Elastodynamics - Fundamentals of wave propagation, scattering, imaging, inversion and computation

by

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Synopsis:

- Preliminaries (tensors, subscript notation, summation convention)
- Elastodynamic wave field and source quantities
- Elastodynamic wave field equations
- Elastodynamic constitutive relations
- Elastic waves radiated by sources, elastodynamic Green's tensors
- Elastodynamic scattering theory (contrast-source formulation)
- The elastodynamic inverse source (imaging) problem
- The elastodynamic inverse scatterer constitution problem
- Elastodynamic wave field computation (space-time integrated field equations)

- **Observational spacetime:** $\mathbb{R}^3 \times \mathbb{R}$
- **Position:** $\boldsymbol{x} \in \mathbb{R}^3$
- **(Elapsed) time:** $t \in \mathbb{R}$

- **Tensor subscript notation x:** $x_m \ (m = 1, 2, 3)$
- **Spatial differentiation $\partial/\partial x_m$:** $\partial_m \ (m = 1, 2, 3)$
- **Temporal differentiation $\partial/\partial t$:** ∂_t (reserved symbol)
- **Kronecker unit tensor $\delta_{m,n}$:** $\delta_{m,n} = 1$ for $m = n$, $\delta_{m,n} = 0$ for $m \neq n$
- **Summation convention (repeated subscripts):** $\delta_{p,n}\delta_{n,q} = \sum_{n=1}^{3} \delta_{p,n}\delta_{n,q} = \delta_{p,q}$
- **Examples:** • $\partial_m x_n = \delta_{m,n}$ • $\delta_{n,n} = 3$

Tensors, subscript notation, summation convention

- **Elastodynamic unit tensors of rank four:**
  - $\Delta^{+}_{i,j,p,q}$
  - $\Delta^{\delta}_{i,j,p,q}$
  - $\Delta^{\Delta}_{i,j,p,q}$

- **Symmetrical unit tensor:** $\Delta^{+}_{i,j,p,q} = (\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})/2$
- **Diagonal unit tensor:** $\Delta^{\delta}_{i,j,p,q} = \delta_{i,j}\delta_{p,q}/3$
- **Deviatoric unit tensor:** $\Delta^{\Delta}_{i,j,p,q} = \Delta^{+}_{i,j,p,q} - \Delta^{\delta}_{i,j,p,q}$

- **Properties:**
  - $\Delta^{\delta}_{i,j,m,n}\Delta^{\delta}_{m,n,p,q} = \Delta^{\delta}_{i,j,p,q}$
  - $\Delta^{\Delta}_{i,j,m,n}\Delta^{\Delta}_{m,n,p,q} = \Delta^{\Delta}_{i,j,p,q}$
  - $\Delta^{\delta}_{i,j,m,n}\Delta^{\Delta}_{m,n,p,q} = 0$
  - $\Delta^{\Delta}_{i,j,m,n}\Delta^{\delta}_{m,n,p,q} = 0$

- $\Longrightarrow$ **MUTUALLY ORTHOGONAL UNIT TENSORS!**

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

Elastodynamic unit tensors of rank four

Elastodynamic field and source quantities		
• $-\tau_{p,q}\ (=-\tau_{q,p})$	: $-$ dynamic stress	intensive[1]
• v_r	: particle velocity	field quantities
• $-e_{i,j}\ (=-e_{j,i})$	: $-$ dynamic strain	extensive[2]
• Φ_k	: mass flow density	field quantities
• f_k	: volume density of force	source
• $h_{i,j}(=h_{j,i})$	: volume density of deformation rate	quantities

- [1] **Area density of power flow:**

  - $S_m^{\text{elast}} = -\Delta^+_{m,p,q,r}\tau_{p,q}v_r = -\tau_{m,q}v_q$

- [2] **Volume density of wave momentum:**

  - $G_n^{\text{elast}} = -\Delta^+_{n,i,j,k}e_{i,j}\Phi_k = -e_{n,j}\Phi_j$

Elastodynamic field and source quantities, power flow, wave momentum

- **WAVEFIELDS:**
  - Changes in space ($\partial_m = \partial/\partial x_m$) **counterbalance**
  - Changes in time ($\partial_t = \partial/\partial t$)

Elastodynamic wave equations	
$-\Delta^{+}_{k,m,p,q}\,\partial_m\,\tau_{p,q} + \partial_t\,\Phi_k = f_k$	equation of motion
$\Delta^{+}_{i,j,n,r}\,\partial_n\,v_r - \partial_t\,e_{i,j} = h_{i,j}$	deformation rate equation

- **Constitutive relations (linear, time-invariant, causal, locally reacting):**

- $\Phi_k(\boldsymbol{x},t) = \rho_{k,r}(\boldsymbol{x},t) \overset{(t)}{*} v_r(\boldsymbol{x},t)$
- $\rho_{k,r}(\boldsymbol{x},t) = 0$ for $t < 0$ **(causality)**
  - $\rho_{k,r}(\boldsymbol{x},t)$ = **inertia relaxation function**
- $\overset{(t)}{*}$ = **time convolution**
- $e_{i,j}(\boldsymbol{x},t) = S_{i,j,p,q}(\boldsymbol{x},t) \overset{(t)}{*} \tau_{p,q}(\boldsymbol{x},t)$
- $S_{i,j,p,q}(\boldsymbol{x},t) = 0$ for $t < 0$ **(causality)**
  - $S_{i,j,p,q}(\boldsymbol{x},t)$ = **compliance relaxation function**

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

Elastodynamic wave equations, constitutive relations

MATERIAL PROPERTIES

- **Lossless** (instantaneously reacting)

$$\rho_{k,r}(\boldsymbol{x},t) = \rho_{k,r}(\boldsymbol{x})\,\boxed{\delta(t)}$$

$$S_{i,j,p,q}(\boldsymbol{x},t) = S_{i,j,p,q}(\boldsymbol{x})\,\boxed{\delta(t)}$$

- **Homogeneous** (spatially shift invariant)

$$\rho_{k,r}(\boldsymbol{x},t) = \rho_{k,r}(t)$$

$$S_{i,j,p,q}(\boldsymbol{x},t) = S_{i,j,p,q}(t)$$

- **Isotropic** (orientation invariant)

$$\rho_{k,r}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\,\boxed{\delta_{k,r}}$$

$$S_{i,j,p,q}(\boldsymbol{x},t) = S^{\delta}(\boldsymbol{x},t)\,\boxed{\Delta^{\delta}_{i,j,p,q}} + S^{\Delta}(\boldsymbol{x},t)\,\boxed{\Delta^{\Delta}_{i,j,p,q}}$$

ELASTIC TENSOR PARTITIONING

- **Dynamic stress** **(dynamic strain: $\tau_{p,q} \Longrightarrow e_{i,j}$)**
  - $\tau_{p,q} = [\tau_{p,q}]^+ = [\tau_{p,q}]^\delta + [\tau_{p,q}]^\Delta$
  - $[\tau_{p,q}]^+ = \Delta^+_{p,q,i,j}\tau_{i,j} = (\tau_{p,q} + \tau_{q,p})/2$
  - $[\tau_{p,q}]^\delta = \Delta^\delta_{p,q,i,j}\tau_{i,j} = (\tau_{i,i}/3)\delta_{p,q}$
  - $[\tau_{p,q}]^\Delta = \Delta^\Delta_{p,q,i,j}\tau_{i,j} = \Delta^+_{p,q,i,j}\tau_{i,j} - \Delta^\delta_{p,q,i,j}\tau_{i,j} = (\tau_{p,q} + \tau_{q,p})/2 - (\tau_{i,i}/3)\delta_{p,q}$
- **Compliance**
  - $$\begin{bmatrix} [e_{i,j}]^\delta \\ [e_{i,j}]^\Delta \end{bmatrix}(\boldsymbol{x}, t) = \begin{bmatrix} [S_{i,j,p,q}]^{\delta,\delta} & [S_{i,j,p,q}]^{\delta,\Delta} \\ [S_{i,j,p,q}]^{\Delta,\delta} & [S_{i,j,p,q}]^{\Delta,\Delta} \end{bmatrix}(\boldsymbol{x}, t) \overset{(t)}{*} \begin{bmatrix} [\tau_{p.q}]^\delta \\ [\tau_{p,q}]^\Delta \end{bmatrix}(\boldsymbol{x}, t)$$

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

ELASTIC WAVE PROPAGATION

- **General structure:** **source $\overset{\text{propagator}}{\Longrightarrow}$ receiver (intensive field)**

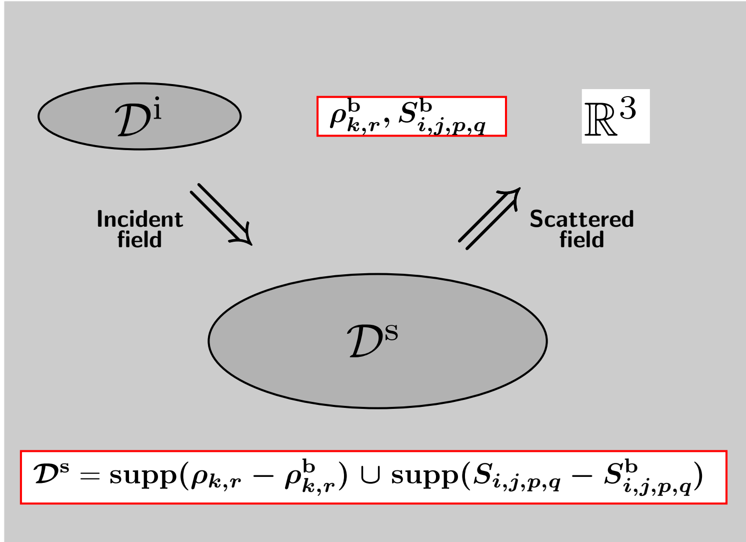
- $[F](\boldsymbol{x},t) = \int_{\boldsymbol{x}'\in\mathrm{supp}(Q)} [G](\rho, S, \boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} [Q](\boldsymbol{x}', t)\mathrm{d}V(\boldsymbol{x}')$

Field	**Green's tensor**	**Sources**
• $[F] = \begin{bmatrix} -[\tau_{p,q}]^{\delta} \\ -[\tau_{p,q}]^{\Delta} \\ v_r \end{bmatrix}$	• $[G] = \begin{bmatrix} [G_{p,q,k}]^{\delta} & [G_{p,q,i,j}]^{\delta,\delta} & [G_{p,q,i,j}]^{\delta,\Delta} \\ [G_{p,q,k}]^{\Delta} & [G_{p,q,i,j}]^{\Delta,\delta} & [G_{p,q,i,j}]^{\Delta,\Delta} \\ [G_{r,k}] & [G_{r,i,j}]^{\delta} & [G_{r,i,j}]^{\Delta} \end{bmatrix}$	• $[Q] = \begin{bmatrix} f_k \\ [h_{i,j}]^{\delta} \\ [h_{i,j}]^{\Delta} \end{bmatrix}$

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

Elastic wave propagation, Green's tensor (propagator)

• **Total field**		support
Wave field	$-\tau_{p,q}, v_r$	$\mathbb{R}^3$
Actual medium	$\rho_{k,r}, S_{i,j,p,q}$	$\mathbb{R}^3$
Sources	$f^{\mathrm{i}}_k, h^{\mathrm{i}}_{i,j}$	$\mathcal{D}^{\mathrm{i}} \subset \mathbb{R}^3$
• **Incident field**		support
Wave field	$-\tau^{\mathrm{i}}_{p,q}, v^{\mathrm{i}}_r$	$\mathbb{R}^3$
Background medium	$\rho^{\mathrm{b}}_{k,r}, S^{\mathrm{b}}_{i,j,p,q}$	$\mathbb{R}^3$
Sources	$f^{\mathrm{i}}_k, h^{\mathrm{i}}_{i,j}$	$\mathcal{D}^{\mathrm{i}} \subset \mathbb{R}^3$
• **Scattered field**		support
Wave field	$-\tau^{\mathrm{s}}_{p,q}, v^{\mathrm{s}}_r$	$\mathbb{R}^3$
Background medium	$\rho^{\mathrm{b}}_{k,r}, S^{\mathrm{b}}_{i,j,p,q}$	$\mathbb{R}^3$
Sources	$f^{\mathrm{s}}_k, h^{\mathrm{s}}_{i,j}$	$\mathcal{D}^{\mathrm{s}} \subset \mathbb{R}^3$



SCATTERED FIELD = TOTAL FIELD − INCIDENT FIELD

Scattering configuration

SCATTERED FIELD = TOTAL FIELD − INCIDENT FIELD

- **Total field**

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau_{p,q} + \rho_{k,r} \overset{(t)}{*} \partial_t v_r = f^{\mathrm{i}}_k$$
$$\Delta^{+}_{i,j,n,r}\partial_n v_r - S_{i,j,p,q} \overset{(t)}{*} \partial_t \tau_{p,q} = h^{\mathrm{i}}_{i,j}$$

- **Incident field**

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{\mathrm{i}}_{p,q} + \rho^{\mathrm{b}}_{k,r} \overset{(t)}{*} \partial_t v^{\mathrm{i}}_r = f^{\mathrm{i}}_k$$
$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{i}}_r - S^{\mathrm{b}}_{i,j,p,q} \overset{(t)}{*} \partial_t \tau^{\mathrm{i}}_{p,q} = h^{\mathrm{i}}_{i,j}$$

- **Scattered field**

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{\mathrm{s}}_{p,q} + \rho^{\mathrm{b}}_{k,r} \overset{(t)}{*} \partial_t v^{\mathrm{s}}_r = f^{\mathrm{s}}_k$$
$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{s}}_r - S^{\mathrm{b}}_{i,j,p,q} \overset{(t)}{*} \partial_t \tau^{\mathrm{s}}_{p,q} = h^{\mathrm{s}}_{i,j}$$

- **Contrast sources**

- $f^{\mathrm{s}}_k = -(\rho_{k,r} - \rho^{\mathrm{b}}_{k,r}) \overset{(t)}{*} \partial_t v_r$
- $h^{\mathrm{s}}_{i,j} = (S_{i,j,p,q} - S^{\mathrm{b}}_{i,j,p,q}) \overset{(t)}{*} \partial_t \tau_{p,q}$

Scattering problem (contrast source formulation)

FORWARD/DIRECT SCATTERING PROBLEM

- **Input:** medium: $\{\rho_{k,r}, S_{i,j,p,q}\}$, background: $\{\rho^{\rm b}_{k,r}, S^{\rm b}_{i,j,p,q}\}$,

  irradiating sources: $\{f^{\rm i}_k, h^{\rm i}_{i,j}\}$
- **Output:** incident field: $\{-\tau^{\rm i}_{p,q}, v^{\rm i}_r\}$, scattered field: $\{-\tau^{\rm s}_{p,q}, v^{\rm s}_r\}$

- $$\left[F^{\rm i,s}\right](\boldsymbol{x},t) = \int_{\boldsymbol{x}'\in{\rm supp}(Q^{\rm i,s})} [G]\left(\rho^{\rm b}, S^{\rm b}, \boldsymbol{x}, \boldsymbol{x}', t\right) \overset{(t)}{*} \left[Q^{\rm i,s}\right](\boldsymbol{x}',t){\rm d}V(\boldsymbol{x}')$$

$$\begin{bmatrix} f^{\rm s}_k \\ h^{\rm s}_{i,j} \end{bmatrix} = \begin{bmatrix} -(\rho_{k,r} - \rho^{\rm b}_{k,r}) \overset{(t)}{*} \partial_t(v^{\rm i}_r + v^{\rm s}_r) \\ (S_{i,j,p,q} - S^{\rm b}_{i,j,p,q}) \overset{(t)}{*} \partial_t(\tau^{\rm i}_{p,q} + \tau^{\rm s}_{p,q}) \end{bmatrix}$$

- $\{\boldsymbol{x}, t\} \in \{{\rm supp}([Q^{\rm s}]), \mathbb{R}\}$ $\Longrightarrow$ integral equation of the 2nd kind

in $[F](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{{\rm supp}([Q^{\rm s}]),\mathbb{R}\}}$ or in $[Q^{\rm s}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{{\rm supp}([Q^{\rm s}]),\mathbb{R}\}}$

Forward/Direct scattering problem (integral-equation formulation)

INVERSE SOURCE (IMAGING) PROBLEM

- **Input:** background medium: $\{\rho^{\rm b}_{k,r}, S^{\rm b}_{i,j,p,q}\}$; irradiating sources: $\{f^{\rm i}_k, h^{\rm i}_{i,j}\}$;

  support(scatterer): $\mathcal{D}^{\rm s}$; observational domain: $\mathcal{D}^{\Omega} \not\subset \mathcal{D}^{\rm s}$;

  incident wave field: $[F]^{\rm i}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathbb{R}^3,\mathbb{R}\}}$;

  scattered wave field: $[F]^{\rm s}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\Omega},\mathbb{R}\}}$ (data)

- **Desired output:** scattering sources: $[Q^{\rm s}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\rm s},\mathbb{R}\}}$ (model)

- **MODEL/DATA EQUATION:**

- $$[F^{\rm s}](\boldsymbol{x},t) = \int_{\boldsymbol{x}'\in{\rm supp}(Q^{\rm s})} [G](\rho^{\rm b}, S^{\rm b}, \boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} [Q^{\rm s}](\boldsymbol{x}',t){\rm d}V(\boldsymbol{x}')$$

NOTE: • $[F^{\rm s}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\Omega},\mathbb{R}\}} \overset{\text{~~1-to-1~~}}{\Longrightarrow} [Q^{\rm s}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\rm s},\mathbb{R}\}}$

$[Q^{\rm s}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\rm s},\mathbb{R}\}} \simeq [Q^{\rm s}]^{[\infty]}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\rm s},\mathbb{R}\}}$ (minimum norm solution)

INVERSE SCATTERER CONSTITUTION PROBLEM

- **Input:** background medium: $\{\rho^{\mathrm{b}}_{k,r}, S^{\mathrm{b}}_{i,j,p,q}\}$; irradiating sources: $\{f^{\mathrm{i}}_{k}, h^{\mathrm{i}}_{i,j}\}$;

  support(scatterer): $\mathcal{D}^{\mathrm{s}}$; observational domain: $\mathcal{D}^{\Omega} \not\subset \mathcal{D}^{\mathrm{s}}$;

  incident wave field: $[F]^{\mathrm{i}}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathbb{R}^3,\mathbb{R}\}}$;

  scattered wave field: $[F]^{\mathrm{s}}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\Omega},\mathbb{R}\}}$ (data)

- **Desired output:** scattering sources: $[Q^{\mathrm{s}}](\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\mathrm{s}},\mathbb{R}\}}$ (model) $\Longrightarrow$

$$[X]\,(\boldsymbol{x},t) = \{\rho_{k,r} - \rho^{\mathrm{b}}_{k,r}, S_{i,j,p,q} - S^{\mathrm{b}}_{i,j,p,q}\}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\mathrm{s}},\mathbb{R}\}}$$ (object contrast)

- $$[F^{\mathrm{s}}]\,(\boldsymbol{x},t) = \int_{\boldsymbol{x}'\in\mathrm{supp}(Q^{\mathrm{s}})} [G]\,(\rho^{\mathrm{b}}, S^{\mathrm{b}}, \boldsymbol{x}, \boldsymbol{x}', t) \overset{(t)}{*} [Q^{\mathrm{s}}]\,(\boldsymbol{x}',t)\mathrm{d}V(\boldsymbol{x}') \overset{\boldsymbol{x}\in\mathcal{D}^{\Omega}}{\Longrightarrow}$$

- **MODEL/DATA EQUATION:** $\Longrightarrow [Q^{\mathrm{s}}]^{[\infty]}(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\mathrm{s}},\mathbb{R}\}}$

  ($=$ minimum norm solution) $\Longrightarrow [F^{\mathrm{s}}]\,(\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathbb{R}^3,\mathbb{R}\}}$

OBJECT EQUATION

• Object equation:

- $[Q^{\mathrm{s}}]^{[\infty]}(\boldsymbol{x},t) = [X](\boldsymbol{x},t) \overset{(t)}{*} \left[[F^{\mathrm{i}}](\boldsymbol{x},t) + [F^{\mathrm{s}}](\boldsymbol{x},t) \right]$ for $\{\boldsymbol{x},t\} \in \{\mathcal{D}^{\mathrm{s}},\mathbb{R}\}$
- $[X](\boldsymbol{x},t) = \left\{ \rho_{k,r} - \rho^{\mathrm{b}}_{k,r}, S_{i,j,p,q} - S^{\mathrm{b}}_{i,j,p,q} \right\} (\boldsymbol{x},t)|_{\{\boldsymbol{x},t\}\in\{\mathcal{D}^{\mathrm{s}},\mathbb{R}\}}$ (object contrast)

• Object equation (properties):

- pointwise equation highly unstable
- reconstructed contrast parameters may be outside physical range

• 'Remedies' (?):

- replace object equation with (some?) minimum norm solution
- put bounds on parameter values (e.g., via

$$\chi \longrightarrow \overline{\chi} = B \ \log \left[\frac{\chi_{\mathrm{max}} - \chi}{\chi - \chi_{\mathrm{min}}} \right] \quad \text{for } \chi_{\mathrm{min}} < \chi < \chi_{\mathrm{max}}$$

and use $\overline{\chi}^{[\infty]}$ to evaluate $\chi^{[\infty]}$)

WAVEFIELD MODELING (ANALYTICAL TOOLS)

TIME LAPLACE TRANSFORMATION (CAUSAL FUNCTIONS)

- **Causality condition:**
  - $\mathrm{supp}[u(\boldsymbol{x},t)] = \mathcal{T} = \{t \in \mathbb{R}, t_0 < t < \infty\}$
- **Time Laplace transformation:**
  - $\hat{u}(\boldsymbol{x},s) = \int_{t\in\mathcal{T}} \exp(-st)u(\boldsymbol{x},t)\mathrm{d}t$ for $s \in \mathcal{L}$
  - $\mathcal{L} = \{s \in \mathbb{R}, s = s_0 + nh, s_0 > 0, h > 0, n = 0,1,2,\ldots\}$ (Lerch sequence)
- **Properties:**
  - $\hat{u}(\boldsymbol{x},s)|_{s\in\mathcal{L}} \overset{1-\mathrm{to}-1}{\Longleftrightarrow} u(\boldsymbol{x},t)|_{t\in\mathcal{T}}$
  - $s\ \hat{u}(\boldsymbol{x},s) \Longleftrightarrow \partial_t u(\boldsymbol{x},t)$
  - $\hat{u}(\boldsymbol{x},s)\ \hat{v}(\boldsymbol{x},s) \Longleftrightarrow u(\boldsymbol{x},t) \overset{(t)}{*} v(\boldsymbol{x},t)$ [$\overset{(t)}{*}$ = time convolution]
- **Application:** analysis and synthesis of linear, time-invariant, causal systems

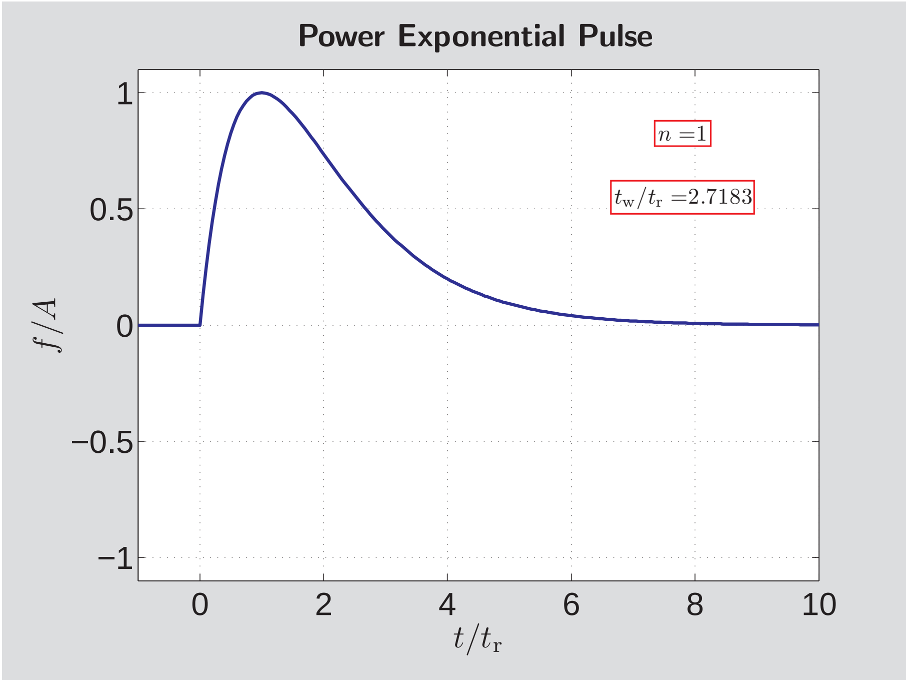
Time Laplace transformation (causal functions)

THE POWER EXPONENTIAL PULSE

- **Pulse shape:**
  - $f(t) = A\left(\frac{t}{t_{\mathrm{r}}}\right)^{\nu} \exp\left[-\nu\left(\frac{t}{t_{\mathrm{r}}} - 1\right)\right] H(t)$
- **Parameters:**
  - pulse amplitude $A = f(t_{\mathrm{r}})$
  - pulse rise time t_{r} $\quad [\partial_t f(t_{\mathrm{r}}) = 0]$
  - pulse time width $t_{\mathrm{w}} = t_{\mathrm{r}}\frac{\Gamma(\nu+1)}{\nu^{\nu+1}}\exp(\nu) \quad \left[A\, t_{\mathrm{w}} = \int_{t=0}^{\infty} f(t)\mathrm{d}t\right]$
- **Time Laplace transform:**
  - $\hat{f}(s) = A\frac{\Gamma(\nu+1)}{(s+\nu/t_{\mathrm{r}})^{\nu+1}}\,\frac{\exp(\nu)}{t_{\mathrm{r}}^{\nu}}$ $\quad$ [$\Gamma(\cdot)$ = Euler gamma function]

The power exponential pulse (a convenient excitation function)

POWER EXPONENTIAL PULSES:



The power exponential pulse

SPATIAL WAVENUMBER REPRESENTATION

- **Spatial wavenumber representation:**
  - $\hat{u}(\boldsymbol{x},s) = \left(\frac{1}{2\pi}\right)^3 \int_{\boldsymbol{k}\in\mathbb{R}^3} \exp(-\mathrm{i}k_m x_m)\, \tilde{u}(\boldsymbol{k},s)\, \mathrm{d}\boldsymbol{k}$ for $\boldsymbol{x} \in \mathbb{R}^3$
- **Properties:**
  - $\tilde{u}(\boldsymbol{k},s)|_{\boldsymbol{k}\in\mathbb{R}^3} \overset{1-\text{to}-1}{\Longleftrightarrow} \hat{u}(\boldsymbol{x},s)|_{\boldsymbol{x}\in\mathbb{R}^3}$
  - $-\mathrm{i}k_m\ \tilde{u}(\boldsymbol{k},s) \Longleftrightarrow \partial_m\, \hat{u}(\boldsymbol{x},s)$
  - $\tilde{u}(\boldsymbol{k},s)\ \tilde{v}(\boldsymbol{k},s) \Longleftrightarrow \hat{u}(\boldsymbol{x},s) \overset{(\boldsymbol{x})}{*} \hat{v}(\boldsymbol{x},s)$ [$\overset{(\boldsymbol{x})}{*}$ = spatial convolution]
- **Application:**

  analysis of wave motion in linear, time-invariant, shift-invariant configurations
- **Dirac delta distribution:** • $\delta(\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^3 \int_{\boldsymbol{k}\in\mathbb{R}^3} \exp(-\mathrm{i}k_m x_m)\mathrm{d}\boldsymbol{k}$

SPATIAL WAVE SLOWNESS REPRESENTATION

- **Spatial wave slowness representation:**
  - $\hat{u}(\boldsymbol{x}, s) = \left(\frac{s}{2\pi}\right)^3 \int_{\boldsymbol{\alpha}\in\mathbb{R}^3} \exp(-\mathrm{i}s\alpha_m x_m)\, \tilde{u}(\boldsymbol{\alpha}, s)\, \mathrm{d}\boldsymbol{\alpha}$ for $\boldsymbol{\alpha} \in \mathbb{R}^3,\; s \in \mathbb{R}, s > 0$
- **Properties:**
  - $\tilde{u}(\boldsymbol{\alpha}, s)|_{\boldsymbol{\alpha}\in\mathbb{R}^3} \overset{1-\text{to}-1}{\Longleftrightarrow} \hat{u}(\boldsymbol{x}, s)|_{\boldsymbol{x}\in\mathbb{R}^3}$
  - $-\mathrm{i}s\alpha_m\; \tilde{u}(\boldsymbol{\alpha}, s) \Longleftrightarrow \partial_m\, \hat{u}(\boldsymbol{x}, s)$
  - $\tilde{u}(\boldsymbol{\alpha}, s)\; \tilde{v}(\boldsymbol{\alpha}, s) \Longleftrightarrow \hat{u}(\boldsymbol{x}, s) \overset{(\boldsymbol{x})}{*} \hat{v}(\boldsymbol{x}, s)$ [$\overset{(\boldsymbol{x})}{*}$ = spatial convolution]
- **Application:**

  Cagniard-like methods (e.g., waves in layered media)
- **Dirac delta distribution:** • $\delta(\boldsymbol{x}) = \left(\frac{s}{2\pi}\right)^3 \int_{\boldsymbol{\alpha}\in\mathbb{R}^3} \exp(-\mathrm{i}s\alpha_m x_m)\mathrm{d}\boldsymbol{\alpha}$

GREEN'S FUNCTION (DISSIPATIVE SCALAR WAVE EQUATION)

- **Dissipative scalar wave equation:**
  - $\partial_m \partial_m G - \frac{1}{c^2}(\alpha + \partial_t)(\beta + \partial_t)G = -\delta(\boldsymbol{x}, t)$
- **Spatial wavenumber representation:**
  - $\hat{G}(\boldsymbol{x}, s) = \left(\frac{1}{2\pi}\right)^3 \int_{\boldsymbol{k}\in\mathbb{R}^3} \exp(-\mathrm{i}k_m x_m)\, \tilde{G}(\boldsymbol{k}, s)\, \mathrm{d}\boldsymbol{k}$ with
  - $\tilde{G}(\boldsymbol{k}, s) = \frac{1}{k_m k_m + (s+\alpha)(s+\beta)/c^2}$
- **Space time Green's function:**
  - $G(\boldsymbol{x}, s) = \frac{U_1(\alpha, \beta, |\boldsymbol{x}|/c, t)}{4\pi|\boldsymbol{x}|}$
  - $U_1(\alpha, \beta, T, t) = \exp[-(\alpha+\beta)t/2]\times$

$$\left\{\delta(t-T) + \frac{(|\beta-\alpha|/2)}{(t^2-T^2)^{1/2}} I_1\left[(|\beta-\alpha|/2)(t^2-T^2)^{1/2}\right]\right\} H(t-T)$$

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

Green's function of the dissipative scalar wave equation

PARTICLE VELOCITY WAVE EQUATION

- **For homogeneous, isotropic, perfectly elastic solid:**
  - $-\partial_m([\tau_{m,k}]^\delta + [\tau_{m,k}]^\Delta) + \rho\partial_t v_k = f_k$
  - $[\partial_i v_j]^\delta - S^\delta \, \partial_t[\tau_{i,j}]^\delta = [h_{i,j}]^\delta$
  - $[\partial_i v_j]^\Delta - S^\Delta \, \partial_t[\tau_{i,j}]^\Delta = [h_{i,j}]^\Delta$
- **Particle velocity wave equation:**
  - $(c_P^2 - c_S^2) \, \partial_k \partial_m v_m + c_S^2 \, \partial_m \partial_m v_k - \partial_t^2 v_k =$
    $\rho^{-1}\partial_t f_k - (\rho S^\delta)^{-1}\partial_m[h_{m,k}]^\delta - (\rho S^\Delta)^{-1}\partial_m[h_{m,k}]^\delta$
- **P-wave speed:**

$$c_P = \left(\frac{2S^\delta + S^\Delta}{3\rho S^\delta S^\Delta}\right)^{1/2}$$

- **S-wave speed:**

$$c_S = \left(\frac{1}{2\rho S^\Delta}\right)^{1/2}$$

For further details, see

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

The particle velocity wave equation

WAVEFIELD MODELING (COMPUTATIONAL TOOLS)

THE SPACE-TIME INTEGRATED FIELD EQUATIONS

- **General form:**
  - $\int_{t\in\mathcal{T}} \mathrm{d}t \int_{\boldsymbol{x}\in\mathcal{D}} [\textbf{LHS}]\,\mathrm{d}V(\boldsymbol{x}) = \int_{t\in\mathcal{T}} \mathrm{d}t \int_{\boldsymbol{x}\in\mathcal{D}} [\textbf{RHS}]\,\mathrm{d}V(\boldsymbol{x})$
- **Ingredients (domain of computation):**
  - $\int_{\boldsymbol{x}\in\mathcal{D}} \partial_m [\cdots]\,\mathrm{d}V(\boldsymbol{x}) = \int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m [\cdots]\,\mathrm{d}V(\boldsymbol{x})$ (3D Gauss)
  - $\int_{t\in\mathcal{T}} \partial_t [\cdots]\,\mathrm{d}t = [\cdots]|_{t\in\partial\mathcal{T}}$ ('1D Gauss')
  - implementation on simplicial grid $\Longrightarrow$
  - $\int_{\boldsymbol{x}\in\mathcal{D}} [\cdots]\,\mathrm{d}V(\boldsymbol{x}) \simeq [\cdots]_{\mathrm{av}} \int_{\boldsymbol{x}\in\mathcal{D}} \mathrm{d}V(\boldsymbol{x})$ (Linear interpolation over $\mathcal{D}$)
  - $\int_{t\in\mathcal{T}} [\cdots]\,\mathrm{d}t \simeq [\cdots]_{\mathrm{av}} \int_{t\in\mathcal{T}} \mathrm{d}t$ (Linear interpolation over $\mathcal{T}$)
  - constitutive coefficients constant over $\mathcal{D}\times\mathcal{T}$

The space-time integrated field equations

THE SPACE-TIME INTEGRATED FIELD EQUATIONS

- **Properties:**
  - Discretized system of equations contains only quantities that are continuous across interfaces of spatial (jump) discontinuities in material
  - For differentiable fields the procedure converges to exact solution as maxdiam$(\mathcal{D}) \downarrow 0$ and maxdiam$(\mathcal{T}) \downarrow 0$
- **Perfectly matched embedding:**
  - $\partial_{x_n} \longmapsto \partial_{\hat{X}_n} = \dfrac{1}{\hat{\chi}_n(x_n, s)}\partial_{x_n} \Longrightarrow$
  - $\hat{X}_n(x_n, s) = \displaystyle\int_{\xi_n = a_n}^{x_n} \hat{\chi}_n(\xi_n, s)\mathrm{d}\xi_n \quad (n = 1, 2, 3)$
  - $\hat{\chi}_n(x_n, s) = \mathbf{1}$ for $n = 1, 2, 3;\ \boldsymbol{x} \in \mathcal{D}$

Time-dependent, causal Cartesian coordinate stretching

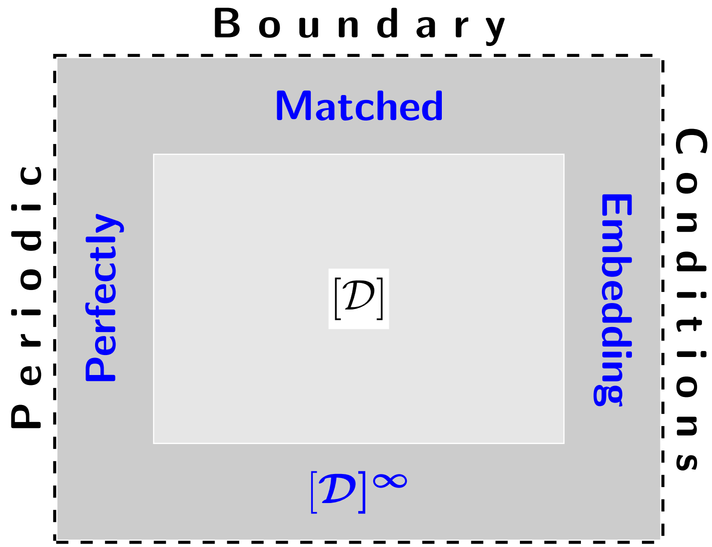
[De Hoop, Remis, Van den Berg, Journal of Computational Physics, 221 (2007) pp. 88–105]

The space-time integrated field equations (continued)

THE PERFECTLY MATCHED EMBEDDING

• Periodic boundary conditions:

- $\text{FIELD}(x_m + L_m) = \text{FIELD}(x_m) \quad (m = 1, 2, 3)$



[De Hoop, Remis, Van den Berg, Journal of Computational Physics, 221 (2007) pp. 88–105]

THANK YOU !

APPENDIX

SPACE-TIME MODEL/DATA EQUATION

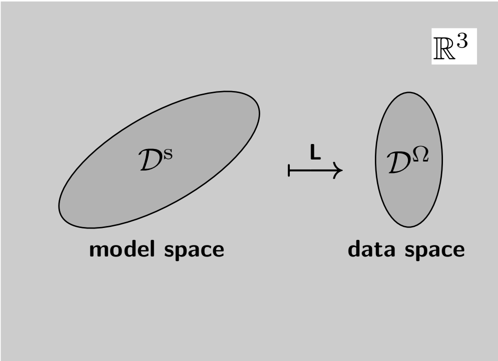
• Configuration:

- $\mathcal{D}^{\rm s} = {\rm supp(model)} \subset \mathbb{R}^3$
- $\mathcal{D}^{\Omega} = {\rm supp(data)} \subset \mathbb{R}^3$

} • $\mathcal{D}^{\Omega} \cap \mathcal{D}^{\rm s} = \emptyset$ ('remote sensing')

- $\boldsymbol{m}(\boldsymbol{x},t) \in \mathcal{D}^{\rm s} \times \mathbb{R}$ (model)
- $\boldsymbol{d}(\boldsymbol{x},t) \in \mathcal{D}^{\Omega} \times \mathbb{R}$ (data)

} • $\{\mathcal{D}^{\rm s} \times \mathbb{R}\} \xrightarrow{\mathsf{L}} \{\mathcal{D}^{\Omega} \times \mathbb{R}\}$ (linear operator)



• Model/data equation:

• $\boldsymbol{d} = \mathsf{L}\boldsymbol{m}$ $\qquad \boldsymbol{d} \xrightarrow{?} \boldsymbol{m}$

Space-time model/data equation

'SOLVING' MODEL/DATA EQUATION

Model/data equation: • $\boldsymbol{d} = \mathbf{L}\boldsymbol{m}$ for $\{\boldsymbol{x}, t\} \in \{\mathcal{D}^{\Omega}, \mathbb{R}\} \Longrightarrow \boldsymbol{d} \xrightarrow{?} \boldsymbol{m}$

Iterative 'solution' (computational procedure):

- $\boldsymbol{m} \in \{\boldsymbol{m}^{[0]}, \boldsymbol{m}^{[1]}, \boldsymbol{m}^{[2]}, \ldots\}$; $\boldsymbol{m}^{[0]} = \mathbf{0}$ (sequence of models)
- $\boldsymbol{r}^{[i]} = \boldsymbol{d} - \mathbf{L}\boldsymbol{m}^{[i]}$ for $i = 0, 1, 2, \ldots$; $\boldsymbol{r}^{[0]} = \boldsymbol{d}$ ($\boldsymbol{r}$ = residual)
- $\langle \cdot, \cdot \rangle$ symmetric inner product in (functional) Hilbert space

Mismatch error: • $ERR^{[i]} = \langle \boldsymbol{r}^{[i]}, \boldsymbol{r}^{[i]} \rangle = \|\boldsymbol{r}^{[i]}\|^2$ ($\|\cdot\|$ = norm ≥ 0)

Improvement condition: $ERR^{[i+1]} < ERR^{[i]}$ for $i = 0, 1, 2, \ldots$

Model update: • $\boldsymbol{m}^{[i+1]} = \boldsymbol{m}^{[i]} + \delta\boldsymbol{m}^{[i]}$ for $i = 0, 1, 2, \ldots$

$\Longrightarrow$ improved model

'Solving' model/data equation

'SOLVING' MODEL/DATA EQUATION

- $ERR^{[i+1]} = ERR^{[i]} - 2\langle \mathbf{L}\delta \boldsymbol{m}^{[i]}, \boldsymbol{r}^{[i]}\rangle + \|\mathbf{L}\delta \boldsymbol{m}^{[i]}\|^2$

$= ERR^{[i]} - 2\langle \delta \boldsymbol{m}^{[i]}, \mathbf{L}^*\boldsymbol{r}^{[i]}\rangle + \|\mathbf{L}\delta \boldsymbol{m}^{[i]}\|^2$ ($\mathbf{L}^*$ = adjoint to $\mathbf{L}$)

Steepest descent: • $\delta \boldsymbol{m}^{[i]} = \alpha^{[i]}\mathbf{L}^*\boldsymbol{r}^{[i]}$

- $\alpha^{[i]} = \dfrac{\|\mathbf{L}^*\boldsymbol{r}^{[i]}\|^2}{\|\mathbf{L}\mathbf{L}^*\boldsymbol{r}^{[i]}\|^2} \Longrightarrow$ • $ERR^{[i+1]} = ERR^{[i]} - \dfrac{\|\mathbf{L}^*\boldsymbol{r}^{[i]}\|^4}{\|\mathbf{L}\mathbf{L}^*\boldsymbol{r}^{[i]}\|^2} < ERR^{[i]} \Longrightarrow$

- $\{\boldsymbol{m}^{[0]}, \boldsymbol{m}^{[1]}, \boldsymbol{m}^{[2]}, \ldots\} \xrightarrow{\text{converges}} \boldsymbol{m}^{[\infty]}$

- $\mathbf{L}^*\mathbf{L}\,\boldsymbol{m}^{[\infty]} = \mathbf{L}^*\boldsymbol{d}$ **BUT** • $\mathbf{L}\,\boldsymbol{m}^{[\infty]} \stackrel{?}{=} \boldsymbol{d}$ **NO!**

De Hoop, M. V., De Hoop, A. T., Proceedings of the Royal Society of London, Series A, 456 (2000) pp. 641–682.