Array-structure based theory of Maxwell wavefields

by

Adrianus T. de Hoop

Delft University of Technology

Laboratory of Electromagnetic Research

Faculty of Electrical Engineering, Mathematics and Computer Science

Mekelweg 4 • 2628 CD Delft • the Netherlands

T: +31 15 2785203 / +31 15 2788613

E: a.t.dehoop@tudelft.nl **W:** www.atdehoop.com

Presentation given at the Workshop 'Pulsed EM Fields'

Delft University of Technology | University of Hong Kong

Delft NL, 2013 March 18–19

Synopsis:

- Wavefield arrays in $(N+1)$-spacetime, Operations on arrays
- The observer in $(N+1)$-spacetime, Subscript notation, summation convention
- Intensive field quantities, extensive field quantities, source quantities
- EM field equations (Maxwell), field/source compatibility relations
- Causality, constitutive relations
- Energy balance, Balance of momentum
- Radiation from sources
- Wavefield reciprocity (time-convolution, time-correlation)
- Conclusion

• WAVEFIELD ARRAYS IN (N+1)-SPACETIME: $\mathbb{R}^N \times \mathbb{R}$

•$\{\Phi, E_k, H_{i,j}, \sigma_{i,j,k}\}(\boldsymbol{x}, t) \in \{$arrays of arraylength$N$; $\boldsymbol{x} \in \mathbb{R}^N, t \in \mathbb{R}\}$

• OPERATIONS ON ARRAYS:

- **antisymmetrization:** $[H_{i,j}]^- \stackrel{\text{def}}{=} (H_{i,j} - H_{j,i})/2 = -[H_{j,i}]^-$
- **symmetrization:** $[H_{i,j}]^+ \stackrel{\text{def}}{=} (H_{i,j} + H_{j,i})/2 = +[H_{j,i}]^+$
- **diagonalization:** $[H_{i,j}]^\delta \stackrel{\text{def}}{=} (H_{k,k}/N)\delta_{i,j}$
- **deviatorization:** $[H_{i,j}]^\Delta \stackrel{\text{def}}{=} [H_{i,j}]^+ - [H_{i,j}]^\delta$
- **circulation:** $[\sigma_{i,j,k}]^\circlearrowleft \stackrel{\text{def}}{=} (\sigma_{i,j,k} + \sigma_{j,k,i} + \sigma_{k,i,j})/3 \ (i \neq j \neq k)$ ($N \geq 3$)

$$= [\sigma_{j,k,i}]^\circlearrowleft = [\sigma_{k,i,j}]^\circlearrowleft$$

Operations on wavefield arrays in (N+1)-spacetime $\mathbb{R}^N \times \mathbb{R}$

• PROPERTIES:

$$\{[\cdot]^-, [\cdot]^+, [\cdot]^\delta, [\cdot]^\Delta\} \in \{\text{ projection operators }\}$$

• projection: repetition ⟶ reproduction

$$\{[\cdot]^-, [\cdot]^+\} \in \{\text{ mutually orthogonal operators }\}$$

$$\{[\cdot]^-, [\cdot]^\delta, [\cdot]^\Delta\} \in \{\text{ mutually orthogonal operators }\}$$

• mutually orthogonal: combination ⟶ annihilation

• orthogonal decomposition: $H_{i,j} = [H_{i,j}]^- + [H_{i,j}]^\delta + [H_{i,j}]^\Delta$

• $[\partial_m \partial_k]^- = 0$, • $[\partial_m \partial_k]^+ = \partial_m \partial_k$

• $\partial_k \partial_m [H_{m,k}]^- = 0$

• $[\partial_k [\partial_i E_j]^-]^\circlearrowleft = 0 \;(i \neq j \neq k; N \geq 3)$

• $(3+1)$-spacetime $\mathbb{R}^3 \times \mathbb{R}$, $(\boldsymbol{x} \in \mathbb{R}^3, t \in \mathbb{R})$: ~~metric~~ (~~*Einstein*~~) affine (Weyl)

(Background) $\mathbb{R}^3$: • HOMOGENEOUS • ISOTROPIC: $\Longrightarrow$ Relativity

Field	Observer	Decomposition: Electric | Magnetic field		
			'Maxwell'	
Electro-	Observer 1 $\Longrightarrow$	• Electric Field 1	$\Longleftrightarrow$	• Magnetic Field 1
Magnetic	$\downarrow v$ • $\uparrow -v$	(special) $\longleftarrow \Updownarrow$	'Lorentz'	$\Updownarrow \longrightarrow$ (relativity)
Field	Observer 2 $\Longrightarrow$	• Electric Field 2	$\Longleftrightarrow$	• Magnetic Field 2
			'Maxwell'	

• $v =$ (constant) relative velocity of Observers 1 & 2

- **Einstein:** All macroscopic physical wavefields are **tensorial** in nature

- **Tensor of rank p in $\mathbb{R}^N$:** p-**d array** of lengths $\underbrace{N \times \ldots \times N}_{(p=0,1,2,3,\ldots)}$

- **SUBSCRIPT NOTATION:** $(m, n = 1, \ldots, N)$
  - $\boldsymbol{x} \rightarrow x_m$: position (1-d array)
  - $\partial_m = \partial/\partial x_m$: spatial derivative (1-d array)
  - $\partial_m x_n = \delta_{m,n}$: Kronecker array (2-d unit array)
    (• $\delta_{m,n} = 1$ if $m = n$, • $\delta_{m,n} = 0$ if $m \neq n$)

- **SUMMATION CONVENTION (repeated subscripts):**
  - $\delta_{p,n}\delta_{n,q} = \sum_{n=1}^{N} \delta_{p,n}\delta_{n,q} = \delta_{p,q}$
  - $\delta_{m,m} = \sum_{m=1}^{N} \delta_{m,m} = N$

Array (Weyl) *structure of Maxwell wavefields*

	Intensive field quantity 'field strength'	**Extensive** field quantity 'flux density'	**Source** quantity 'volume density of' 'current'
Electric	E_k	D_k	J_k
Magnetic	$[H_{i,j}]^- = -[H_{j,i}]^-$	$[B_{i,j}]^- = -[B_{j,i}]^-$	$[K_{i,j}]^- = -[K_{j,i}]^-$
	'PRODUCT'	**'PRODUCT'**	
	$S_m = [H_{m,k}]^- E_k$ area density of EM power flow	$G_i = [B_{i,j}]^- D_j$ volume density of EM wave momentum	

- $\{E_k, D_k, J_k\}$ (1-d), $\{[H_{i,j}]^-, [B_{i,j}]^-, [K_{i,j}]^-\}$ (2-d anti-symmetric)

Maxwell (EM) wavefield and source quantities

- **WAVEFIELDS:**
  - Changes in space ($\partial_m = \partial/\partial x_m$) **counterbalance**
  - Changes in time ($\partial_t = \partial/\partial t$)

EM field equations (Maxwell)

- $\partial_m [H_{m,k}]^- + \partial_t D_k = J_k$ (Maxwell 1)
- $[\partial_i E_j]^- + \partial_t [B_{i,j}]^- = [K_{i,j}]^-$ (Maxwell 2)

- **FIELD/SOURCE COMPATIBILITY RELATIONS:**

- $\partial_k \partial_m [H_{m,k}]^- = 0$

$\Longrightarrow$ • $\partial_t(\partial_k D_k) = \partial_k J_k$

- $[\partial_k [\partial_i E_j]^-]^{\circlearrowleft} = 0 \; (i \neq j \neq k; N \geq 3)$

$\Longrightarrow$ • $\partial_t [\partial_k [B_{i,j}]^-]^{\circlearrowleft} = [\partial_k [K_{i,j}]^-]^{\circlearrowleft} \; (i \neq j \neq k; N \geq 3)$

EM field equations and field/source compatibility relations

• VOLUME DENSITY OF ELECTRIC CHARGE:

- $\rho \stackrel{\text{def}}{=\!=} -\partial_t^{-1}\partial_k J_k$ (0-d array)

$\Longrightarrow$ • $\partial_k D_k = -\rho$

• VOLUME DENSITY OF MAGNETIC CHARGE:

- $\sigma_{k,i,j} \stackrel{\text{def}}{=\!=} -\partial_t^{-1}[\partial_k[K_{i,j}]^-]^{\circlearrowleft}\ (i \neq j \neq k; N \geq 3)$

(completely antisymmetric 3-d array)

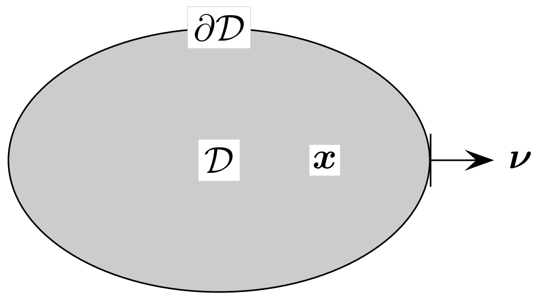
$\Longrightarrow$ • $[\partial_k[B_{i,j}]^-]^{\circlearrowleft} = -\sigma_{k,i,j}(i \neq j \neq k; N \geq 3)$

Volume densities of electric and magnetic charge

- **CAUSALITY (UNIQUENESS OF TIME EVOLUTION $=$ UNIQUENESS OF INITIAL-VALUE PROBLEM):**
- **EM field**$(\boldsymbol{x}, t_0)$ for $\boldsymbol{x} \in \mathbb{R}^3$ $\overset{\text{Field equations}}{\Longrightarrow \Longrightarrow}$
  EM field$(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^3, t > t_0$ (uniquely)
- **'Maxwell'**: # Equations $<$ # Unknowns $\Longrightarrow$
- CONSTITUTIVE RELATIONS
  (simplest case: homogeneous, isotropic, lossless):
- $D_k = \epsilon E_k$ (ϵ = electric permittivity)
- $[B_{i,j}]^- = \mu^- [H_{i,j}]^-$ (μ^- = magnetic permeability)
- $\Longrightarrow$ **'Maxwell'**: # Equations $=$ # Unknowns

• EM ENERGY BALANCE (local & global over domain $\mathcal{D}$):

EM field equations at $(\boldsymbol{x}, t)$	
• (Maxwell 1)$_k$	E_k
• (Maxwell 2)$_{i,j}$	$[H_{i,j}]^-$
• EM energy balance	$+$

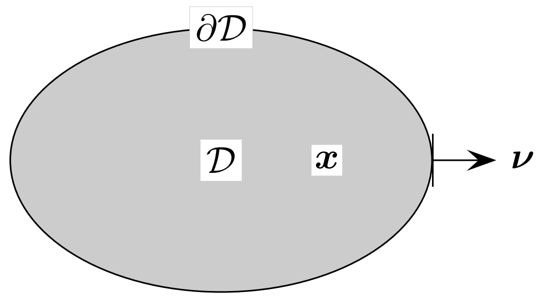


• $\partial_m S_m + \partial_t w = P$ (local)	
• $S_m = [H_{m,k}]^- E_k$	area density of EM power flow (Poynting vector)
• $w = E_k D_k/2 + [H_{i,j}]^-[B_{i,j}]^-/2$	volume density of stored EM energy
• $P = J_k E_k + [K_{i,j}]^-[H_{i,j}]^-$	volume density of EM power delivered by sources
'GAUSS' $\Longrightarrow$	• $\int_{\partial\mathcal{D}} S_m \nu_m \mathrm{d}A + \partial_t \int_{\mathcal{D}} w \mathrm{d}V = \int_{\mathcal{D}} P \mathrm{d}V$ (global)

EM energy balance (local & global) ($\Longleftrightarrow$ Thermodynamics)

• BALANCE OF EM MOMENTUM (local & global over domain $\mathcal{D}$):

EM field equations at $(\boldsymbol{x}, t)$	
• (Maxwell 1)$_k$	$[B_{i,k}]^-$
• (Maxwell 2)$_{i,j}$	D_j
• balance of EM momentum	$+$



• $-\partial_i T_{i,j} + \partial_t G_j = f_j$ (local)	
$T_{i,j} =$ •	Maxwell stress (−'EM radiation pressure')
$G_i =$ •	volume density of EM momentum
$f_j =$ •	volume density of force exerted by the sources
'GAUSS' $\Longrightarrow$	• $-\int_{\partial\mathcal{D}} \nu_i T_{i,j} \mathrm{d}A + \partial_t \int_{\mathcal{D}} G_j \mathrm{d}V = \int_{\mathcal{D}} f_j \mathrm{d}V$ (global)

Balance of EM momentum (local & global) ($\Longleftrightarrow$ Mechanics)

- **EM RADIATION FROM SOURCES:**

- **Maxwell 1**
- **Maxwell 2**

- Elimination of $[H_{i,j}]^{-}$ $+$ use of
- $\partial_k E_k = \epsilon^{-1}\partial_t^{-1}(\partial_m J_m)$ (compatibility relation) $\Longrightarrow$

- **ELECTRIC-FIELD WAVE EQUATION:**

$$(\partial_m \partial_m) E_k - c^{-2}\partial_t^2 E_k = -F_k$$

- $c = (\epsilon\,\mu^{-})^{-1/2}$ (EM wavespeed)
- $F_k = \mu^{-}\,\partial_t J_k - \epsilon^{-1}\partial_t^{-1}\partial_k(\partial_m J_m) - \partial_m[K_{m,k}]^{-}$

• WAVE EQUATION (HOMOGENEOUS MEDIUM):

- [WAVE OPERATOR] FIELD $= -$ SOURCE
  - WAVE OPERATOR $= \partial_m\partial_m - c^{-2}\partial_t^2$
  - SOURCE $= \delta(\boldsymbol{x},t) \overset{(\boldsymbol{x})(t)}{*\;*}$ SOURCE
- [WAVE OPERATOR] GREEN $= -\delta(\boldsymbol{x},t)$

$\Longrightarrow$ • **SOLUTION:** $\Longrightarrow$ • FIELD = GREEN $\overset{(\boldsymbol{x})(t)}{*\;*}$ SOURCE

- GREEN $= \dfrac{\delta(t-|\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|}$ for $\boldsymbol{x} \neq \mathbf{0}$ (N=3)(3D)

- $\overset{(\boldsymbol{x})(t)}{*\;*}\ \partial_m\partial_t = \partial_m\partial_t \overset{(\boldsymbol{x},t)}{*}$

Wave equation and its solution (homogeneous medium)

- **RADIATED FIELD = GREEN $\overset{(\boldsymbol{x})}{*}\overset{(t)}{*}$ SOURCE**

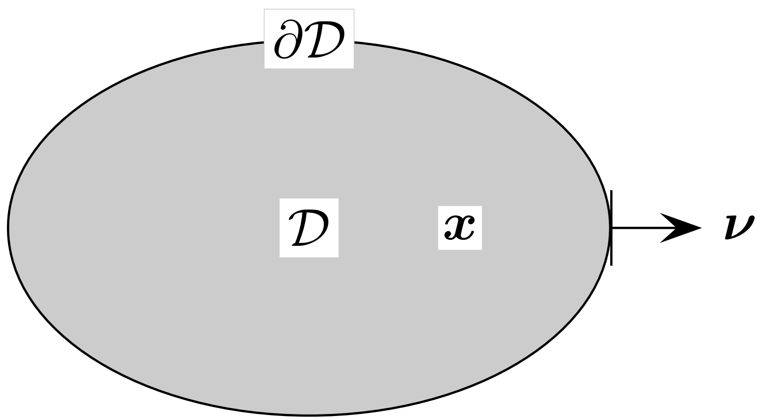
- **ELECTRIC-CURRENT 'POTENTIAL':**
  - $A_k = G \overset{(\boldsymbol{x})}{*}\overset{(t)}{*} J_k$

- **MAGNETIC-CURRENT 'POTENTIAL':**
  - $[\Psi_{i,j}]^- = G \overset{(\boldsymbol{x})}{*}\overset{(t)}{*} [K_{i,j}]^-$

Electric-current and magnetic-current potentials

• WAVEFIELD RECIPROCITY (interaction of two STATES):

Reciprocity $(\boldsymbol{x} \in \mathcal{D} \cup \partial\mathcal{D},\ t \in \mathbb{R})$			
State	Field strengths	Flux densities	Sources
A	$E_k^A, [H_{i,j}^A]^-$	$D_k^A, [B_{i,j}^A]^-$	$J_k^A, [K_{i,j}^A]^-$
B	$E_k^B, [H_{i,j}^B]^-$	$D_k^B, [B_{i,j}^B]^-$	$J_k^B, [K_{i,j}^B]^-$



EM Wavefield reciprocity (States^A and ^{B})

• AREA DENSITY OF TIME-CONVOLUTION INTERACTION FLOW:

- $S_m^{AB}(\boldsymbol{x},t) = [H_{m,k}^A]^-(\boldsymbol{x},t) \overset{(t)}{*} E_k^B(\boldsymbol{x},t)$
- $\overset{(t)}{*}$ = time convolution

• VOLUME DENSITY OF STORED TIME-CONVOLUTION INTERACTION:

- $\partial_m S_m^{AB}(\boldsymbol{x},t) = Q^{AB}(\boldsymbol{x},t)$
- $Q^{AB} = (J_k^A - \partial_t D_k^A) \overset{(t)}{*} E_k^B + ([K_{m,k}^B]^- - \partial_t[B_{m,k}^B]^-) \overset{(t)}{*} [H_{m,k}^A]^-$

• AREA DENSITY OF TIME-CORRELATION INTERACTION FLOW:

- $S_m^{AB^*}(\boldsymbol{x},t) = [H_{m,k}^A]^-(\boldsymbol{x},t) \overset{(t)}{*} E_k^{B^*}(\boldsymbol{x},t)$
- $\overset{(t)}{*}$ = time convolution
- * = time reversal $(t^* = -t, \partial_t = -\partial_t^*)$

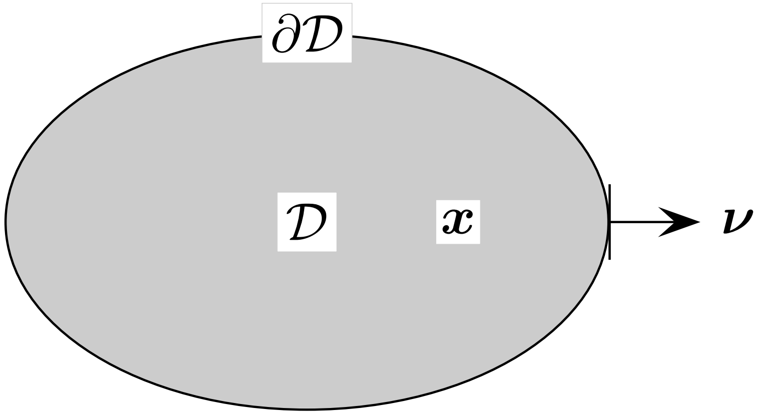
• VOLUME DENSITY OF STORED TIME-CORRELATION INTERACTION:

- $\partial_m S_m^{AB^*}(\boldsymbol{x},t) = Q^{AB^*}(\boldsymbol{x},t)$
- $Q^{AB^*} = (J_k^A - \partial_t D_k^A) \overset{(t)}{*} E_k^{B^*} + ([K_{m,k}^{B^*}]^- - \partial_t [B_{m,k}^{B^*}]^-) \overset{(t)}{*} [H_{m,k}^A]^-$

EM Wavefield reciprocity (local time-correlation interaction States A and B^*)

• TIME-CONVOLUTION EM RECIPROCITY (LOCAL):

• $\partial_m(S_m^{AB} - S_m^{BA}) = Q^{AB} - Q^{BA}$

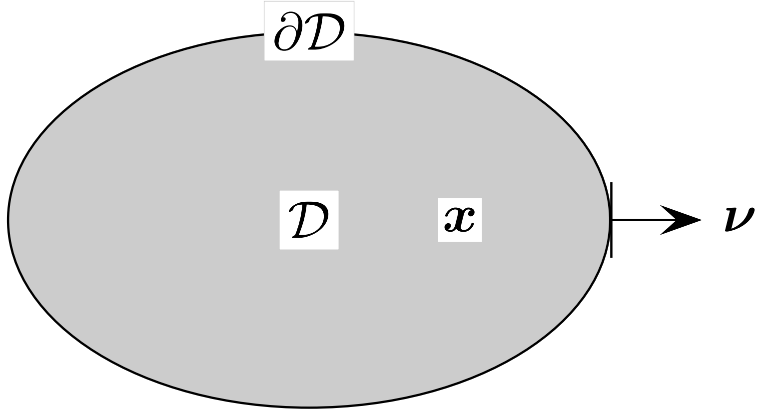


• GLOBAL: ('GAUSS')

• $$\int_{\partial\mathcal{D}} \nu_m(S_m^{AB} - S_m^{BA})\mathrm{d}A = \int_{\mathcal{D}} (Q^{AB} - Q^{BA})\mathrm{d}V$$

• TIME-CORRELATION EM RECIPROCITY (LOCAL):

- $\partial_m(S_m^{AB^*} + S_m^{B^*A}) = Q^{AB^*} + Q^{B^*A}$



• GLOBAL: ('GAUSS')

- $$\int_{\partial\mathcal{D}} \nu_m(S_m^{AB^*} + S_m^{B^*A})\mathrm{d}A = \int_{\mathcal{D}} (Q^{AB^*} + Q^{B^*A})\mathrm{d}V$$

• ARRAY-STRUCTURE BASED EM THEORY:

- Involves only elementary mathematics ($\partial_m, \partial_t, \overset{(t)}{*}, \overset{(\boldsymbol{x})}{*}$, 'GAUSS')
- Does away with standard ~~VECTOR CALCULUS~~
- Does away with the condition of ~~RIGHT-HANDEDNESS~~ in orientation of spatial reference frame
- Reformulates the notion of the '~~MAGNETIC MONOPOLE~~' ($=$ 3-d array)
- Considers SpaceTime as an 'affine' space (Weyl), not a 'metric' space (Einstein) $\Longrightarrow$ ~~4D LORENTZ 'METRIC'~~

• ARRAY-STRUCTURE BASED EM THEORY:

- Has implications for 'string theory' (Dirac) in quantum electrodynamics (no 'vector potential' as the basic ingredient)
- Maps straightforwardly on 'array' handling in computational electromagnetics
- Can be generalized to $(N+1)$-spacetime, $(N>3)$ $\Longrightarrow$ theoretical cosmology
- Yields straightforward axiomatic approach to (special) relativity (Lorentz transformations)

THE END

THANK YOU FOR

YOUR ATTENTION