Appendix B

Integral-transformation methods

In a number of canonical acoustic, elastodynamic and electromagnetic radiation and scattering problems, integral-transformation methods are standard tools of analysis. Their application proves to be most useful in problems associated with configurations whose properties (though not the wave fields occurring in them) are *shift invariant* in time and/or in one or more of the spatial coordinates. As far as the time coordinate is concerned, we have, in addition, to take into account the property of *causality*. By the latter we mean that changes in the time behaviour of the sources that generate the wave field, are only allowed to manifest themselves in changes in the time behaviour of the generated wave field after some positive (or, in the limit, zero) elapse of time. The causality condition can mathematically most easily be accounted for by the use of the *one-sided Laplace transformation*. For this reason, the one-sided Laplace tranformation is the most appropriate one as the integral transformation with respect to time. A similar argument does not apply to the variations of the wave field in space. Here, it is of importance that one should be able to handle fields in unbounded domains. For this feature, the *Fourier transformation* is the most appropriate one, and hence the (N-dimensional, with $N \geqslant 1$) Fourier transformation will be employed as the integral transformation with respect to one or more of the spatial variables. Since the transformations are of the same structure for any Cartesian tensor function of position and time, the different formulas will be given only for scalar functions.

B.1 Laplace transformation of a causal time function

Assume that the sources that generate the wave field are switched on at the instant t_0. In view of the causality condition, the interest in the behaviour of the field is then in the interval (Figure B.1-1)

$$\mathcal{T} = \{t \in \mathcal{R}; t > t_0\} . \tag{B.1-1}$$

Furthermore, we shall denote by $\mathcal{T}'$ the complement of $\mathcal{T}$ and the instant t_0 in $\mathcal{R}$. Hence,

$$\mathcal{T}' = \{t \in \mathcal{R}; t < t_0\} . \tag{B.1-2}$$

Occasionally, we shall denote the instant t_0 (the common boundary of $\mathcal{T}$ and $\mathcal{T}'$) by $\partial\mathcal{T}$, i.e.

$$\partial\mathcal{T} = \{t \in \mathcal{R}; t = t_0\} . \tag{B.1-3}$$

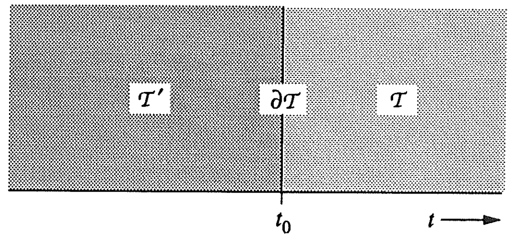

Figure B.1-1 Time intervals $\mathcal{T}$ and $\mathcal{T}'$, and their common boundary $\partial\mathcal{T}$; $\mathcal{T}\cup\partial\mathcal{T}\cup\mathcal{T}'=\mathcal{R}$.

The one-sided Laplace transform of some physical quantitiy $f=f(\boldsymbol{x},t)$, defined in $\mathcal{T}$ and in some as yet unspecified domain in space, is then given by

$$\hat{f}(\boldsymbol{x},s) = \int_{t\in\mathcal{T}} \exp(-st)\, f(\boldsymbol{x},t)\, \mathrm{d}t\,. \tag{B.1-4}$$

Now, causality is enforced by extending the range of f by the value zero when $t\in\mathcal{T}'$, and requiring that Equation (B.1-4), considered as an integral equation to be solved for $f(\boldsymbol{x},t)$, at given $\hat{f}(\boldsymbol{x},s)$, has a unique solution, viz. the value zero when $t\in\mathcal{T}'$ and the reproduction of the function that we started with when $t\in\mathcal{T}$. It can be shown that this requirement can be met by a proper choice of the transformation parameter s. Because of the practical reason that in physics all quantities have bounded values, we shall mostly restrict ourselves to functions f that are bounded. Then, the right-hand side of Equation (B.1-4) is a convergent integral if s is either *real and positive* (which choice has its advantages in the theory of a number of specific wave propagation problems), or *complex* with $\mathrm{Re}(s) > 0$ (which choice leads, in the limiting case where $s = \mathrm{j}\omega$, where j is the imaginary unit and ω is real and positive, to the well-known frequency-domain analysis with complex time factor $\exp(\mathrm{j}\omega t)$, ω being the angular frequency of the relevant frequency component). Due to the analyticity of the Laplace transformation kernel $\exp(-st)$, $\hat{f}(\boldsymbol{x},s)$ is an analytic function of s in $\{s\in\mathcal{C};\mathrm{Re}(s)>0\}$. By *Lerch's theorem* the uniqueness of Equation (B.1-4) when considered as an integral equation is guaranteed if $\hat{f}$ is given at a sequence of points $\{s_n\in\mathcal{R};s_n = s_0 + nh\}$ with s_0 sufficiently large, real and positive, h real and positive, and $n = 0,1,2,\dots$. In fact, Lerch's theorem holds under the more general condition that

$$\int_{t'=t_0}^{t} f(\boldsymbol{x},t')\, \mathrm{d}t'$$

is continuous (see Widder 1946; Doetsch 1950; Schouten 1961a). Unfortunately, there is no explicit inversion formula known that is based on this result, although there exist numerical procedures to solve Equation (B.1-4), when invoked on a sequence of points compatible with Lerch's theorem, that work quite well.

By taking the extended definition of f, Equation (B.1-4) can also be written as

$$\hat{f}(\boldsymbol{x},s) = \int_{t\in\mathcal{R}} \exp(-st)\, f(\boldsymbol{x},t)\, \mathrm{d}t\,. \tag{B.1-5}$$

In elucidating the properties of the Laplace transformation it is often advantageous to use Equation (B.1-5) rather than Equation (B.1-4).

In a number of cases encountered in the theory of the propagation of impulsive waves, the transformation from the s-domain (also denoted as the complex frequency domain) back to the time domain is carried out either by direct inspection or by inspection after applying some elementary rules of the Laplace transformation. For this reason, some of these rules are surveyed below.

The Laplace transform of $f = f(\boldsymbol{x},t)$ is also often symbolised by the notation

$$\hat{f}(\boldsymbol{x},s) = \mathcal{L}(f;\boldsymbol{x},s) \,. \tag{B.1-6}$$

Behaviour as $|s| \to \infty$

The asymptotic behaviour of the time Laplace transform as the transform parameter goes to infinity is found to be

$$|\exp(st_0)\,\hat{f}(\boldsymbol{x},s)| = o(1) \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad \mathrm{Re}(s) \geqslant s_0 > 0 \,. \tag{B.1-7}$$

Differentiation with respect to time

Let $f = f(\boldsymbol{x},t)$ denote a function that is defined when $t \in \mathcal{T}$ and that is equal to zero when $t \in \mathcal{T}'$, while being defined on some domain in space. Then, the (one-sided) Laplace transform of the derivative $\partial_t f$ of f is found as

$$\int_{t=t_0}^{\infty} \exp(-st)\partial_t f(\boldsymbol{x},t)\,\mathrm{d}t = -\exp(-st_0)\,\lim_{t \downarrow t_0} f(\boldsymbol{x},t) + s\hat{f}(\boldsymbol{x},s) \,. \tag{B.1-8}$$

Equation (B.1-8) can also be interpreted as

$$\begin{aligned} s\hat{f}(\boldsymbol{x},s) &= \int_{t \in \mathcal{R}} \exp(-st)\partial_t\,[\chi_{\mathcal{T}}(t)\,f(\boldsymbol{x},t)]\,\mathrm{d}t \\ &= \int_{t \in \mathcal{R}} \exp(-st)\,[\partial_t \chi_{\mathcal{T}}(t)]\,f(\boldsymbol{x},t)\,\mathrm{d}t + \int_{t \in \mathcal{R}} \exp(-st)\chi_{\mathcal{T}}(t)\partial_t f(\boldsymbol{x},t)\,\mathrm{d}t \\ &= \exp(-st_0)\,\lim_{t \downarrow t_0} f(\boldsymbol{x},t) + \int_{t \in \mathcal{R}} \exp(-st)\chi_{\mathcal{T}}(t)\partial_t f(\boldsymbol{x},t)\,\mathrm{d}t \,, \end{aligned} \tag{B.1-9}$$

where $\chi_{\mathcal{T}}(t)$ denotes the characteristic set of the interval $\mathcal{T}$ (Figure B.1-2):

$$\chi_{\mathcal{T}}(t) = \{1, \tfrac{1}{2}, 0\} \qquad \text{for} \quad t \in \{\mathcal{T}, \partial\mathcal{T}, \mathcal{T}'\} \,, \tag{B.1-10}$$

and where the first term on the right-hand side in Equation (B.1-9) accounts for the presence of an impulse function $\delta(t - t_0)$ operative at $t = t_0$, whose strength equals the jump in f when passing the instant $t = t_0$ in the direction of increasing t. Upon incorporating the latter contribution in the definition of the time derivative of f, the rule applies that the s-domain equivalent of the operation of time differentiation is the multiplication by a factor of s.

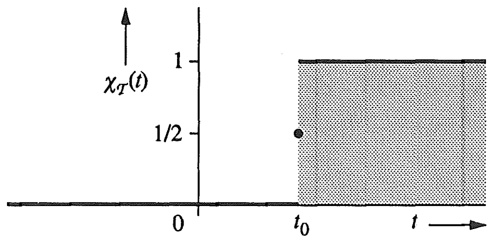

Figure B.1-2 Characteristic function $\chi_{\mathcal{T}}(t)$ of the set $\mathcal{T}$ shown in Figure B.1-1.

Equation (B.1-9) exemplifies that the transformation rules find their simplest expression when the domain of f to be considered is $\mathcal{R}$ rather than $\mathcal{T}$.

Time convolution

Let $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ denote two functions that are defined on $\mathcal{R}$ in time and on some domain in space. Then, the *time convolution* $\mathrm{C}_t(f_1 f_2;x,t)$ of f_1 and f_2 is defined as

$$\mathrm{C}_t(f_1,f_2;x,t) = \int_{t'\in\mathcal{R}} f_1(x,t')\,f_2(x,t-t')\,\mathrm{d}t$$

$$= \int_{t'\in\mathcal{R}} f_1(x,t-t')\,f_2(x,t')\,\mathrm{d}t = \mathrm{C}_t(f_2,f_1;x,t)\,. \tag{B.1-11}$$

Equation (B.1-11) shows that the convolution is a symmetrical functional of the two constituent functions. Taking the Laplace transform of Equation (B.1-11) results in

$$\hat{\mathrm{C}}_t(f_1,f_2;x,s) = \hat{f}_1(x,s)\hat{f}_2(x,s)\,. \tag{B.1-12}$$

For Equation (B.1-12) to be valid, there must exist at least one value of s for which the two definition integrals for $\hat{f}_1(x,s)$ and $\hat{f}_2(x,s)$ converge simultaneously. In the majority of cases this happens in a strip parallel to the imaginary axis in the complex s plane.

The time convolution of $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ is also often symbolised by the notation

$$\mathrm{C}_t(f_1,f_2;x,t) = (f_1 \overset{(t)}{*} f_2)(x,t)\,. \tag{B.1-13}$$

Time correlation

Let $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ denote two functions that are defined on $\mathcal{R}$ in time and on some domain in space. Then, the *time correlation* $\mathrm{R}_t(f_1,f_2;x,t)$ of f_1 and f_2 is defined as

$$\mathrm{R}_t(f_1,f_2;x,t) = \int_{t'\in\mathcal{R}} f_1(x,t')\,f_2(x,t'-t)\,\mathrm{d}t'$$

$$= \int_{t'\in\mathcal{R}} f_1(x,t'+t)\,f_2(x,t')\,\mathrm{d}t' = \mathrm{R}_t(f_2,f_1;x,-t)\,. \tag{B.1-14}$$

Equation (B.1-14) shows that the correlation is not a symmetrical functional of the two constituent functions. Taking the Laplace transform of Equation (B.1-14) results in

$$\hat{\mathrm{R}}_t(f_1,f_2;x,s) = \hat{f}_1(x,s)\hat{f}_2(x,-s)\,. \tag{B.1-15}$$

For Equation (B.1-15) to be valid, there must exist at least one value of s for which the two definition integrals for $\hat{f}_1(x,s)$ and $\hat{f}_2(x,-s)$ converge simultaneously. In the majority of cases this only occurs for imaginary values of s.

Time reversal

Let $f=f(x,t)$ denote a function that is defined on $\mathcal{R}$ in time and on some domain in space. Then, the time reversed $\mathrm{J}_t(f)(x,t)$ of f is defined as

$$\mathrm{J}_t(f)(x,t) = f(x,-t)\,. \tag{B.1-16}$$

Taking the Laplace transform of Equation (B.1-16) results in

$$\hat{\mathrm{J}}_t(f)(x,s) = \hat{f}(x,-s)\,. \tag{B.1-17}$$

For causal time functions, $\hat{f} = \hat{f}(x,s)$ is analytic in some right half $\mathrm{Re}(s) > s_0$ of the complex s plane; then, $\hat{\mathrm{J}}_t(f)(x,s)$ is analytic in the left half $\mathrm{Re}(s) < -s_0$ of the complex s plane.

From Equations (B.1-11), (B.1-14) and (B.1-16) it follows that

$$\mathrm{R}_t(f_1,f_2;x,t) = \mathrm{C}_t(f_1,\mathrm{J}_t(f_2);x,t) = (f_1 \overset{(t)}{*} \mathrm{J}_t(f_2))(x,t)\,, \tag{B.1-18}$$

which upon taking the Laplace transform leads again to Equation (B.1-15).

Bromwich inversion integral

The inverse Laplace transformation can be carried out explicitly by evaluating the following inversion integral (the *Bromwich integral*) in the complex s plane:

$$\frac{1}{2\pi\mathrm{j}}\int_{s\in\mathrm{Br}} \exp(st)\hat{f}(x,s)\,\mathrm{d}s = \chi_{\mathcal{T}}(t)\,f(x,t)\,, \tag{B.1-19}$$

where the path of integration $\mathrm{Br} = \{s\in\mathcal{C};\mathrm{Re}(s) = \sigma_0 \geqslant s_0\}$ (which is parallel to the imaginary s axis) is situated in the right half of the complex s plane where $\hat{f}$ is analytic (Figure B.1-3).

The result when $t\in\partial\mathcal{T}$ holds on the assumption that the integration in the left-hand side is carried out as a Cauchy principal-value integral around "infinity", i.e.

$$\int_{s=\sigma_0-\mathrm{j}\infty}^{\sigma_0+\mathrm{j}\infty} \ldots\,\mathrm{d}s = \lim_{\Omega\to\infty}\int_{s=\sigma_0-\mathrm{j}\Omega}^{\sigma_0+\mathrm{j}\Omega} \ldots\,\mathrm{d}s\,. \tag{B.1-20}$$

In many cases the integration with respect to s can be done analytically by employing theorems of the theory of functions of a complex variable, amongst which are Cauchy's theorem and Jordan's lemma.

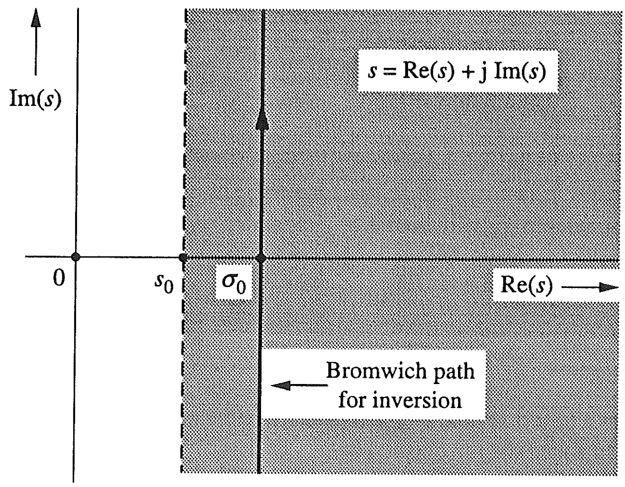

Figure B.1-3 Bromwich path of integration in the complex s plane used for the explicit inversion of the one-sided Laplace transformation.

Cauchy's theorem

Let C be a piecewise smooth simply closed curve in the complex s plane and let $\hat{f} = \hat{f}(x,s)$ be a function that is defined on C and in its interior as well as on some domain in space. Then, *Cauchy's theorem* states that

$$\frac{1}{2\pi j} \oint_{s \in C} \hat{f}(x,s)\, ds = 0 \qquad \text{(B.1-21)}$$

if $\hat{f}$ is analytic in the interior of C and continuous up to C.

Jordan's lemma

Let Γ^- be a circular arc of radius Δ and centre at s_0 on the real s axis, and located in the left half-plane $\mathrm{Re}(s) \leqslant s_0$ (note that the equality sign is included). Let, similarly, Γ^+ be a circular arc of radius Δ and centre at s_0 on the real s axis, and located in the right half-plane $\mathrm{Re}(s) \geqslant s_0$ (note that the equality sign is included). Then, Jordan's lemma states that (Figures B.1-4 and B.1-5)

$$\frac{1}{2\pi j} \int_{s \in \Gamma^-} \exp(st)\hat{f}(x,s)\, ds = o(1) \quad \text{as} \quad \Delta \to \infty \quad \text{if} \quad t > 0 \qquad \text{(B.1-22)}$$

and

$$\frac{1}{2\pi j} \int_{s \in \Gamma^+} \exp(st)\hat{f}(x,s)\, ds = o(1) \quad \text{as} \quad \Delta \to \infty \quad \text{if} \quad t < 0 \qquad \text{(B.1-23)}$$

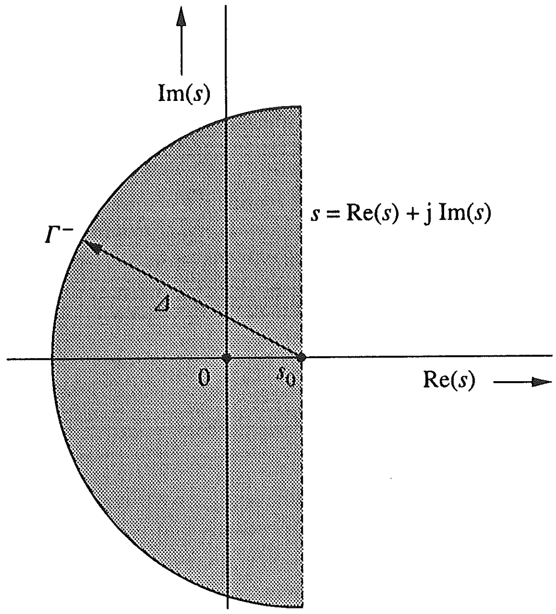

Figure B.1-4 Circular arc Γ^- in the closed left half-plane $\mathrm{Re}(s) \leqslant s_0$ on which Jordan's lemma applies for $t > 0$.

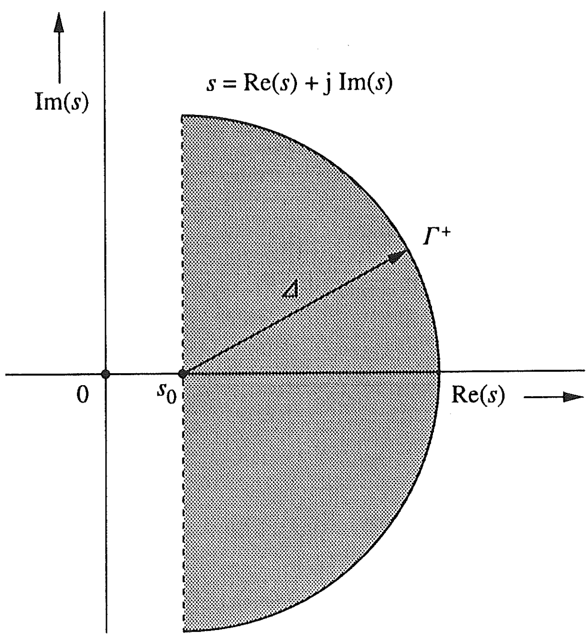

Figure B.1-5 Circular arc Γ^+ in the closed right half-plane $\mathrm{Re}(s) \geqslant s_0$ on which Jordan's lemma applies for $t < 0$.

provided that $\hat{f}(x,s) = o(1)$ as $|s - s_0| \to \infty$.

The Schouten–Van der Pol theorem

A theorem that finds interesting applications in the theory of wave propagation, and more generally in mathematical physics, is the *Schouten–Van der Pol theorem*, which was published independently in 1934 by Schouten and Van der Pol (see Schouten 1934, 1961b; Van der Pol 1934, 1960; Van der Pol and Bremmer 1950). It finds its main applications in the analysis of causal wave functions of time. Let $f = f(x,t)$ be such a wave function and let $\hat{f} = \hat{f}(x,s)$ be its one-sided Laplace transform. Then, writing τ for the time variable of integration,

$$\hat{f}(x,s) = \int_{\tau=t_0}^{\infty} \exp(-s\tau)\, f(x,\tau)\, \mathrm{d}\tau \qquad \text{for} \quad \mathrm{Re}(s) > s_0\,, \tag{B.1-24}$$

in which $\{s \in \mathcal{C}; \mathrm{Re}(s) > s_0\}$ is the right half of the complex s plane where $\hat{f}$ is analytic. Now, in a number of cases, the s-domain solution to a problem that is related to the one of which $\hat{f} = \hat{f}(x,s)$ is the s-domain solution is found as $\hat{\Phi} = \hat{\Phi}(x,s) = \hat{f}[x, \phi(s)]$, i.e. $\hat{\Phi}(x,s)$ follows from $\hat{f} = \hat{f}(x,s)$ by replacing in the latter s by some function $\phi(s)$, where $\phi(s)$ is analytic in the same half-plane as $\hat{f}$. Under these circumstances, it follows from Equation (B.1-24) that

$$\hat{\Phi}(x,s) = \int_{\tau=t_0}^{\infty} \exp[-\phi(s)\tau]\, f(x,\tau)\, \mathrm{d}\tau \qquad \text{for} \quad \mathrm{Re}(s) > s_0\,. \tag{B.1-25}$$

In view of Lerch's uniqueness theorem there now exists a causal function of time $\Phi = \Phi(x,t)$ that has $\hat{\Phi}$ as its one-sided Laplace transform. If, in addition, $\exp[-\phi(s)\tau]$ admits an integral representation of the Laplace type, $\Phi = \Phi(x,t)$ can be obtained by inspection (Schouten–Van der Pol theorem). For such a case, let

$$\exp[-\phi(s)\tau] = \int_{t \in \mathcal{T}_\psi} \exp(-st)\psi(t,\tau)\, \mathrm{d}t \qquad \text{for} \quad \mathrm{Re}(s) > s_0\,, \tag{B.1-26}$$

where $\mathcal{T}_\psi$ is the support of $\psi = \psi(t,\tau)$ in t. (This support may be unbounded to the right.) Substitution of Equation (B.1-26) in Equation (B.1-25) and an interchange of the order of integration lead to

$$\hat{\Phi}(x,s) = \int_{t \in \mathcal{T}_\psi} \exp(-st)\, \mathrm{d}t \int_{\tau=t_0}^{\infty} \psi(t,\tau)\, f(x,\tau)\, \mathrm{d}\tau \qquad \text{for} \quad \mathrm{Re}(s) > s_0\,. \tag{B.1-27}$$

Equation (B.1-27) is of the form

$$\hat{\Phi}(x,s) = \int_{t \in \mathcal{T}_\psi} \exp(-st)\Phi(x,t)\, \mathrm{d}t \qquad \text{for} \quad \mathrm{Re}(s) > s_0\,. \tag{B.1-28}$$

In view of Lerch's uniqueness theorem, $\Phi = \Phi(x,t)$ is found by inspection from Equations (B.1-27) and (B.1-28) as

$$\Phi = \left[\int_{\tau=t_0}^{\infty} \psi(t,\tau)\, f(x,\tau)\, \mathrm{d}\tau \right] \chi_{\mathcal{T}_\psi}(t)\,, \tag{B.1-29}$$

where $\chi_{\mathcal{T}_\psi}$ is the characteristic function of the interval $\mathcal{T}_\psi$.

In a number of cases, the function $\psi = \psi(t,\tau)$ occurring in Equation (B.1-26) can be obtained by evaluating the Bromwich inversion integral

$$\psi(t,\tau) = \frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} \exp(st) \exp[-\phi(s)t] \,\mathrm{d}s \tag{B.1-30}$$

analytically.

Exercises

Exercise B.1-1

Let $f_1 = f_1(x,t)$ be defined on the interval $\mathcal{T}_1 = \{t \in \mathcal{R}; t > t_1\}$ and be zero on the interval $\mathcal{T}_1' = \{t \in \mathcal{R}; t < t_1\}$. Similarly, let $f_2 = f_2(x,t)$ be defined on the interval $\mathcal{T}_2 = \{t \in \mathcal{R}, t > t_2\}$ and be zero on the interval $\mathcal{T}_2' = \{t \in \mathcal{R}; t < t_2\}$. Give the expression for the time convolution of f_1 and f_2, taking into account the ranges where f_1 and f_2 differ from zero.

Answer:

$$\mathrm{C}_t(f_1, f_2; x,t) = \begin{cases} 0 \quad \text{when} \quad t < t_1 + t_2 \,, \\[2ex] \displaystyle\int_{t'=t_1}^{t-t_2} f_1(x,t') f_2(x,t-t') \,\mathrm{d}t' \quad \text{when} \quad t > t_1 + t_2 \,, \\[2ex] \displaystyle\int_{t'=t_2}^{t-t_1} f_1(x,t-t') f_2(x,t') \,\mathrm{d}t' \quad \text{when} \quad t > t_1 + t_2 \,. \end{cases} \tag{B.1-31}$$

Exercise B.1-2

Let $f_1 = f_1(x,t)$ be defined on the interval $\mathcal{T}_1 = \{t \in \mathcal{R}; t > t_1\}$ and be zero on the interval $\mathcal{T}_1' = \{t \in \mathcal{R}; t < t_1\}$. Similarly, let $f_2 = f_2(x,t)$ be defined on the interval $\mathcal{T}_2 = \{t \in \mathcal{R}, t > t_2\}$ and be zero on the interval $\mathcal{T}_2' = \{t \in \mathcal{R}; t < t_2\}$. Give the expression for the time correlation of f_1 and f_2, taking into account the ranges where f_1 and f_2 differ from zero.

Answer:

$$\mathrm{R}_t(f_1, f_2; x,t) = \begin{cases} \displaystyle\int_{t'=\max(t_1, t_2+t)}^{\infty} f_1(x,t') f_2(x,t'-t) \,\mathrm{d}t' \,, \\[2ex] \displaystyle\int_{t'=\max(t_1-t, t_2)}^{\infty} f_1(x,t'+t) f_2(x,t') \,\mathrm{d}t' \,. \end{cases} \tag{B.1-32}$$

Exercise B.1-3

Show that from Equations (B.1-14) and (B.1-15), and the Bromwich inversion integral Equation (B.1-19), for $t = 0$ it follows that

$$\int_{t'\in\mathcal{R}} f_1(x,t')\, f_2(x,t')\, \mathrm{d}t' = (2\pi\mathrm{j})^{-1} \int_{s\in\mathrm{Br}} \hat{f}_1(x,s)\hat{f}_2(x,-s)\, \mathrm{d}s\,. \tag{B.1-33}$$

This result is known as *Plancherel's theorem.*

Exercise B.1-4

Give the result of Exercise B.1-3 for $f_1 = f_2 = f$ (*Parseval's theorem*).

Answer:

$$\int_{t'\in\mathcal{R}} f(x,t')\, f(x,t')\, \mathrm{d}t' = (2\pi\mathrm{j})^{-1} \int_{s\in\mathrm{Br}} \hat{f}(x,s)\hat{f}(x,-s)\, \mathrm{d}s\,. \tag{B.1-34}$$

Exercise B.1-5

Show that the left-hand side of Equation (B.1-19) yields the value zero if $t\in\mathcal{T}'$. (*Hint*: Use Cauchy's theorem, Jordan's lemma, Equation (B.1-7), and the fact that $\hat{f}$ is analytic in the right half $\mathrm{Re}(s) > \sigma_0$ of the complex s plane.)

Exercise B.1-6

Determine the Laplace transform $\hat{f} = \hat{f}(s)$ of the following class of functions: $f(t) = \exp[\alpha(t-t_0)][(t-t_0)^n/n!]\chi_{\mathcal{T}}(t)$. Here, n is a non-negative integer, α is a complex number, and $\mathcal{T} = \{t\in\mathcal{R};t>t_0\}$.

Answer:

$$\hat{f}(s) = (s-\alpha)^{-n-1}\exp(-st_0) \qquad \text{for} \quad \mathrm{Re}(s) > \mathrm{Re}(\alpha)\,.$$

Exercise B.1-7

Substitute the result of Exercise B.1-6 in Equation (B.1-19) and verify the result. (*Hint*: Close the path of integration in Equation (B.1-19) by supplementing it by a semi-circle "at infinity" in the half-plane $\mathrm{Re}(s) < \mathrm{Re}(\alpha)$, make use of Equation (B.1-7) and apply Jordan's lemma and the residue theorem.)

Exercise B.1-8

Determine the Laplace transform $\hat{f} = \hat{f}(s)$ of the function $f = f(t)$ that is defined as follows: $f_\Delta(t) = \{0,\Delta^{-1},0\}$ for $\{t<0,\ 0<t<\Delta,\ t>\Delta\}$ and has unit area under its graphical representation.

Answer:

$$f(s) = [1-\exp(-s\Delta)]/s\Delta\,.$$

Exercise B.1-9

In the limit $\Delta \to 0$ the function f_Δ defined in Exercise B.1-8 approaches the (one-dimensional) Dirac delta distribution (a generalised function) $\delta(t)$. Determine the Laplace transform $\hat{\delta}(s)$ of $\delta(t)$ by taking the limit $\Delta \to 0$ in the answer of Exercise B.1-8.

Answer:

$$\hat{\delta}(s) = 1 \qquad \text{for all} \quad s \in \mathcal{C}.$$

Exercise B.1-10

Determine the Laplace transform $\hat{f} = \hat{f}(s)$ of the Heaviside unit step function $f(t) = H(t)$, with $H(t) = \{0, \tfrac{1}{2}, 1\}$ for $\{t<0,\ t=0,\ t>0\}$.

Answer:

$$\hat{f}(s) = s^{-1} \qquad \text{for} \quad \mathrm{Re}(s) > 0\,.$$

Exercise B.1-11

Determine the Laplace transform $\hat{f} = \hat{f}(s)$ of the right-periodic function

$$f(t) = \sum_{n=0}^{\infty} \phi(t - nT)[H(t - nT) - H(t - nT - T)]\,,$$

in which $H(t)$ is the Heaviside unit step function.

Answer: $\hat{f}(s) = \hat{\phi}(s,T)/[1 - \exp(-sT)]$, where

$$\hat{\phi}(s,T) = \int_{t=0}^{T} \exp(-st)\phi(t)\,\mathrm{d}t\,.$$

Exercise B.1-12

Let $\hat{f} = \hat{f}(s)$ be the Laplace transform of $f = f(t)$ with support $\mathcal{T}$. Show that

$$\int_{t\in\mathcal{T}} f(t)\,\mathrm{d}t = \hat{f}(0)\,. \qquad \text{(B.1-35)}$$

(This result can be useful in the evaluation of certain integrals.)

Exercise B.1-13

Let us denote the function $f = f(x,t)$ as causal if its support is $\{t\in\mathcal{R};t \geqslant 0\}$ and as anti-causal if its support is $\{t\in\mathcal{R};t \leqslant 0\}$. Under what conditions is the time convolution $\mathrm{C}_t(f_1,f_2;x,t)$ of $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ (a) causal, (b) anti-causal? Under what conditions is the time correlation $\mathrm{R}_t(f_1,f_2;x,t)$ of $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ (c) causal, (d) anti-causal?

Answer: (a) if f_1 is causal and f_2 is causal; (b) if f_1 is anti-causal and f_2 is anti-causal; (c) if f_1 is causal and f_2 is anti-causal; (d) if f_1 is anti-causal and f_2 is causal.

B.2 Spatial Fourier transformation

Since, depending on circumstances, particularly on the number of spatial dimensions in which a configuration is shift invariant in space, the Fourier transformation is carried out with respect to a single, to several, or to all spatial dimensions, the main properties of the spatial Fourier transformation will be discussed for an arbitrary number $N \geqslant 1$ of dimensions. Also, depending on circumstances, particularly on whether the functions to be transformed have a bounded or an unbounded support (= domain in which they differ from zero), the transformation can be carried out over a bounded subdomain $\mathcal{D}$ of $\mathcal{R}^N$ or over the entire space $\mathcal{R}^N$. Usually, since most wave-field configurations are time invariant even if not shift invariant in space, the spatial Fourier transformations are carried out after the time Laplace transform has already been taken.

Let us consider the scalar wave-field quantity $f = f(\boldsymbol{x},t)$ that is defined in some bounded or unbounded domain $\mathcal{D}$ in N-dimensional Euclidean space $\mathcal{R}^N$ and let $\hat{f} = \hat{f}(\boldsymbol{x},s)$ denote its time Laplace transform. The *spatial Fourier transformation* of $\hat{f}$ over the domain $\mathcal{D}$ is then defined as

$$\tilde{f}(\mathrm{j}\boldsymbol{k},s) = \int_{\boldsymbol{x}\in\mathcal{D}} \exp(\mathrm{j}k_q x_q)\hat{f}(\boldsymbol{x},s)\,\mathrm{d}V\,, \tag{B.2-1}$$

where $\mathrm{d}V = \mathrm{d}x_1 \ldots \mathrm{d}x_N$ is the elementary volume in $\mathcal{R}^N$. Let, further, $\partial\mathcal{D}$ denote the boundary surface of $\mathcal{D}$ and let $\mathcal{D}'$ denote the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^N$ (Figure B.2-1).

Extending the definition of $\hat{f}(\boldsymbol{x},s)$ by the value zero for $\boldsymbol{x}\in\mathcal{D}'$, Equation (B.2-1) can also be written as

$$\tilde{f}(\mathrm{j}\boldsymbol{k},s) = \int_{\boldsymbol{x}\in\mathcal{R}^N} \exp(\mathrm{j}k_q x_q)\hat{f}(\boldsymbol{x},s)\,\mathrm{d}V\,. \tag{B.2-2}$$

In Equations (B.2-1) and (B.2-2), $\boldsymbol{k}$ is denoted as the *angular wave vector*; in terms of its Cartesian components we have

$$\boldsymbol{k} = k_1\boldsymbol{i}(1) + \ldots + k_N\boldsymbol{i}(N)\,. \tag{B.2-3}$$

Furthermore,

$$k_q x_q = k_1 x_1 + \ldots + k_N x_N\,. \tag{B.2-4}$$

In our general analysis, we take $\boldsymbol{k}\in\mathcal{R}^N$. Then a sufficient condition for the convergence of the definition integrals is the absolute integrability of $\hat{f}(\boldsymbol{x},s)$ over the domain $\mathcal{D}$. Since then for unbounded domains $\hat{f}(\boldsymbol{x},s)$ necessarily goes to zero as $|\boldsymbol{x}|\to\infty$, we denote this class of functions also as localised (in space). The Fourier transform domain is also denoted as the angular wave-vector domain or the spectral domain. The transformation from the angular wave-vector domain back to the spatial domain is usually carried out by employing the *Fourier inversion integral*

$$(2\pi)^{-N}\int_{\boldsymbol{k}\in\mathcal{R}^N} \exp(-\mathrm{j}k_q x_q)\tilde{f}(\mathrm{j}k_q x_q)\,\mathrm{d}V = \chi_{\mathcal{D}}(\boldsymbol{x})\hat{f}(\boldsymbol{x},s)\,, \tag{B.2-5}$$

where $\chi_{\mathcal{D}}(\boldsymbol{x})$ denotes the characteristic set of the domain $\mathcal{D}$ (Figure B.2-2):

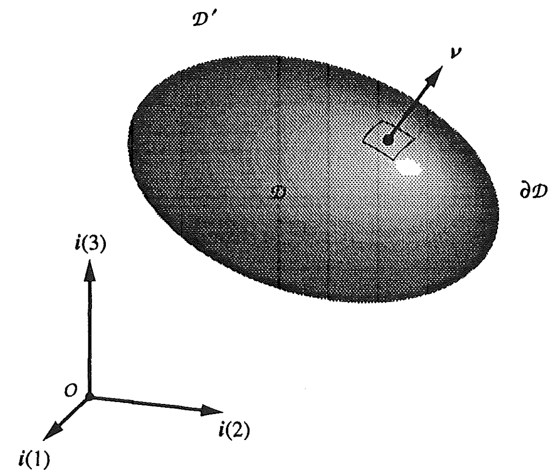

Figure B.2-1 Domain $\mathcal{D}$, with boundary surface $\partial\mathcal{D}$ and the complement $\mathcal{D}'$ of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$; ν is the unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$.

$$\chi_{\mathcal{D}} = \begin{cases} 1 & \text{for } x\in\mathcal{D}, \\ \tfrac{1}{2} & \text{for } x\in\partial\mathcal{D}, \\ 0 & \text{for } x\in\mathcal{D}'. \end{cases} \tag{B.2-6}$$

The result for $x\in\partial\mathcal{D}$ holds on the assumption that $\partial\mathcal{D}$ has a unique tangent plane, while the integral on the left-hand side has to be interpreted as a Cauchy principal-value integral around "infinity". In a number of cases the integrations with respect to some (or all) k_q can be carried out analytically by employing theorems of the theory of functions of a complex variable (amongst which are Cauchy's theorem and Jordan's lemma). Next, some elementary rules for the Fourier transformation will be discussed.

The Fourier transform of $\hat{f} = \hat{f}(x,s)$ is also often symbolised by the notation

$$\tilde{f}(\mathrm{j}k,s) = \mathcal{F}(f;\mathrm{j}k,s)\,. \tag{B.2-7}$$

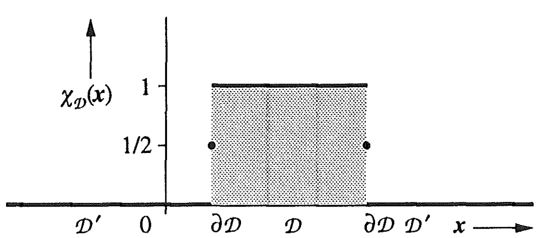

Figure B.2-2 Characteristic function $\chi_{\mathcal{D}}(x)$ of the set $\mathcal{D}$.

Behaviour as $|k| \to \infty$

The asymptotic behaviour of the Fourier transform as the transform parameter goes to infinity is found to be

$$\tilde{f}(jk,s) = o(1) \quad \text{as} \quad |k| \to \infty \quad \text{for} \quad k \in \mathcal{R}^N . \tag{B.2-8}$$

The result of Equation (B.2-8) is known as the *Riemann–Lebesgue lemma*.

Differentiation with respect to the spatial coordinates

Let $f = f(x,t)$ denote a function that is defined on $\mathcal{D}$ and that is equal to zero on $\mathcal{D}'$. Let, further, $\hat{f} = \hat{f}(x,s)$ denote its time Laplace transform. Then, the spatial Fourier transform of $\partial_p \hat{f}$ is found as

$$\int_{x \in \mathcal{D}} \exp(jk_q x_q) \partial_p \hat{f}(x,s)\, dV$$

$$= \int_{x \in \mathcal{D}} \{\partial_p [\exp(jk_q x_q) \hat{f}(x,s)] - [\partial_p \exp(jk_q x_q)]\, \hat{f}(x,s)\}\, dV$$

$$= \int_{x \in \partial\mathcal{D}} \exp(jk_q x_q) \hat{f}(x,s) \nu_p\, dA - jk_p \tilde{f}(jk,s) , \tag{B.2-9}$$

where Gauss' theorem has been used to arrive at the integral over $\partial\mathcal{D}$, ν_p denotes the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$, and the value of $\hat{f}$ on $\partial\mathcal{D}$ is the limiting value approaching $\partial\mathcal{D}$ via $\mathcal{D}$. Equation (B.2-9) can be rewritten as

$$-jk_p \tilde{f}(jk,s) = \int_{x \in \mathcal{R}^N} \exp(jk_q x_q) \partial_p [\chi_{\mathcal{D}}(x) \hat{f}(x,s)]\, dV$$

$$= \int_{x \in \mathcal{R}^N} \exp(jk_q x_q) [\partial_p \chi_{\mathcal{D}}(x)]\, \hat{f}(x,s)\, dV + \int_{x \in \mathcal{R}^N} \exp(jk_q x_q) \chi_{\mathcal{D}}(x) [\partial_p \hat{f}(x,s)]\, dV$$

$$= - \int_{x \in \partial\mathcal{D}} \exp(jk_q x_q) \nu_p \hat{f}(x,s)\, dA + \int_{x \in \mathcal{D}} \exp(jk_q x_q) \partial_p \hat{f}(x,s)\, dA , \tag{B.2-10}$$

where the property has been used that $\partial_p \chi_{\mathcal{D}}(x)$ has a spatial unit pulse behaviour in the direction of the unit vector along the normal to $\partial\mathcal{D}$ when this would point toward $\mathcal{D}$ (which explains the minus sign in front of the integral over $\partial\mathcal{D}$ in Equation (B.2-10)).

Spatial convolution

Let $\hat{f}_1 = \hat{f}_1(x,s)$ and $\hat{f}_2 = \hat{f}_2(x,s)$ denote two time Laplace transform domain functions that are defined on $\mathcal{R}^N$. Then, the *spatial convolution* $C_x(\hat{f}_1, \hat{f}_2; x, s)$ of $\hat{f}_1$ and $\hat{f}_2$ is defined as

$$\mathrm{C}_x(\hat{f}_1,\hat{f}_2;x,s) = \int_{x'\in\mathcal{R}^N} \hat{f}_1(x',s)\hat{f}_2(x-x',s)\,\mathrm{d}V$$

$$= \int_{x'\in\mathcal{R}^N} \hat{f}_1(x-x',s)\hat{f}_2(x',s)\,\mathrm{d}V = \mathrm{C}_x(\hat{f}_2,\hat{f}_1;x,s)\,. \tag{B.2-11}$$

Equation (B.2-11) shows that the convolution is a symmetrical functional of the two constituent functions. Taking the Fourier transform of Equation (B.2-11) results in

$$\tilde{\mathrm{C}}_x(\hat{f}_1,\hat{f}_2;\mathrm{j}k,s) = \tilde{f}_1(\mathrm{j}k,s)\,\tilde{f}_2(\mathrm{j}k,s)\,. \tag{B.2-12}$$

For Equation (B.2-12) to be valid, there must exist at least one value of k for which the two definition integrals for $\tilde{f}_1 = \tilde{f}_1(\mathrm{j}k,s)$ and $\tilde{f}_2 = \tilde{f}_2(\mathrm{j}k,s)$ converge simultaneously. For absolutely integrable functions this is the case for real values of k.

The spatial convolution of $\hat{f}_1 = \hat{f}_1(x,s)$ and $\hat{f}_2 = \hat{f}_2(x,s)$ is often symbolised by the notation

$$\mathrm{C}_x(\hat{f}_1,\hat{f}_2;x,s) = (\hat{f}_1 \overset{(x)}{*} \hat{f}_2)(\mathrm{j}k,s)\,. \tag{B.2-13}$$

Spatial correlation

Let $\hat{f}_1 = \hat{f}_1(x,s)$ and $\hat{f}_2 = \hat{f}_2(x,s)$ denote two time Laplace transform domain functions that are defined on $\mathcal{R}^N$. Then, the *spatial correlation* $\mathrm{R}_x(f_1,f_2;x,s)$ of $\hat{f}_1$ and $\hat{f}_2$ is defined as

$$\mathrm{R}_x(\hat{f}_1,\hat{f}_2;x,s) = \int_{x'\in\mathcal{R}^N} \hat{f}_1(x',s)\hat{f}_2(x'-x,s)\,\mathrm{d}V$$

$$= \int_{x'\in\mathcal{R}^N} \hat{f}_1(x'+x,s)\hat{f}_2(x',s)\,\mathrm{d}V = \mathrm{R}_x(\hat{f}_2,\hat{f}_1;-x)\,. \tag{B.2-14}$$

Equation (B.2-14) shows that the correlation is not a symmetrical functional of the two constituent functions. Taking the Fourier transform of Equation (B.2-14) results into

$$\tilde{\mathrm{R}}_x(\hat{f}_1,\hat{f}_2;\mathrm{j}k,s) = \tilde{f}_1(\mathrm{j}k,s)\,\tilde{f}_2(-\mathrm{j}k,s)\,. \tag{B.2-15}$$

For Equation (B.2-15) to be valid, there must exist at least one value of k for which the two definition integrals for $\hat{f}_1(\mathrm{j}k,s)$ and $\hat{f}_2(-\mathrm{j}k,s)$ converge simultaneously. For absolutely integrable functions this is the case for real values of k.

Spatial inversion with respect to a point

Let $\hat{f} = \hat{f}(x,s)$ denote a time Laplace transform domain function that is defined on $\mathcal{R}^N$ in space. Then, the spatially inverted $\mathrm{J}_x(\hat{f})(x,s)$ of $\hat{f}$ is defined as

$$\mathrm{J}_x(\hat{f})(x,s) = \hat{f}(-x,s)\,. \tag{B.2-16}$$

Taking the Fourier transform of Equation (B.2-16) results into

$$\tilde{\mathrm{J}}_x(\hat{f})(\mathrm{j}k,s) = \tilde{f}(-\mathrm{j}k,s)\,. \tag{B.2-17}$$

From Equations (B.2-11), (B.2-14) and (B.2-16) it follows that

$$\mathrm{R}_x(\hat{f}_1,\hat{f}_2;x,s) = \mathrm{C}_x(\hat{f}_1,\mathrm{J}_x(\hat{f}_2);x,s)\,, \tag{B.2-18}$$

which upon taking the Fourier transform leads again to Equation (B.2-15).

Numerical evaluation

It is standard practice to numerically evaluate Fourier transforms with the aid of Fast Fourier Transform algorithms (FFT-algorithms). In them, the properties of trigonometric functions of a single variable when sampled at 2^N equidistant abscissa (N = non-negative integer) are properly combined to yield an algorithm that is considerably faster than any of the ones that would be based on standard numerical integration formulas (see, for example, Brigham 1974).

Exercises

Exercise B.2-1

Show that from Equations (B.2-14) and (B.2-15) and the Fourier inversion theorem Equation (B.2-5) for $\boldsymbol{x} = 0$ it follows that

$$\int_{\boldsymbol{x}'\in\mathcal{R}^N} \hat{f}_1(\boldsymbol{x}',s)\hat{f}_2(\boldsymbol{x}',s)\,\mathrm{d}V = (2\pi)^{-N}\int_{\boldsymbol{k}\in\mathcal{R}^N} \tilde{f}_1(\mathrm{j}\boldsymbol{k},s)\,\tilde{f}_2(-\mathrm{j}\boldsymbol{k},s)\,\mathrm{d}V\,. \tag{B.2-19}$$

This result is known as *Plancherel's theorem.*

Exercise B.2-2

Give the result of Equation (B.2-19) for $\hat{f}_1 = \hat{f}_2$ (*Parseval's theorem*).

Answer:

$$\int_{\boldsymbol{x}'\in\mathcal{R}^N} \hat{f}(\boldsymbol{x}',s)\hat{f}(\boldsymbol{x}',s)\,\mathrm{d}V = (2\pi)^{-N}\int_{\boldsymbol{k}\in\mathcal{R}^N} \tilde{f}(\mathrm{j}\boldsymbol{k},s)\,\tilde{f}(-\mathrm{j}\boldsymbol{k},s)\,\mathrm{d}V\,. \tag{B.2-20}$$

Exercise B.2-3

Determine the Fourier transform $\tilde{\chi}_{\mathcal{D}} = \tilde{\chi}_{\mathcal{D}}(\mathrm{j}\boldsymbol{k})$ of the characteristic function $\chi_{\mathcal{D}} = \chi_{\mathcal{D}}(\boldsymbol{x})$ of the ellipsoid $\mathcal{D} = \{\boldsymbol{x}\in\mathcal{R}^3\,;0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 < 1\}$ in $\mathcal{R}^3$, where a_1, a_2 and a_3 are real numbers. (*Hint*: Replace in the definition integral of the Fourier transform over $\mathcal{D}$ the variables of integration by $y_1 = x_1/a_1$, $y_2 = x_2/a_2$, $y_3 = x_3/a_3$, and introduce in the resulting integral spherical coordinates with O as centre and $\boldsymbol{k}$ as polar axis.)

Answer:

$$\tilde{\chi}_{\mathcal{D}} = (4\pi a_1 a_2 a_3/3) * 3\Upsilon^{-3}[-\Upsilon\cos(\Upsilon) + \sin(\Upsilon)],$$

in which

$$\Upsilon = \left[(k_1a_1)^2 + (k_2a_2)^2 + (k_3a_3)^2\right]^{1/2} > 0 .$$

Exercise B.2-4

Determine the Fourier transform $\tilde{\chi}_{\mathcal{D}} = \tilde{\chi}_{\mathcal{D}}(\mathrm{j}\boldsymbol{k})$ of the characteristic function $\chi_{\mathcal{D}} = \chi_{\mathcal{D}}(\boldsymbol{x})$ of the rectangular block $\mathcal{D} = \{\boldsymbol{x}\in\mathcal{R}^3; -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3\}$ in $\mathcal{R}^3$, where a_1, a_2 and a_3 are real positive numbers.

Answer:

$$\tilde{\chi}_{\mathcal{D}} = 8a_1a_2a_3 * \frac{\sin(k_1a_1)}{k_1a_1}\frac{\sin(k_2a_2)}{k_2a_2}\frac{\sin(k_3a_3)}{k_3a_3} .$$

Exercise B.2-5

Determine the Fourier transform $\tilde{f} = \tilde{f}(\mathrm{j}\boldsymbol{k})$ of the function $f = f(\boldsymbol{x})$ that is defined as follows: $f_\Delta = \Delta^{-3}$ for $\{\boldsymbol{x}\in\mathcal{R}^3; -\Delta/2 < x_1 < \Delta/2,\ -\Delta/2 < x_2 < \Delta/2,\ -\Delta/2 < x_3 < \Delta/2\}$ and $f_\Delta = 0$ elsewhere.

Answer:

$$\tilde{f}_\Delta = \Delta^{-3}\frac{\sin(k_1\Delta)}{k_1}\frac{\sin(k_2\Delta)}{k_2}\frac{\sin(k_3\Delta)}{k_3} .$$

Exercise B.2-6

In the limit $\Delta\to 0$ the function f_Δ defined in Exercise B.2-5 approaches the (three-dimensional) Dirac delta distribution (a generalised function) $\delta(\boldsymbol{x})$. Determine the Fourier transform $\tilde{\delta}(\mathrm{j}\boldsymbol{k})$ of $\delta(\boldsymbol{x})$ by taking the limit $\Delta\to 0$ in the answer of Exercise B.2-5.

Answer: $\tilde{\delta}(\mathrm{j}\boldsymbol{k}) = 1$ for all $\boldsymbol{k}\in\mathcal{R}^3$.

Exercise B.2-7

Let $\tilde{f} = \tilde{f}(\mathrm{j}\boldsymbol{k})$ be the Fourier transform of $f = f(\boldsymbol{x})$ defined over the domain $\mathcal{D}$. Show that

$$\int_{\boldsymbol{x}\in\mathcal{D}} f(\boldsymbol{x})\,\mathrm{d}V = \tilde{f}(\boldsymbol{0}) . \qquad \text{(B.2-21)}$$

(This result can be useful in the evaluation of certain integrals.)

B.3 The Kramers–Kronig causality relations

The Kramers–Kronig causality relations express the behaviour of the Laplace transform of a causal time function defined on the interval $\{t\in\mathcal{R}; t > 0\}$ in case the complex Laplace-transform parameter s approaches, via the half-plane $\mathrm{Re}(s) > 0$, the imaginary axis of the s plane. The relations are in particular of importance in describing the causal behaviour of matter insofar as this behaviour expresses itself in constitutive relaxation functions. Let $f = f(\boldsymbol{x},t)$ denote a causal space–time function and let $\hat{f} = \hat{f}(x,s)$ denote its Laplace transform according to Equation (B.1-4). The complex function $\hat{f} = \hat{f}(x,s)$ is analytic in the complex variable s and regular in the half-plane $\mathrm{Re}(s) > 0$. We assume that its limiting value as $s \to \mathrm{j}\omega$, where j is the imaginary unit and $\omega \in \mathcal{R}$, exists and is given by

$$\hat{f}(x,\mathrm{j}\omega) = \int_{t=0}^{\infty} \exp(-\mathrm{j}\omega t)\, f(x,t)\, \mathrm{d}t\,. \tag{B.3-1}$$

(A sufficient condition for the existence of the right-hand side of Equation (B.3-1) is the existence of

$$\int_{t=0}^{\infty} |f(x,t)|\, \mathrm{d}t\,,$$

i.e. the absolute integrability of $f(x,t)$.) For any real-valued $f(x,t)$ we now decompose $\hat{f}(x,\mathrm{j}\omega)$ into its real and imaginary parts according to (note the sign in front of j, which is chosen in accordance with international convention)

$$\hat{f}(x,\mathrm{j}\omega) = f'(x,\omega) - \mathrm{j}\, f''(x,\omega)\,. \tag{B.3-2}$$

Since

$$\exp(-\mathrm{j}\omega t) = \cos(\omega t) - \mathrm{j}\, \sin(\omega t)\,, \tag{B.3-3}$$

Equation (B.3-1) leads to

$$f'(x,\omega) = \int_{t=0}^{\infty} \cos(\omega t)\, f(x,t)\, \mathrm{d}t\,, \tag{B.3-4}$$

$$f''(x,\omega) = \int_{t=0}^{\infty} \sin(\omega t)\, f(x,t)\, \mathrm{d}t\,. \tag{B.3-5}$$

From Equations (B.3-4) and (B.3-5) it follows that, since $\cos(-\omega t) = \cos(\omega t)$ and $\sin(-\omega t) = -\sin(\omega t)$,

$$f'(x,-\omega) = f'(x,\omega) \tag{B.3-6}$$

and

$$f''(x,-\omega) = -f''(x,\omega)\,, \tag{B.3-7}$$

respectively. Hence, f' is an even function of ω and f'' is an odd function of ω. In their turn, these symmetry relations in ω have their consequences for the Fourier representation of the space–time function f that corresponds to $\hat{f}$. Writing the Bromwich inversion integral given in Equation (B.1-19) as an integral over the imaginary axis in the complex s plane (i.e. taking $s = \mathrm{j}\omega$, with the consequence that $\mathrm{d}s = \mathrm{j}\, \mathrm{d}\omega$), we arrive at the Fourier representation

$$\begin{aligned} f(x,t) &= (2\pi)^{-1} \int_{\omega=-\infty}^{\infty} \exp(-\mathrm{j}\omega t) \hat{f}(x,\mathrm{j}\omega)\, \mathrm{d}\omega \\ &= (2\pi)^{-1} \int_{\omega=-\infty}^{\infty} [\cos(\omega t) + \mathrm{j}\, \sin(\omega t)]\, [f'(x,\omega) - \mathrm{j}\, f''(x,\omega)]\, \mathrm{d}\omega \\ &= \pi^{-1} \int_{\omega=0}^{\infty} \cos(\omega t)\, f'(x,\omega)\, \mathrm{d}\omega + \pi^{-1} \int_{\omega=0}^{\infty} \sin(\omega t)\, f''(x,\omega)\, \mathrm{d}\omega\,, \end{aligned} \tag{B.3-8}$$

where Equations (B.3-6) and (B.3-7) have been used. Now, the right-hand side of Equation (B.3-8) must vanish for $t < 0$, since the left-hand side was defined to be zero in the interval $\{t \in \mathcal{R}; t < 0\}$. Hence,

$$\pi^{-1} \int_{\omega=0}^{\infty} \cos(\omega t) f'(x,\omega)\, \mathrm{d}\omega + \pi^{-1} \int_{\omega=0}^{\infty} \sin(\omega t) f''(x,\omega)\, \mathrm{d}\omega = 0 \quad \text{for } t < 0 . \tag{B.3-9}$$

Upon writing $t = -t'$, where $t' > 0$, we therefore obtain

$$\pi^{-1} \int_{\omega=0}^{\infty} \cos(\omega t') f'(x,\omega)\, \mathrm{d}\omega - \pi^{-1} \int_{\omega=0}^{\infty} \sin(\omega t') f''(x,\omega)\, \mathrm{d}\omega = 0 \quad \text{for } t' > 0 . \tag{B.3-10}$$

Using Equation (B.3-10) in Equation (B.3-8), the following alternative expressions are obtained:

$$f(x,t) = (2/\pi) \int_{\omega=0}^{\infty} \cos(\omega t) f'(x,\omega)\, \mathrm{d}\omega \qquad \text{for} \quad t > 0 , \tag{B.3-11}$$

or

$$f(x,t) = (2/\pi) \int_{\omega=0}^{\infty} \sin(\omega t) f''(x,\omega)\, \mathrm{d}\omega \qquad \text{for} \quad t > 0 . \tag{B.3-12}$$

Note that the representations (B.3-11) and (B.3-12) only apply to causal space–time functions that are defined on the interval $\{t \in \mathcal{R}; t > 0\}$. Since the left-hand sides of Equations (B.3-11) and (B.3-12) represent one and the same function in the interval $\{t \in \mathcal{R}; t > 0\}$, their right-hand sides must do so, too, and hence $f'(x,\omega)$ and $f''(x,\omega)$ must be interrelated. This interrelation shows up by further investigating the properties of $\hat{f}(x, \mathrm{j}\omega)$ in the complex s plane, which is done next.

From the causality of the function $f(x,t)$ under investigation it follows that $\hat{f}(x,s)$ is an analytic function of the complex variable s, that is regular in the right half $\mathrm{Re}(s) > 0$ of the complex s plane. Let $s = \mathrm{j}\Omega$ be a point on the imaginary s-axis and let $\mathcal{C}$ denote the closed path of integration in the complex s plane as shown in Figure B.3-1.

The part $\mathcal{C}_\Delta$ of $\mathcal{C}$ is the semi circle with centre at $s = \mathrm{j}\Omega$ and radius $\Delta > 0$; the part $\mathcal{C}_\delta$ of $\mathcal{C}$ is the semi circle with centre at $s = \mathrm{j}\Omega$ and radius $\delta > 0$. The remaining part of $\mathcal{C}$ lies along the imaginary s axis. It is assumed that $\delta < \Delta$. Application of Cauchy's theorem then yields

$$\int_{s \in \mathcal{C}} \frac{\hat{f}(x,s)}{s - \mathrm{j}\Omega}\, \mathrm{d}s = 0 , \tag{B.3-13}$$

since $\hat{f}$ and $(s - \mathrm{j}\Omega)^{-1}$ are regular in the bounded domain interior to $\mathcal{C}$. In view of Equation (B.1-7) for $t_0 = 0$, the contribution from $\mathcal{C}_\Delta$ to the integral is estimated as

$$\left| \int_{s \in \mathcal{C}_\Delta} \frac{\hat{f}(x,s)}{s - \mathrm{j}\Omega}\, \mathrm{d}s \right| = o(1) \qquad \text{as} \quad \Delta \to \infty , \tag{B.3-14}$$

while a local Taylor expansion of $\hat{f}(x,s)$ around $s = \mathrm{j}\Omega$ on $\mathcal{C}_\delta$ and a subsequent evaluation of the resulting integral along $\mathcal{C}_\delta$ lead to

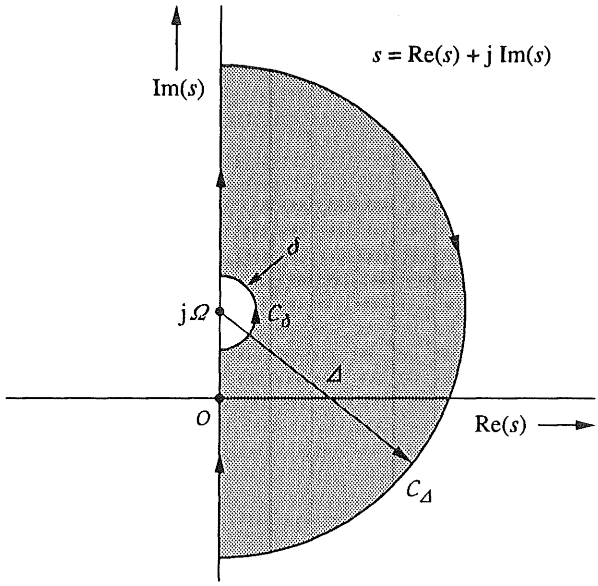

Figure B.3-1 Closed contour of integration in the complex s plane used to establish the Kramers–Kronig causality relations.

$$\int_{s \in C_\delta} \frac{\hat{f}(x,s)}{s - \mathrm{j}\Omega} \mathrm{d}s \rightarrow \pi \mathrm{j} \hat{f}(x,\mathrm{j}\Omega) \quad \text{as} \quad \delta \rightarrow 0 . \tag{B.3-15}$$

Taking both limits $\Delta \rightarrow \infty$ and $\delta \rightarrow 0$ in Equation (B.3-13), there results

$$⨍_{\omega=-\infty}^{\infty} \frac{\hat{f}(x,\mathrm{j}\omega)}{\omega - \Omega} \mathrm{d}\omega + \pi \mathrm{j} \hat{f}(x,\mathrm{j}\Omega) = 0 , \tag{B.3-16}$$

in which $⨍$ denotes the *Cauchy principal value* of the relevant integral, which is defined as

$$⨍_{\omega=-\infty}^{\infty} \ldots \mathrm{d}\omega = \lim_{\Delta \rightarrow \infty, \delta \rightarrow 0} \left[\int_{\omega=\Omega-\Delta}^{\Omega-\delta} \ldots \mathrm{d}\omega + \int_{\omega=\Omega+\delta}^{\Omega+\Delta} \ldots \mathrm{d}\omega \right] . \tag{B.3-17}$$

Next, Equation (B.3-2) is used, and Equation (B.3-16) is decomposed into its real and imaginary parts. The result is

$$f''(x,\Omega) = -\frac{1}{\pi} ⨍_{\omega=-\infty}^{\infty} \frac{f'(x,\omega)}{\omega - \Omega} \mathrm{d}\omega \quad \text{for} \quad \Omega \in \mathcal{R} \tag{B.3-18}$$

and

$$f'(x,\Omega) = \frac{1}{\pi} ⨍_{\omega=-\infty}^{\infty} \frac{f''(x,\omega)}{\omega - \Omega} \mathrm{d}\omega \quad \text{for} \quad \Omega \in \mathcal{R} . \tag{B.3-19}$$

Equations (B.3-18) and (B.3-19) are known as the Kramers–Kronig causality relations. They interrelate the quantities $f'(x,\omega)$ and $f''(x,\omega)$. (Note that these quantities have only been defined for real values of ω.) Without any recourse to complex-function theory the relations (B.3-18) and (B.3-19) define the *Hilbert transformation* and express that $f'(x,\omega)$ and $f''(x,\omega)$ form a pair of *Hilbert transforms.*

Several alternative forms of Equations (B.3-18) and (B.3-19) exist. First, it is observed that

$$\int_{\omega=-\infty}^{\infty} \frac{1}{\omega - \Omega}\,\mathrm{d}\omega = 0 \qquad \text{for} \quad \Omega \in \mathcal{R}. \tag{B.3-20}$$

This follows from

$$\int_{s\in C} \frac{1}{s - \mathrm{j}\Omega}\,\mathrm{d}s = 0\,, \tag{B.3-21}$$

together with

$$\int_{s\in C_\Delta} \frac{1}{s - \mathrm{j}\Omega}\,\mathrm{d}s \to -\pi\mathrm{j} \qquad \text{as} \quad \Delta \to \infty\,, \tag{B.3-22}$$

and

$$\int_{s\in C_\delta} \frac{1}{s - \mathrm{j}\Omega}\,\mathrm{d}s \to \pi\mathrm{j} \qquad \text{as} \quad \delta \to 0\,. \tag{B.3-23}$$

With the aid of Equation (B.3-20), Equations (B.3-18) and (B.3-19) can be rewritten as

$$f''(x,\Omega) = -\frac{1}{\pi}\int_{\omega=-\infty}^{\infty} \frac{f'(x,\omega) - f'(x,\Omega)}{\omega - \Omega}\,\mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}, \tag{B.3-24}$$

and

$$f'(x,\Omega) = \frac{1}{\pi}\int_{\omega=-\infty}^{\infty} \frac{f''(x,\omega) - f''(x,\Omega)}{\omega - \Omega}\,\mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}. \tag{B.3-25}$$

In Equations (B.3-24) and (B.3-25) the integrals around $\omega = \Omega$ are no longer principal-value integrals but proper integrals.

Secondly, the symmetry properties given in Equations (B.3-6) and (B.3-7) can be used to rewrite Equations (B.3-18) and (B.3-19) as

$$f''(x,\Omega) = -\frac{2}{\pi}\int_{\omega=0}^{\infty} \frac{f'(x,\omega)\Omega}{\omega^2 - \Omega^2}\,\mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}, \tag{B.3-26}$$

and

$$f'(x,\Omega) = \frac{2}{\pi}\int_{\omega=0}^{\infty} \frac{f''(x,\omega)\omega}{\omega^2 - \Omega^2}\,\mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}. \tag{B.3-27}$$

Furthermore, it follows from Equation (B.3-20) that

$$\int_{\omega=0}^{\infty} \frac{1}{\omega^2 - \Omega^2} \, \mathrm{d}\omega = 0 \qquad \text{for} \quad \Omega \in \mathcal{R}. \tag{B.3-28}$$

With the aid of Equation (B.3-28), Equations (B.3-26) and (B.3-27) can be rewritten as

$$f''(x,\Omega) = -\frac{2}{\pi} \int_{\omega=0}^{\infty} \frac{[f'(x,\omega) - f'(x,\Omega)]\Omega}{\omega^2 - \Omega^2} \, \mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}, \tag{B.3-29}$$

and

$$f'(x,\Omega) = \frac{2}{\pi} \int_{\omega=0}^{\infty} \frac{f''(x,\omega)\omega - f''(x,\Omega)\Omega}{\omega^2 - \Omega^2} \, \mathrm{d}\omega \qquad \text{for} \quad \Omega \in \mathcal{R}. \tag{B.3-30}$$

Equations (B.3-29) and (B.3-30) are known as the *Bode relations* (see Bode 1959).

The rewriting of the causality relations in a form that avoids principal-value integrals has its advantages in the numerical evaluation of the relevant integrals.

Exercises

Exercise B.3-1

Show, by considering the Laplace transform $\hat{f} = \hat{f}(s)$ of $f = f(t) = \exp(-\alpha t)H(t)$ with $\alpha > 0$ (see Exercise B.1-6) at $s = \mathrm{j}\omega$, that $f'(\omega) = \alpha/(\omega^2 + \alpha^2)$ and $f''(\omega) = \omega/(\omega^2 + \alpha^2)$ are a pair of Hilbert transforms. (*Hint*: Observe that $\hat{f}(s) = (s + \alpha)^{-1}$ and hence $\hat{f}(\mathrm{j}\omega) = (\mathrm{j}\omega + \alpha)^{-1}$ = $(-\mathrm{j}\omega + \alpha)/(\omega^2 + \alpha^2)$.)

Exercise B.3-2

From the results of Exercise B.3-1 it follows that for $\hat{f}(s)$ = $(s + \alpha)^{-1}$ we have, by direct evaluation of the integral,

$$\int_{\omega=-\infty}^{\infty} f'(\omega) \, \mathrm{d}\omega = \pi \, ,$$

irrespective of the value of α, provided that $\alpha > 0$. Hence, if $\alpha \downarrow 0, f' = f'(\omega)$ approaches, apart from a constant factor, the Dirac delta distribution, since $f'(\omega) \to 0$ as $\alpha \downarrow 0$ if $\omega \neq 0$, and

$$\int_{\omega=-\infty}^{\infty} f'(\omega) \, \mathrm{d}\omega = \pi \, .$$

This leads to the well-known representation of the Dirac delta distribution

$$\mathrm{Re}\left(\frac{1}{\pi s}\right) \to \delta(\omega) \qquad \text{as} \quad s \to \mathrm{j}\omega \quad \text{via} \quad \mathrm{Re}(s) > 0 \, . \tag{B.3-31}$$

B.4 Fourier series and Poisson's summation formula

The Fourier series is a particular representation of a quantity that is periodic in one or more of its variables. The representation employs expansion functions that reflect the pertinent periodicity; the corresponding expansion coefficients follow from the specification of the relevant quantity over a single period. In this section the expansion and the expression for the expansion coefficients will be given for a function that is periodic in time. The discussion about functions that are periodic in one or more of the spatial coordinates runs along similar lines.

Let $f = f(t)$ be a possibly complex function of the real variable t and let f be periodic in t with period $T > 0$. Then

$$f(t) = f(t+T) \qquad \text{for all} \quad t \in \mathcal{R}\,. \tag{B.4-1}$$

From Equation (B.4-1) it follows, by repeatedly replacing t by $t + T$, that

$$f(t) = f(t+nT) \qquad \text{for all} \quad t \in \mathcal{R} \quad \text{and} \quad n = 0, \pm 1, \pm 2, \text{ etc.} \tag{B.4-2}$$

The *Fourier series* associated with $f = f(t)$ now follows by representing f as an expansion in terms of the sequence of functions $\{\exp(\mathrm{j}2\pi mt/T); m = 0, \pm 1, \pm 2, \text{etc.}\}$, each specimen of which is periodic in t with the period T. The representation will be derived from the condition that, for given $N \geqslant 0$, the expression

$$E = \frac{1}{T}\int_{t=0}^{T} \left| f(t) - \sum_{m=-N}^{N} a(m)\exp(\mathrm{j}2\pi mt/T)\right|^2 \mathrm{d}t \geqslant 0 \tag{B.4-3}$$

is minimised. To arrive at the desired expression for $a(m)$, the expression is rewritten as

$$E = \sum_{m=-N}^{N} \left| a(m) - \frac{1}{T}\int_{t=0}^{T} \exp(-\mathrm{j}2\pi mt/T)\, f(t)\,\mathrm{d}t \right|^2$$

$$+ \frac{1}{T}\int_{t=0}^{T} |f(t)|^2 \mathrm{d}t - \sum_{m=-N}^{N} \frac{1}{T^2}\left| \int_{t=0}^{T} \exp(-\mathrm{j}2\pi mt/T)\, f(t)\,\mathrm{d}t \right|^2 , \tag{B.4-4}$$

where the property

$$\frac{1}{T}\int_{t=0}^{T} \exp(-\mathrm{j}2\pi mt/T)\exp(\mathrm{j}2\pi nt/T)\,\mathrm{d}t = \delta(m,n)\,, \tag{B.4-5}$$

in which $\delta(m,n)$ is the Kronecker symbol: $\delta(m,n) = 1$ if $m = n$ and $\delta(m,n) = 0$ if $m \neq n$, has been used. As Equation (B.4-3) shows, the right-hand side of Equation (B.4-4) is non-negative for any choice of the coefficients $a(m)$. Hence, E is minimised if we take

$$a(m) = \frac{1}{T}\int_{t=0}^{T} \exp(-\mathrm{j}2\pi mt/T)\, f(t)\,\mathrm{d}t \qquad \text{for} \quad m = 0, \pm 1, \pm 2, \ldots, \pm N\,. \tag{B.4-6}$$

With this expression, Equation (B.4-4) leads to

$$E = \frac{1}{T}\int_{t=0}^{T} |f(t)|^2 \mathrm{d}t - \sum_{m=-N}^{N} |a(m)|^2\,. \tag{B.4-7}$$

Since, again, $E > 0$, the choice of Equation (B.4-6) implies

$$\sum_{m=-N}^{N} |a(m)|^2 \leqslant \frac{1}{T}\int_{t=0}^{T} |f(t)|^2 \mathrm{d}t \qquad \text{for any} \quad N \geqslant 0\,, \tag{B.4-8}$$

which result is denoted as *Bessel's inequality*. If the class of functions to which $f = f(t)$ belongs is such that $E \to 0$ as $N \to \infty$ the sequence $\{\exp(-\mathrm{j}2\pi mt/T);\ m = 0, \pm 1, \pm 2, \text{etc.}\}$ is denoted as *complete* with respect to this class. In this case, the limit $N \to \infty$ can be taken and we end up with

$$\frac{1}{T}\int_{t=0}^{T} |f(t)|^2 \mathrm{d}t = \sum_{m=-\infty}^{\infty} |a(m)|^2\,, \tag{B.4-9}$$

which result is denoted as the *completeness relation*. In this limit it follows from Equation (B.4-3) since $E \to 0$, that

$$f(t) = \sum_{m=-\infty}^{\infty} a(m) \exp(\mathrm{j}2\pi mt/T)\,, \tag{B.4-10}$$

with

$$a(m) = \frac{1}{T}\int_{t=0}^{T} \exp(-\mathrm{j}2\pi mt/T)\, f(t)\, \mathrm{d}t \qquad \text{for} \quad m = 0, \pm 1, \pm 2, \text{etc.} \tag{B.4-11}$$

Poisson's summation formula

Poisson's summation formula deals with the representation of a periodic function that arises from repeating periodically a given transient function in time (or a localised function in space). Let the pertinent transient function be $\phi = \phi(t)$ and consider the function

$$f(t) = \sum_{n=-\infty}^{\infty} \phi(t + nT)\,, \tag{B.4-12}$$

where $T > 0$. Since, apparently, $f(t + T) = f(t)$ for all t, f is periodic in t with period T. As such, f will, assuming that f is square integrable over a period, admit a Fourier series representation of the type

$$f(t) = \sum_{m=-\infty}^{\infty} a(m) \exp(\mathrm{j}2\pi mt/T)\,, \tag{B.4-13}$$

with (see Equation (B.4-6))

$$a(m) = \frac{1}{T}\int_{t'=0}^{T} \exp(-\mathrm{j}2\pi mt'/T)\left[\sum_{n=-\infty}^{\infty} \phi(t' + nT)\right] \mathrm{d}t' \ \text{ for } m = 0, \pm 1, \pm 2, \text{etc.} \tag{B.4-14}$$

The expression for $a(m)$ is now rewritten as

$$\begin{aligned} a(m) &= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{t'=0}^{T} \exp(-\mathrm{j}2\pi mt'/T)\phi(t' + nT)\, \mathrm{d}t' \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{t''=nT}^{(n+1)T} \exp(-\mathrm{j}2\pi mt''/T)\phi(t'')\, \mathrm{d}t'' \end{aligned}$$

$$= \frac{1}{T} \int_{t''=-\infty}^{\infty} \exp(-j2\pi m t''/T)\phi(t'')\,dt'' . \tag{B.4-15}$$

However,

$$\int_{t''=-\infty}^{\infty} \exp(-j2\pi m t''/T)\phi(t'')\,dt'' = \hat{\phi}(j2\pi m/T) , \tag{B.4-16}$$

where $\hat{\phi} = \hat{\phi}(s)$ denotes the two-sided Laplace transform of $\phi = \phi(t)$, i.e.

$$\hat{\phi}(s) = \int_{t''=-\infty}^{\infty} \exp(-st'')\phi(t'')\,dt'' . \tag{B.4-17}$$

With this, the expression

$$a(m) = \frac{1}{T} \hat{\phi}(j2\pi m/T) \tag{B.4-18}$$

results, and hence, by taking $t = 0$ in Equations (B.4-12) and (B.4-13), we arrive at

$$\sum_{n=-\infty}^{\infty} \phi(nT) = \sum_{m=-\infty}^{\infty} a(m) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \hat{\phi}(j2\pi mt/T)$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{t''=-\infty}^{\infty} \exp(-j2\pi m t''/T)\phi(t'')\,dt'' . \tag{B.4-19}$$

Equation (B.4-19) is *Poisson's summation formula.* This formula is of importance to some numerical integration procedures.

Exercises

Exercise B.4-1

Let $\{\phi(m) = \phi(m;t); m = 0, \pm 1, \pm 2, \text{etc.}\}$ be a given sequence of real- or complex-valued functions that is defined in the (bounded or unbounded) interval $\mathcal{T}$. Assume, further, that the sequence is orthogonal and normalised (or orthonormal), i.e.

$$\int_{t \in \mathcal{T}} \phi(m;t)\phi(n;t)\,dt = \delta(m,n) , \tag{B.4-20}$$

where $\delta(m,n)$ is the Kronecker symbol: $\delta(m,n) = 1$ if $m = n$, $\delta(m,\text{n}) = 0$ if $m \neq n$. Let, now, $f = f(t)$ be a given, real- or complex-valued, function, also defined on the interval $\mathcal{T}$. (a) Derive an expression for the expansion coefficients $\{a(m); m = 0, \pm 1, \pm 2, \text{etc.}\}$ such that, for given $N \geqslant 0$,

$$E = \int_{t \in \mathcal{T}} \left| f(t) - \sum_{m=-N}^{N} a(m)\phi(m;t) \right|^2 dt \geqslant 0 \tag{B.4-21}$$

is minimised. (*Hint*: Rewrite Equation (B.4-21) as

$$E = \sum_{m=-N}^{N} \int_{t \in \mathcal{T}} \left| a(m) - \int_{t \in \mathcal{T}} \phi^*(m;t) f(t)\, \mathrm{d}t \right|^2$$

$$+ \int_{t \in \mathcal{T}} |f(t)|^2\, \mathrm{d}t - \sum_{m=-N}^{N} \left| \int_{t \in \mathcal{T}} \phi^*(m;t) f(t)\, \mathrm{d}t \right|^2 , \tag{B.4-22}$$

observe that the right-hand side is non-negative, from which it follows that E is minimised if

$$a(m) = \int_{t \in \mathcal{T}} \phi^*(m;t) f(t)\, \mathrm{d}t \qquad \text{for} \quad m = 0, \pm 1, \pm 2, \text{etc.} \tag{B.4-23}$$

Here, the asterisk denotes the complex conjugate. If the sequence $\{\phi(m;t); m = 0, \pm 1, \pm 2, \text{etc.}\}$ is complete in the class of functions $f = f(t)$ considered, $E \to 0$ as $N \to \infty$. (b) Show that, under these conditions,

$$\int_{t \in \mathcal{T}} |f(t)|^2\, \mathrm{d}t = \sum_{m=-\infty}^{\infty} |a(m)|^2 \tag{B.4-24}$$

(completeness relation). (*Hint*: Observe that substitution of Equation (B.4-23) in Equation (B.4-22) leads to

$$E = \int_{t \in \mathcal{T}} |f(t)|^2\, \mathrm{d}t - \sum_{m=-N}^{N} |a(m)|^2 \geqslant 0 , \tag{B.4-25}$$

which leads to

$$\sum_{m=-N}^{N} |a(m)|^2 \leqslant \int_{t \in \mathcal{T}} |f(t)|^2\, \mathrm{d}t , \tag{B.4-26}$$

and consider the limit $N \to \infty$.)

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