic physical quantization ingredient)

Tensorial EM theory: implications

• WAVEFIELD QUANTITY = TENSOR (EINSTEIN)

• SUBSCRIPT NOTATION

TENSOR	rank 0	rank 1	rank 2	
Array notation (3D)	$[a]$	$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$	$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Subscript notation (Einstein)	a	a_i	$a_{i,j}$	$\delta_{m,n}$

- $\boldsymbol{x} \to x_n$: position vector (tensor of rank 1) • t : time
- $\partial_m = \partial/\partial x_m$ • $\partial_t = \partial/\partial t$ (reserved symbol)
- $\partial_m x_n = \delta_{m,n}$: Kronecker (unit) tensor (tensor of rank 2)

• SUMMATION CONVENTION (repeated subscripts):

- $\delta_{p,n}\delta_{n,q} = \sum_{n=1}^{N} \delta_{p,n}\delta_{n,q} = \delta_{p,q}$ • $\delta_{m,m} = \sum_{m=1}^{N} \delta_{m,m} = N \quad (N = 3)$

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Tensorial EM theory: Einstein subscript notation and summation convention

	Intensive field quantity 'field strength'	Extensive field quantity 'flux density'	Source quantity 'volume density of' 'current'
Electric	E_k	D_k	J_k
Magnetic	$H_{i,j} = -H_{j,i}$	$B_{i,j} = -B_{j,i}$	$K_{i,j} = -K_{j,i}$
	'PRODUCT'	'PRODUCT'	
	$S_m = H_{m,k} E_k$ area density of EM power	$G_i = B_{i,j} D_j$ volume density of EM momentum	

- $\{H_{i,j}, B_{i,j}, K_{i,j}\}$ (anti-symmetric, rank 2)

EM field and source quantities

- **WAVEFIELDS:**
  - Changes in space ($\partial_m = \partial/\partial x_m$) **counterbalance**
  - Changes in time ($\partial_t = \partial/\partial t$)

EM field equations (Maxwell)

- $\partial_m H_{m,k} + \partial_t D_k = J_k$
- $\partial_i E_j - \partial_j E_i + \partial_t B_{i,j} = K_{i,j}$

• FIELD/SOURCE COMPATIBILITY RELATIONS

(= existence conditions for solution of field equations):

- $\partial_k \partial_m H_{m,k} = \partial_m \partial_k H_{m,k} = -\partial_m \partial_k H_{k,m}$ = - $\partial_k \partial_m H_{m,k} = 0$

  $\Longrightarrow$ • $\partial_t(\partial_k D_k) = \partial_k J_k$

- $\partial_k(\partial_i E_j - \partial_j E_i) + \partial_i(\partial_j E_k - \partial_k E_j) + \partial_j(\partial_k E_i - \partial_i E_k) = 0 \ (i \neq j \neq k)$

  $\Longrightarrow$ • $\partial_t(\partial_k B_{i,j} + \partial_i B_{j,k} + \partial_j B_{k,i}) = \partial_k K_{i,j} + \partial_i K_{j,k} + \partial_j K_{k,i} \ (i \neq j \neq k)$

EM field equations and field/source compatibility relations

• VOLUME DENSITY OF ELECTRIC CHARGE:

- $\rho = -\partial_t^{-1}\partial_k J_k$ (scalar)

$\Longrightarrow$ • $\partial_k D_k = -\rho$

• VOLUME DENSITY OF MAGNETIC CHARGE:

- $\sigma_{k,i,j} = -\partial_t^{-1}(\partial_k K^-_{i,j} + \partial_i K^-_{j,k} + \partial_j K^-_{k,i})\ (i \neq j \neq k)$

(completely antisymmetric tensor of rank 3)

$\Longrightarrow$ • $\partial_k B^-_{i,j} + \partial_i B^-_{j,k} + \partial_j B^-_{k,i} = -\sigma_{k,i,j}\ (i \neq j \neq k)$

Volume densities of electric and magnetic charge

• CAUSALITY (UNIQUENESS OF TIME EVOLUTION = UNIQUENESS OF INITIAL-VALUE PROBLEM):

- **EM field**$(\boldsymbol{x}, t_0)$ for $\boldsymbol{x} \in \mathbb{R}^3 \overset{\text{Field equations}}{\Longrightarrow \Longrightarrow}$
  EM field$(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3, t > t_0$ (uniquely)

- **'Maxwell'**: # Equations $\neq$ # Unknowns $\Longrightarrow$

• CONSTITUTIVE RELATIONS (simplest case):

- $D_k = \epsilon E_k$ (ϵ = electric permittivity)

- $B_{i,j} = \mu H_{i,j}$ (μ = magnetic permeability)

- $\Longrightarrow$ **'Maxwell'**: # Equations = # Unknowns

Causality, uniqueness of initial-value problem

• (More or less) COMPLETE EM COURSE:

Part	Degree of difficulty
• Field equations	*
• Energy balance	*
• Balance of field momentum ('radiation pressure')	***
• Field/source/contrast-in-medium reciprocity relations	*
• Radiation from sources	**

in 4 PAGES ! ($\Longrightarrow$ Conference Proceedings URSI EMT-S Berlin 2010)

• ALSO:

- Lorentz covariance (special relativity)

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• EM RADIATION FROM SOURCES:

• Elimination of D_k, $B_{i,j}$ and $\left\{\begin{matrix} H_{i,j} \\ E_k \end{matrix}\right\}$ from the field equations $\Longrightarrow$

$\left\{\begin{matrix} \text{vector} \\ \text{tensor} \end{matrix}\right\}$ wave equation for $\left\{\begin{matrix} E_k \\ H_{i,j} \end{matrix}\right\}$: [• $\mathsf{D} = \mathsf{D}(\epsilon, \mu, \partial_m, \partial_t)$ • $c = (\epsilon\,\mu)^{-1/2}$]

$$\bullet\ (\partial_m\partial_m - c^{-2}\partial_t^2)\begin{bmatrix} E_r \\ H_{p,q} \end{bmatrix} = -\begin{bmatrix} \mathsf{D}^{EJ}_{r,k} & \mathsf{D}^{EK}_{r,i,j} \\ \mathsf{D}^{KJ}_{p,q,k} & \mathsf{D}^{HK}_{p,q,i,j} \end{bmatrix}\begin{bmatrix} J_k \\ K_{i,j} \end{bmatrix}\left(\overset{(\boldsymbol{x})}{*}\overset{(t)}{*}\ \delta(\boldsymbol{x}, t)\right)$$

$$\bullet\ (\partial_m\partial_m - c^{-2}\partial_t^2)\, G(\boldsymbol{x}, t) = -\delta(\boldsymbol{x}, t) \quad \bullet\ G(\boldsymbol{x}, t) = \frac{\delta(t - |\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|} \text{ for } |\boldsymbol{x}| > 0$$

$$\bullet\ \begin{bmatrix} E_r \\ H_{p,q} \end{bmatrix} = \begin{bmatrix} \mathsf{D}^{EJ}_{r,k} & \mathsf{D}^{EK}_{r,i,j} \\ \mathsf{D}^{KJ}_{p,q,k} & \mathsf{D}^{HK}_{p,q,i,j} \end{bmatrix}\begin{bmatrix} J_k \\ K_{i,j} \end{bmatrix}\overset{(\boldsymbol{x})}{*}\overset{(t)}{*}\ G(\boldsymbol{x}, t)$$

EM radiation from sources

- **TENSORIAL EM THEORY:**
- Leads to (considerable) simplifications
- Involves only elementary mathematics ($\partial_t\partial_x = \partial_x\partial_t, \overset{(t)}{*}, \overset{(\boldsymbol{x})}{*}$)
- Does away with standard ~~VECTOR CALCULUS~~
- Does away with the condition of ~~RIGHT-HANDEDNESS~~ in orientation of spatial reference frame
- Considers SpaceTime as an 'affine' space (not a 'metric' space)
- Reformulates the notion of the '~~MAGNETIC MONOPOLE~~' (= tensor of rank 3)
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- Directly matches array handling in computational coding

Tensorial EM theory: CONCLUSION

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