Electromagnetic field theory - A modern tensorial approach

by

Adrianus T. de Hoop

Delft University of Technology

Laboratory of Electromagnetic Research

Faculty of Electrical Engineering, Mathematics and Computer Science

Mekelweg 4 • 2628 CD Delft • the Netherlands

T: +31 15 2785203 / +31 15 2788613

E: a.t.dehoop@tudelft.nl **W:** www.atdehoop.com

Presentation given at:

URSI Electromagnetic Theory Symposium 2010, Berlin, August 16–19

Synopsis:

- EM fields in (affine) $(3+1)$ spacetime
- Intensive field quantities, extensive field quantities, source quantities
- EM field equations and field/source compatibility relations
- Causality and EM constitutive relations
- EM energy theorem
- Time-domain EM field radiated by sources
- Green's tensor of the electric-field vector wave equation
- Time-domain EM reciprocity relations
- Conclusion

- $(3+1)$-spacetime $(x \in \mathbb{R}^3, t \in \mathbb{R})$: ~~metric~~ affine space

$\mathbb{R}^3$: • HOMOGENEOUS • ISOTROPIC

Field	Observer	Decomposition		
			'Maxwell'	
Electro-	Observer 1 $\Longrightarrow$	• Electric Field 1	$\Longleftrightarrow$	• Magnetic Field 1
Magnetic	$\downarrow v$ • $\uparrow -v$	(special) $\longleftarrow$ $\Updownarrow$	'Lorentz'	$\Updownarrow$ $\longrightarrow$ (relativity)
Field	Observer 2 $\Longrightarrow$	• Electric Field 2	$\Longleftrightarrow$	• Magnetic Field 2
			'Maxwell'	

- $v =$ (constant) relative velocity

- **Einstein:** All macroscopic physical wavefields are **tensorial** in nature
- **Tensor of rank p in $\mathbb{R}^N$:** p-dimensional array of length $\underbrace{N \times \ldots \times N}_{p}$

• SUBSCRIPT NOTATION: $(n = 1, \ldots, N)$

- $\boldsymbol{x} \to x_n$: position vector (tensor of rank 1)
- $\partial_m = \partial/\partial x_m$: spatial derivative (tensor of rank 1)
- $\partial_m x_n = \delta_{m,n}$: Kronecker (unit) tensor (tensor of rank 2)
  (• $\delta_{m,n} = 1$ if $m = n$, • $\delta_{m,n} = 0$ if $m \neq n$)

• SUMMATION CONVENTION (repeated subscripts):

- $\delta_{p,n}\delta_{n,q} = \sum_{n=1}^{N} \delta_{p,n}\delta_{n,q} = \delta_{p,q}$
- $\delta_{m,m} = \sum_{m=1}^{N} \delta_{m,m} = N$

Tensorial nature of physical wavefields (Einstein)

	Intensive field quantity 'field strength'	Extensive field quantity 'flux density'	Source quantity 'volume density of' 'current'
Electric	E_k	D_k	J_k
Magnetic	$H_{i,j} = -H_{j,i}$	$B_{i,j} = -B_{j,i}$	$K_{i,j} = -K_{j,i}$
	'PRODUCT'	'PRODUCT'	
	$S_m = H_{m,k} E_k$ area density of EM power	$G_i = B_{i,j} D_j$ volume density of EM momentum	

- $\{H_{i,j}, B_{i,j}, K_{i,j}\}$ (anti-symmetric, rank 2)

EM field and source quantities

- **WAVEFIELDS:**
  - Changes in space ($\partial_m = \partial/\partial x_m$) **counterbalance**
  - Changes in time ($\partial_t = \partial/\partial t$)

EM field equations (Maxwell)

- $\partial_m H_{m,k} + \partial_t D_k = J_k$
- $\partial_i E_j - \partial_j E_i + \partial_t B_{i,j} = K_{i,j}$

- **FIELD/SOURCE COMPATIBILITY RELATIONS:**

- $\partial_k \partial_m H_{m,k} = \partial_m \partial_k H_{m,k} = -\partial_m \partial_k H_{k,m}$ = - $\partial_k \partial_m H_{m,k} = 0$

  $\Longrightarrow$ • $\partial_t(\partial_k D_k) = \partial_k J_k$

- $\partial_k(\partial_i E_j - \partial_j E_i) + \partial_i(\partial_j E_k - \partial_k E_j) + \partial_j(\partial_k E_i - \partial_i E_k) = 0 \ (i \neq j \neq k)$

  $\Longrightarrow$ • $\partial_t(\partial_k B_{i,j} + \partial_i B_{j,k} + \partial_j B_{k,i}) = \partial_k K_{i,j} + \partial_i K_{j,k} + \partial_j K_{k,i} \ (i \neq j \neq k)$

EM field equations and field/source compatibility relations

• VOLUME DENSITY OF ELECTRIC CHARGE:

• $\boxed{\rho = -\partial_t^{-1}\partial_k J_k}$ (scalar)

$\Longrightarrow$ • $\partial_k D_k = -\rho$

• VOLUME DENSITY OF MAGNETIC CHARGE:

• $\boxed{\sigma_{k,i,j} = -\partial_t^{-1}(\partial_k K^-_{i,j} + \partial_i K^-_{j,k} + \partial_j K^-_{k,i})\ (i \neq j \neq k)}$

(completely antisymmetric tensor of rank 3)

$\Longrightarrow$ • $\partial_k B^-_{i,j} + \partial_i B^-_{j,k} + \partial_j B^-_{k,i} = -\sigma_{k,i,j}\ (i \neq j \neq k)$

Volume densities of electric and magnetic charge

• CAUSALITY (UNIQUENESS OF TIME EVOLUTION = UNIQUENESS OF INITIAL-VALUE PROBLEM):

- **EM field**$(\boldsymbol{x}, t_0)$ for $\boldsymbol{x} \in \mathbb{R}^3 \overset{\text{Field equations}}{\Longrightarrow \Longrightarrow}$
  EM field$(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3, t > t_0$ (uniquely)
- **'Maxwell'**: # Equations $\neq$ # Unknowns $\Longrightarrow$

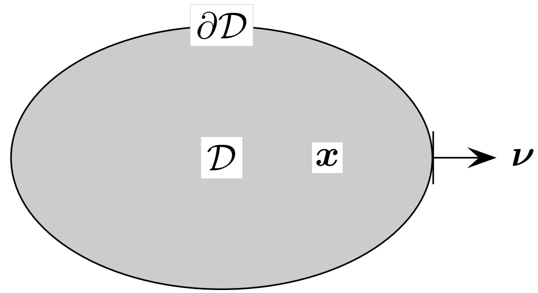
• CONSTITUTIVE RELATIONS (simplest case):

- $D_k = \epsilon E_k$ (ϵ = electric permittivity)
- $B_{i,j} = \mu H_{i,j}$ (μ = magnetic permeability)
- $\Longrightarrow$ **'Maxwell'**: # Equations $=$ # Unknowns

Causality, uniqueness of initial-value problem

• EM ENERGY BALANCE (local & global over domain $\mathcal{D}$):

EM field equations at $(\boldsymbol{x}, t)$	
• $\boldsymbol{\partial_m} H_{m,k} + \boldsymbol{\partial_t} D_k = J_k$	E_k
• $\boldsymbol{\partial_i} E_j] \boldsymbol{\partial_j} E_i + \mu \boldsymbol{\partial_t} H_{i,j} = K_{i,j}$	$H_{i,j}/2$
• EM energy balance	$+$

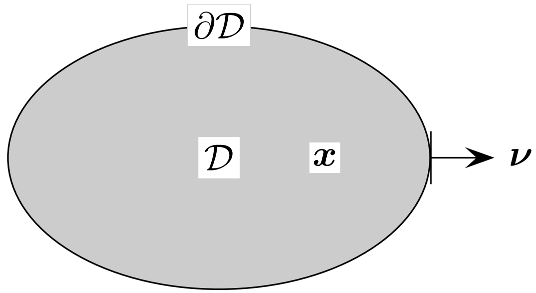


• $\partial_m S_m + \partial_t w = P$ (local)	
• $S_m = H_{m,k} E_k$	area density of EM power flow (Poynting vector)
• $w = E_k D_k/2 + H_{i,j} B_{i,j}/4$	volume density of stored EM energy
• $P = J_k E_k + K_{i,j} H_{i,j}/2$	volume density of EM power delivered by sources
'GAUSS' $\Longrightarrow$ • $\int_{\partial\mathcal{D}} S_m \nu_m \mathrm{d}A + \partial_t \int_{\mathcal{D}} w \mathrm{d}V = \int_{\mathcal{D}} P \mathrm{d}V$ (global)	

EM energy balance (local & global)

• BALANCE OF EM MOMENTUM (local & global over domain $\mathcal{D}$):

EM field equations at $(\boldsymbol{x}, t)$	
• $\partial_m H_{m,k} + \partial_t D_k = J_k$	$B_{i,k}$
• $\partial_i E_j - \partial_j E_i + \partial_t B_{i,j} = K_{i,j}$	D_j
• balance of EM momentum	$+$



	• $-\partial_i T_{i,j} + \partial_t G_j = f_j$ (local)
$T_{i,j} =$ •	Maxwell stress tensor ('EM radiation pressure')
$G_i =$ •	volume density of EM momentum
$f_j =$ •	volume density of force exerted by the sources
'GAUSS' $\Longrightarrow$	• $-\int_{\partial\mathcal{D}} \nu_i T_{i,j} \mathrm{d}A + \partial_t \int_{\mathcal{D}} G_j \mathrm{d}V = \int_{\mathcal{D}} f_j \mathrm{d}V$ (global)

Balance of EM momentum (local & global)

- **MAXWELL STRESS TENSOR:**

$$T_{i,j} = E_i D_j - (E_k D_k/2)\delta_{i,j} - H_{i,k} B_{j,k} + (H_{m,k} B_{m,k}/4)\delta_{i,j}$$

- **VOLUME DENSITY OF EM MOMENTUM:**

$$G_i = B_{i,j} D_j$$

- **VOLUME DENSITY OF FORCE EXERTED BY THE SOURCES:**

$$f_j = B_{j,k} J_k + K_{j,k} D_k - E_j \rho + H_{m,k} \sigma_{m,k,j}/2$$

• MAXWELL STRESS TENSOR (AUXILIARY RELATIONS D_k):

- $D_j(\partial_i E_j - \partial_j E_i) =$

  $\partial_i(E_j D_j/2) - \partial_j(E_i D_j) + E_i \partial_j D_j]$

  $\partial_i(E_j D_j/2) - \partial_j(E_i D_j) + E_i \rho$

• MAXWELL STRESS TENSOR (AUXILIARY RELATIONS $B^{-}_{i,j}$):

- $B^{-}_{i,k}\partial_m H_{m,k} \quad = \quad \partial_m(B_{i,k}H_{m,k}) - H_{m,k}\partial_m B_{i,k}$
- $H_{m,k}\partial_m B_{i,k} \quad\quad H_{m,k}(-\partial_i B_{k,m} - \partial_k B_{m,i}) + \partial_t^{-1}(\partial_m K_{i,k} + \partial_i K_{k,m} + \partial_k K_{m,i})$

$$\partial_i(H_{m,k}B_{m,k}/2) - B_{m,k}\partial_k B_{m,i} - H_{m,k}\sigma_{m,i,k}$$

$$= \partial_i(H_{m,k}B_{m,k}/2) - B^{-}_{m,k}\partial_m B_{i,k} - H_{m,k}\sigma_{m,i,k}$$

$$\Longrightarrow$$

- $B_{i,k}\partial_m H_{m,k} \quad = \quad \partial_i(H_{m,k}B_{m,k}/4) - H_{m,k}\sigma_{m,i,k}/2$

Maxwell stress tensor (auxiliary relations $\boldsymbol{B}^{-}_{i,j}$)

• EM RADIATION FROM SOURCES:

EM field equations at $(\boldsymbol{x}, t)$	
• $\partial_m H_{m,k} + \epsilon\,\partial_t E_k = J_k$	$\mu\,\partial_t$
• $\partial_i E_j - \partial_j E_i + \mu\,\partial_t H_{i,j} = K_{i,j}$	$\Uparrow$

- Elimination of $H^-_{i,j}$ + use of
- $\partial_k E_k = \epsilon^{-1}\partial_t^{-1}(\partial_m J_m)$ (compatibility relation) $\Longrightarrow$

• ELECTRIC-FIELD VECTOR WAVE EQUATION:

- $(\partial_m\partial_m)E_k - c^{-2}\partial_t^2 E_k = -F_k$

- $c = (\epsilon\,\mu)^{-1/2}$ (EM wavespeed)
- $F_k = \mu\,\partial_t J_k - \epsilon^{-1}\partial_t^{-1}\partial_k(\partial_m J_m) - \partial_m K_{m,k}$

• ELECTRIC-FIELD VECTOR WAVE EQUATION:

- $(\partial_m\partial_m)E_k - c^{-2}\partial_t^2 E_k = -F_k$
  $= -\delta(\boldsymbol{x},t) \overset{(\boldsymbol{x})}{*} \overset{(t)}{*} F_k$

- $F_k = \mu\,\partial_t J_k - \epsilon^{-1}\partial_t^{-1}\partial_k(\partial_m J_m) - \partial_m K_{m,k}$

$\Longrightarrow$ **SOLUTION:**

- $E_k = G \overset{(\boldsymbol{x},t)}{*} F_k$
- $(\partial_m\partial_m)G - c^{-2}\partial_t^2 G = -\delta(\boldsymbol{x},t)$
- $G = \dfrac{\delta(t - |\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|}$ for $\boldsymbol{x} \neq \boldsymbol{0}$ (3D)

- $\overset{(\boldsymbol{x})}{*} \overset{(t)}{*} \partial_m\partial_t = \partial_m\partial_t \overset{(\boldsymbol{x},t)}{*}$ $\Downarrow$

The electric-field vector wave equation and its solution

- **THE RADIATED FIELD:** (• $E_k = G \overset{(\boldsymbol{x})(t)}{* \;\; *} F_k \Longrightarrow$)

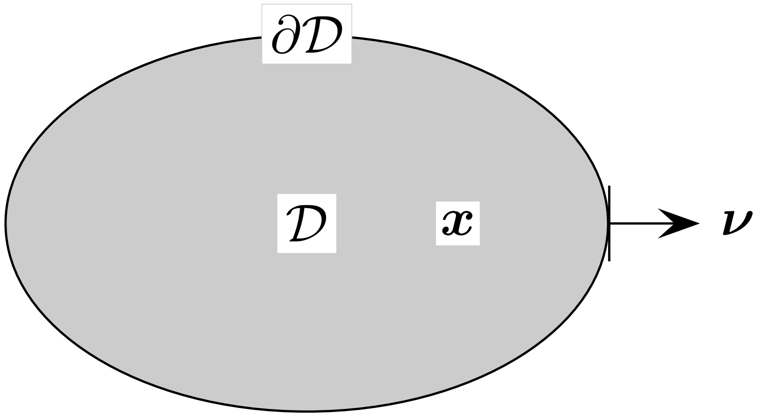
- $E_k = \mu\, \partial_t A_k - \epsilon^{-1} \partial_t^{-1} \partial_k \partial_m A_m - \partial_m \Psi_{m,k}$
- $H_{i,j} = \epsilon\, \partial_t \Psi_{i,j}$ - $\mu^{-1} \partial_t^{-1} \partial_m [\partial_m \Psi_{i,j} + \partial_i \Psi_{j,m} + \partial_j \Psi_{m,i}] - \partial_i A_j + + \partial_j A_i$

- **VECTOR POTENTIAL:**
  - $A_k = G \overset{(\boldsymbol{x})(t)}{* \;\; *} J_k$
- **TENSOR POTENTIAL:**
  - $\Psi_{i,j} = G \overset{(\boldsymbol{x})(t)}{* \;\; *} K_{i,j}$
- **SCALAR GREEN'S FUNCTION:**
  - $G = \dfrac{\delta(t - |\boldsymbol{x}|/c)}{4\pi |\boldsymbol{x}|}$ for $\boldsymbol{x} \neq \mathbf{0}$ (3D)

The radiated field (vector and tensor potentials)

• FIELD RECIPROCITY (interaction of two STATES):

Reciprocity $(\boldsymbol{x} \in \mathcal{D} \cup \partial\mathcal{D},\ t \in \mathbb{R})$			
State	Field	Medium	Source
A	$E_k^A, H_{i,j}^A$	ϵ^A, μ^A	$J_k^A, K_{i,j}^A$
B	$E_k^B, H_{i,j}^B$	ϵ^B, μ^B	$J_k^B, K_{i,j}^B$



EM field reciprocity

• TIME-CONVOLUTION FIELD RECIPROCITY:

State	EM field equations ($\boldsymbol{x} \in \mathcal{D} \cup \partial\mathcal{D},\ t \in \mathbb{R}$)	
A	• $\partial_m H^A_{m,k}(t) + \epsilon^A \partial_t E^A_k(t) = J^A_k(t)$	$\overset{(t)}{*} E^B_k(t)$
B	• $\partial_i E^B_j(t) - \partial_j E^B_i(t) + \mu^B \partial_t H^B_{i,j}(t) = K^B_{i,j}(t)$	$\overset{(t)}{*} H^A_{i,j}(t)/2$
	• $-$ $A \rightleftharpoons B$	

• LOCAL:

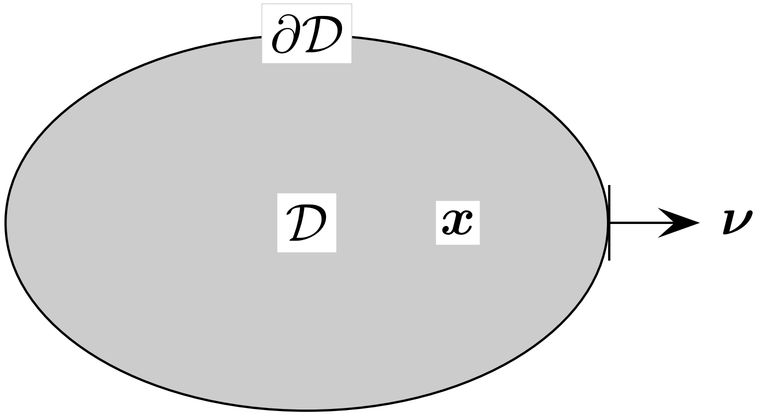
$$\partial_m[\,H^A_{m,k}(t) \overset{(t)}{*} E^B_k(t) - H^B_{m,k}(t) \overset{(t)}{*} E^A_k(t)\,] +$$

$$\partial_t(\,\epsilon^A - \epsilon^B\,)E^A_k(t) \overset{(t)}{*} E^B_k(t) + \partial_t(\,\mu^B - \mu^A\,)H^B_{i,j}(t) \overset{(t)}{*} H^A_{i,j}(t)/2 =$$

$$J^A_k(t) \overset{(t)}{*} E^B_k(t) + K^B_{i,j}(t) \overset{(t)}{*} H^A_{i,j}(t)/2 - J^B_k(t) \overset{(t)}{*} E^A_k(t) - K^A_{i,j}(t) \overset{(t)}{*} H^B_{i,j}(t)/2$$

Time-convolution EM field reciprocity (local)

• TIME-CONVOLUTION FIELD RECIPROCITY:



• GLOBAL: ('GAUSS')

• $$\int_{\partial\mathcal{D}} \nu_m [H^A_{m,k}(t) \overset{(t)}{*} E^B_k(t) - H^B_{m,k}(t) \overset{(t)}{*} E^A_k(t)] \mathrm{d}A +$$
$$\int_{\mathcal{D}} \{(\epsilon^A - \epsilon^B)\partial_t[E^A_k(t) \overset{(t)}{*} E^B_k(t)] + (\mu^B - \mu^A)\partial_t[H^B_{i,j}(t) \overset{(t)}{*} H^A_{i,j}(t)]/2\}\mathrm{d}V =$$
$$\int_{\mathcal{D}} [J^A_k(t) \overset{(t)}{*} E^B_k(t) + K^B_{i,j}(t) \overset{(t)}{*} H^A_{i,j}(t)/2 - J^B_k(t) \overset{(t)}{*} E^A_k(t) - K^A_{i,j}(t) \overset{(t)}{*} H^B_{i,j}(t)/2]\mathrm{d}V$$

Time-convolution EM field reciprocity (global)

• TIME-CORRELATION FIELD RECIPROCITY:

State	EM field equations ($\boldsymbol{x} \in \mathcal{D} \cup \partial\mathcal{D},\ t \in \mathbb{R}$)	
A	• $\partial_m H^A_{m,k}(t) + \epsilon^A \partial_t E^A_k(t) = J^A_k(t)$	$\overset{(t)}{*}\ E^B_k(-t)$
B	• $\partial_i E^B_j(-t) - \partial_j E^B_i(-t) - \mu^B \partial_t H^B_{i,j}(-t) = K^{-;B}_{i,j}(-t)$	$\overset{(t)}{*}\ H^A_{i,j}(t)/2$
	• $+\ A \rightleftharpoons B$	

• LOCAL:

• $\partial_m[\, H^A_{m,k}(t) \overset{(t)}{*} E^B_j(-t) + H^B_{m,k}(-t) \overset{(t)}{*} E^A_j(t)\,] +$

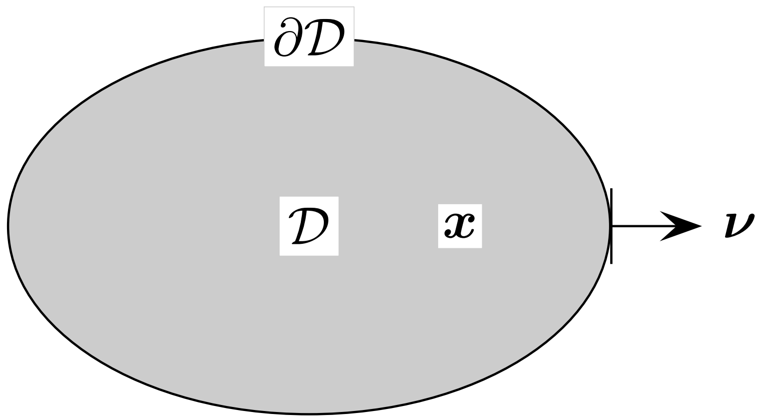
$(\epsilon^A - \epsilon^B)\partial_t[E^A_j(t) \overset{(t)}{*} E^B_j(-t)] + (\mu^A - \mu^B)\partial_t[H^B_{i,j}(-t) \overset{(t)}{*} H^A_{i,j}(t)]/2 =$

$J^A_k(t) \overset{(t)}{*} E^B_k(-t) + K^B_{i,j}(-t) \overset{(t)}{*} H^A_{i,j}(t)/2 +$

$J^B_k(-t) \overset{(t)}{*} E^A_k(t) + K^A_{i,j}(t) \overset{(t)}{*} H^B_{i,j}(-t)/2$

Time-correlation EM field reciprocity (local)

• TIME-CORRELATION FIELD RECIPROCITY:



• GLOBAL: ('GAUSS')

• $$\int_{\partial\mathcal{D}} \nu_m[H^A_{m,k}(t) \overset{(t)}{*} E^B_j(-t) + H^B_{m,k}(-t) \overset{(t)}{*} E^A_j(t)]\mathrm{d}A +$$

$$\int_{\mathcal{D}} [(\epsilon^A - \epsilon^B)\partial_t[E^A_j(t) \overset{(t)}{*} E^B_j(-t)] + (\mu^A - \mu^B)\partial_t[H^B_{i,j}(-t) \overset{(t)}{*} H^A_{i,j}(t)/2]\mathrm{d}V =$$

$$\int_{\mathcal{D}} [J^A_k(t) \overset{(t)}{*} E^B_k(-t) + K^B_{i,j}(-t) \overset{(t)}{*} H^A_{i,j}(t)/2 +$$

$$J^B_k(-t) \overset{(t)}{*} E^A_k(t) + K^A_{i,j}(t) \overset{(t)}{*} H^B_{i,j}(-t)/2]\mathrm{d}V$$

Time-correlation EM field reciprocity (global)

• LORENTZ TRANSFORMATION (COVARIANCE CRITERION) :

$$\overline{x}_\mu = x_\mu$$
$$\overline{x}_N = \alpha(x_N - w_N t)$$
$$\overline{t} = \gamma\, x_N + \alpha\, t$$

$\Longrightarrow$

$$\partial_\mu = \overline{\partial}_\mu$$
$$\partial_N = \alpha\, \overline{\partial}_N + \gamma\, \overline{\partial}_t$$
$$\partial_t = \alpha\, \overline{\partial}_t - \alpha\, w_N\, \overline{\partial}_N$$

• $\{\alpha, \gamma\}$ follow from:

- **EM field equations** $\mathcal{O}(\boldsymbol{x}, t)$
- **Compatibility relations** $\mathcal{O}(\boldsymbol{x}, t)$
- $\{\epsilon, \mu\}$

$\Longleftrightarrow$

$\Longleftrightarrow$

- $\overline{\textbf{EM field equations}}$ $\overline{\mathcal{O}}(\overline{\boldsymbol{x}}, \overline{t})$
- $\overline{\textbf{Compatibility relations}}$ $\overline{\mathcal{O}}(\overline{\boldsymbol{x}}, \overline{t})$
- $\{\epsilon, \mu\}$

Lorentz transformation (covariance criterion)

• LORENTZ
• TRANSFORMATION

Space & Time
$\overline{x}_\mu = x_\mu$ $\overline{x}_N = \alpha(x_N - w_N t)$ $\overline{t} = \alpha(t - w_N x_N/c^2)$
$\alpha = (1 - w_N^2/c^2)^{-1/2}$
$c = (\epsilon\,\mu)^{-1/2}$

Field quantities
$\overline{E}_\mu = \alpha(E_\mu + B_{\mu,N} w_N)$ $\overline{E}_N = E_N$
$\overline{H}_{\mu,\nu} = H_{\mu,\nu}$ $\overline{H}_{\mu,N} = \alpha(H_{\mu,N} + D_\mu w_N)$
Source quantities
$\overline{J}_\mu = J_\mu$ $\overline{J}_N = \alpha(J_N + \rho\, w_N)$
$\overline{\rho} = \alpha(\rho + \epsilon\,\mu J_N w_N)$
$\overline{K}_{\mu,\nu} = \alpha(K_{\mu,\nu} + \sigma_{\mu,\nu,N} w_N)$ $\overline{K}_{\mu,N} = K_{\mu,N}$
$\overline{\sigma}_{\mu,\nu,N} = \alpha(\sigma_{\mu,\nu,N} + \epsilon\,\mu K_{\mu,\nu} w_N)$

$\epsilon\,\mu J_N w_N$

$\epsilon\,\mu K_{\mu,\nu} w_N$

Relativistic corrections to Reynolds' transport theorem

Lorentz transformation (space, time, field & source quantities)

- **FULL TENSORIAL EM THEORY:**
- Involves only elementary mathematics ($\partial_m \partial_n = \partial_n \partial_m, \overset{(t)}{*}, \overset{(\boldsymbol{x})}{*}$)
- Does away with standard ~~VECTOR CALCULUS~~
- Does away with the condition of ~~RIGHT-HANDEDNESS~~ in orientation of spatial reference frame
- Reformulates the notion of the '~~MAGNETIC MONOPOLE~~' (= tensor of rank 3)
- Considers SpaceTime as an 'affine' space (not a 'metric' space) $\Longrightarrow$ ~~4D LORENTZ 'METRIC'~~
- Has implications for 'string theory' (Dirac) in quantum electrodynamics (no 'vector potential' as the basic ingredient)
- Maps straightforwardly on 'array' handling in computational electromagnetics